2 (ii) $y(x) = P(x)\,w(x)$

$\Rightarrow$

$$y' = Pw' + P'w$$
$$y'' = Pw'' + 2P'w' + P''w$$

$$\Rightarrow \quad Pw'' + (2P' - x^3 P)\,w' + \left(P'' - x^3 P' + \frac{P}{x^2}\right) w = 0$$

To get equation of the form $w'' = Q(x)\,w$, we must have

$$2P' - x^3 P = 0$$

$\Rightarrow$

$$\frac{P'}{P} = \frac{x^3}{2}$$

$$\Rightarrow \quad \log P = \frac{x^4}{8}$$

$$\Rightarrow \quad P = \exp\left(\frac{x^4}{8}\right).$$

Consider $Q(x) = -\dfrac{P''}{P} + x^3 \dfrac{P'}{P} + \dfrac{1}{x^2}$

$$= -\left(\frac{P'}{P}\right)' - \left(\frac{P'}{P}\right)^2 + x^3 \frac{P'}{P} + \frac{1}{x^2}$$

$$= -\frac{3}{2}x^2 - \frac{x^6}{4} + \frac{x^6}{2} + \frac{1}{x^2}$$

$$= \frac{x^6}{4} - \frac{3}{2}x^2 + \frac{1}{x^2}$$

Olver's theorem provides two solutions w_1, w_2 for appropriate $f(x)$, $g(x)$.

We choose $f(x) = \dfrac{x^6}{4} - \dfrac{3}{2}x^2 + \dfrac{1}{x^2} > 0$ for large enough x.

$$g(x) \equiv 0.$$

The solutions are

$$w_1(x) = \left(\frac{x^6}{4} - \frac{3}{2}x^2 + \frac{1}{x^2}\right)^{-1/4} \exp\left\{\int^x \sqrt{\frac{x^6}{4} - \frac{3}{2}x^2 - \frac{1}{x^2}}\, dx\right\} \left(1 + \varepsilon_1(x)\right)$$

$$w_2(x) = \left(\frac{x^6}{4} - \frac{3}{2}x^2 + \frac{1}{x^2}\right)^{-1/4} \exp\left\{-\int^x \sqrt{\frac{x^6}{4} - \frac{3}{2}x^2 - \frac{1}{x^2}}\, dx\right\} \left(1 + \varepsilon_2(x)\right)$$

where $|\varepsilon_2(x)| < \exp\left(\frac{1}{2} \mathcal{V}_{x,\infty}(F)\right) - 1$

with $\mathcal{V}_{x,\infty}(F) = \int_x^\infty |F'|\, dx$

where

$$F'(x) = \left(f(x)\right)^{-1/4} \left(\left(f(x)\right)^{-1/4}\right)''$$

$$= -\frac{1}{4} f^{-1/4} \left(f' \cdot f^{-5/4}\right)'$$

$$= -\frac{1}{4} f^{-1/4} \left(f'' f^{-5/4} - \frac{5}{4}(f')^2 f^{-9/4}\right)$$

$$= -\frac{1}{4} f'' f^{-3/2} + \frac{5}{16} (f')^2 f^{-5/2}$$

$$\therefore |F'| \leq \frac{1}{4}\left|\frac{f''}{f^{3/2}}\right| + \frac{5}{16}\left|\frac{(f')^2}{f^{5/2}}\right|$$

Assume $x \geq 2 \Rightarrow x^4 > 12$.

$$\Rightarrow x^6 > 12x^2$$

$$\Rightarrow \frac{x^6}{8} > \frac{3}{2}x^2 \quad \therefore \frac{x^6}{8} - \frac{3}{2}x^2 + \frac{1}{x^2} > 0$$

$$\Rightarrow f = \frac{x^6}{8} + \frac{x^6}{8} - \frac{3}{2}x^2 + \frac{1}{x^2} > \frac{x^6}{8}$$

Also

$$f' = \frac{3}{2}x^5 - 3x - \frac{2}{x^3} < \frac{3}{2}x^5$$

$$f'' = \frac{15}{2}x^4 + \frac{1}{x^4}\left(-3x^4 + 6\right) < \frac{15}{2}x^4$$

$$\therefore f'' f^{-3/2} < \frac{15}{2}x^4 . 2^{9/2} x^{-9} = 15 \times 2^{7/2} x^{-5}$$

$$(f')^2 f^{-5/2} < \frac{9}{4} x^{10} . 2^{15/2} x^{-15} = 9 \times 2^{11/2} x^{-5}$$

$\therefore$

$$|F'| \leq \frac{5}{4} \cdot 2^{7/2} \left(\cancel{3} + 9\right) x^{-5}$$

$$= 15 \times 2^{7/2} x^{-5}$$

$$< 200 \times x^{-5}$$

$$\therefore \quad V_{x,\infty}(F) < 200 \int_x^{\infty} x^{-5}\, dx$$

$$= 200 \left[\frac{x^{-4}}{-4} \right]_x^{\infty}$$

$$= 50 \, x^{-4}$$

$\therefore$ It is finite.

We use $e^{\delta} - 1 < 2\delta$, for $\delta < \frac{1}{2}$, to deduce that

$$|\varepsilon_2(x)| \leq 2 \times 25 \, x^{-4} = 50 \, x^{-4}$$