6

2. (i) Let $t = \frac{1}{x}$, $y_x = \frac{dt}{dx} y_t = \frac{-1}{x^2} y_t$

$$\Rightarrow y_{xx} = \frac{1}{x^4} y_{tt} + \frac{2}{x^3} y_t$$

$$\therefore \frac{1}{x^4} y_{tt} + \frac{2}{x^3} y_t - x^3 \cdot \frac{-1}{x^2} y_t + \frac{1}{x^2} y = 0$$

$$\Rightarrow y_{tt} + 2x\, y_t + x^5 y_t + x^2 y = 0$$

$$\Leftrightarrow y_{tt} + \left(\frac{2}{t} + \frac{1}{t^5}\right) y_t + \frac{1}{t^2} y = 0$$

Since $t\left(\frac{2}{t} + \frac{1}{t^5}\right) = 2 + \frac{2}{t^4}$ not analytic at $t = 0$,

this or equivalently $x = \infty$ is an irregular singular point.

So we use $y(x) = e^{S(x)}$.

$$\Rightarrow S'' + S'^2 - x^3 S' + \frac{1}{x^2} = 0$$

$$\Leftrightarrow$$

$$(S')^2 - x^3 S' = -\frac{1}{x^2} - S''$$

Notice that the leading order equation $S'^2 - x^3 S' + \frac{1}{x^2} = 0$ has two possible solutions:

$$\frac{1}{2}\left\{x^3 \pm \sqrt{x^6 - \frac{4}{x^2}}\right\} \approx \begin{cases} \frac{x^3}{2}\left(1 + 1 - \frac{2}{x^8} + \cdots\right) \\ \frac{x^3}{2}\left(1 - 1 + \frac{2}{x^8} + \cdots\right) \end{cases}$$

$$\sim \begin{cases} x^3 & \text{as } x \to \infty \\ \frac{1}{x^5} & \text{"} \end{cases}$$

The first case gives

$$S' = x^3 - \frac{1}{x^2 S'} - \frac{S''}{S'}$$ (we divide by S' since it is not zero to leading order)

$$\sim x^3 - \frac{1}{x^5} - \frac{S''}{S'}$$ (Note: $\log S' \sim \log(x^3)$)

$$\Rightarrow S \sim \frac{x^4}{4} - 3\log(x) + \text{const}$$

The second case gives $\quad -x^3 S' = -\dfrac{1}{x^2} - S'^2 - S''$

$$\Rightarrow \quad S' = \frac{1}{x^5} + \frac{S'^2}{x^3} + \frac{S''}{x^3}$$

$$\sim \frac{1}{x^5}$$

$$\Rightarrow \quad S \sim -\frac{x^{-4}}{4} + \text{const}$$

$\therefore$ In summary, we get

$$y_1(x) \sim c_1 \, x^{-3} \exp\left(\frac{x^4}{4}\right) \qquad \text{as } x \to +\infty.$$

$$y_2(x) \sim c_2 \left(1 - \frac{1}{4x^4} + \dots\right) \qquad \text{as } x \to +\infty$$

Note that $y_1(x)$ may include a multiple of $y_2(x)$ beyond all orders.