MAPT 2009
Assignment #2

Q.1.(i)

$$y(x) = \frac{1}{2\pi}\int_\gamma e^{ikx}\,\hat{y}(k)\,dk$$

$\Rightarrow$

$$y'(x) = \frac{1}{2\pi}\int_\gamma e^{ikx}\,ik\,\hat{y}(k)\,dk$$

$$y''(x) = \frac{1}{2\pi}\int_\gamma e^{ikx}\,(ik)^2\,\hat{y}(k)\,dk$$

$$y'''(x) = \frac{1}{2\pi}\int_\gamma e^{ikx}\,(ik)^3\,\hat{y}(k)\,dk$$

and

$$x\,y(x) = \frac{1}{2\pi}\int_\gamma \frac{ix}{i}\,e^{ikx}\,\hat{y}(k)\,dk$$

$$= \frac{-i}{2\pi}\int_\gamma \frac{d}{dk}\left(e^{ikx}\right)\hat{y}(k)\,dk$$

$$= \frac{i}{2\pi}\int_\gamma e^{ikx}\,\frac{d\hat{y}(k)}{dk}\,dk\,,$$

by integration by parts (assuming γ is chosen s.t. the product $e^{ikx}\hat{y}(k)$ vanishes at both ends of γ.)

$\therefore$ Eqn (1.1) $\Rightarrow$ $\quad i\dfrac{d\hat{y}}{dk} = -i\,k^3\,\hat{y}$

$\Rightarrow \quad \log \hat{y} = -\dfrac{k^4}{4} + \text{const}$, take const $= 0$ w.l.o.g.

$\Rightarrow \quad \hat{y}(k) = \exp\left(-k^4/4\right)$

$$\therefore \quad y(x) = \frac{1}{2\pi}\int_\gamma \exp\left(ikx - k^4/4\right)dk$$

We require γ such that $e^{ikx - k^4/4} \to 0$ as $|k| \to \infty$.

$\therefore$ Letting $k = |k| e^{i\theta}$, we need

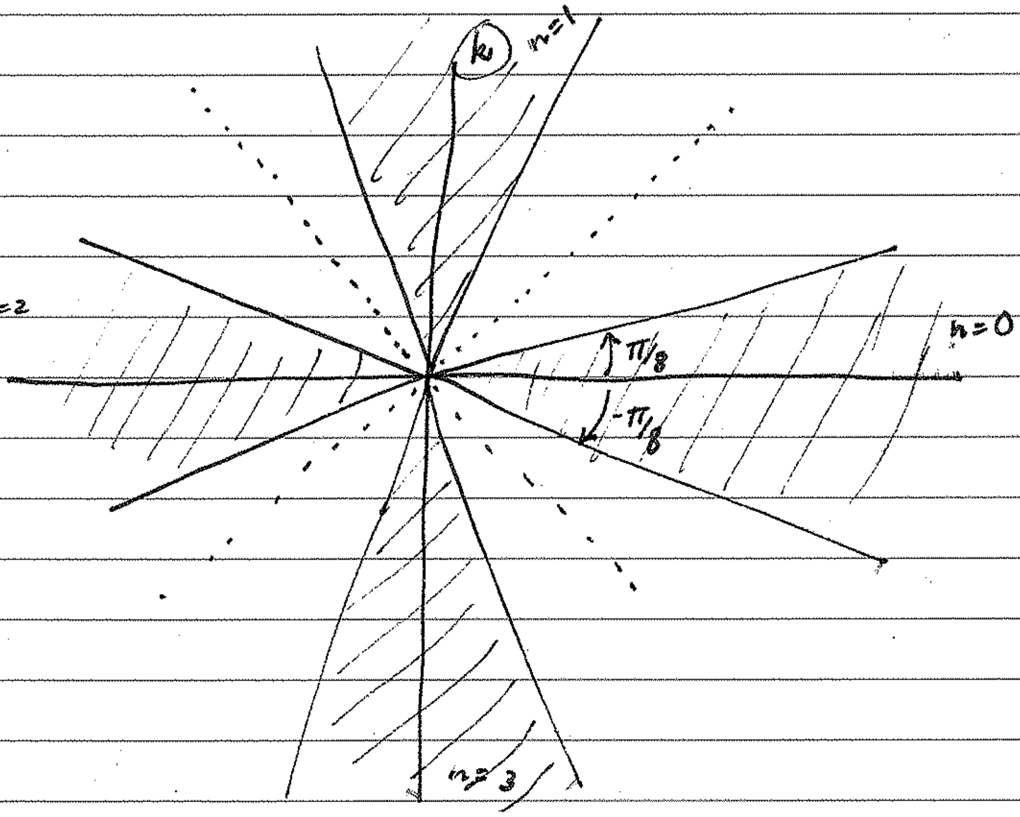
$$\operatorname{Re}\left(ikx - k^4/4\right) < 0$$

$$\Rightarrow \quad -\cos(4\theta) < 0$$

$$\Rightarrow \quad \cos(4\theta) > 0$$

$$\Rightarrow \quad -\frac{\pi}{2} + 2n\pi < 4\theta < \frac{\pi}{2} + 2n\pi, \quad n \in \mathbb{Z}$$

$$\Rightarrow \quad -\frac{\pi}{8} + \frac{n\pi}{2} < \theta < \frac{\pi}{8} + \frac{n\pi}{2}$$



Ends of γ must lie in two of the shaded sectors.