Q1 (ii) Let $k = i x^{1/3} \lambda$.

$\Rightarrow$
$$ikx - k^4/4 = i \cdot i x^{1/3} \lambda \cdot x - x^{4/3} \lambda^4/4$$
$$= -x^{4/3} (\lambda + \lambda^4/4).$$

$\Rightarrow$
$$Y_0(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} i x^{1/3} . \exp\left(x^{4/3} \rho(\lambda)\right) \, d\lambda$$

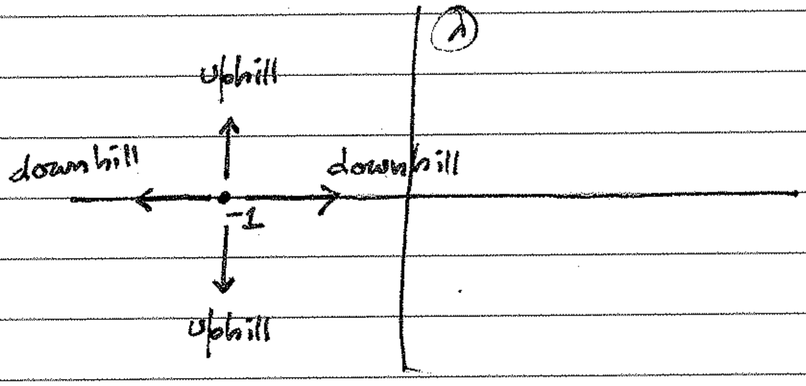
where
$$\rho(\lambda) = -\left(\lambda + \lambda^4/4\right)$$

$$\rho'(\lambda) = -\left(1 + \lambda^3\right) \Rightarrow \text{saddle points at } \lambda = -1, \omega, 1/\omega$$
where $\omega = \exp(-i\pi/3)$.

We use the saddle point method through the saddle point at $\lambda = -1$.

Let $\lambda = -1 + \xi + i\eta$, $\xi, \eta << 1$
$$\rho(\lambda) = -\left(-1 + \xi + i\eta + \frac{1}{4}\left(1 - 4\xi - 4i\eta + 6(\xi^2 + 2i\xi\eta - \eta^2) + \dots\right)\right)$$

$$= \frac{3}{4} - \frac{3}{2}\xi^2 + \frac{3}{2}\eta^2 - 3i\xi\eta + O\left(\xi^3, \eta^3, \xi^2\eta, \xi\eta^2\right)$$



Take the descent path along the real λ-axis through $\lambda = -1$.

Note that
$$\rho(\lambda) = \frac{3}{4} - \frac{3}{2}(\lambda+1)^2 + O\left((\lambda+1)^3\right).$$

$$\therefore Y_0(x) \sim \frac{i x^{1/3}}{2\pi} \int_{-\infty}^{\infty} \exp\left(x^{4/3}\left(\frac{3}{4} - \frac{3}{2}(\lambda+1)^2\right)\right) d\lambda$$

$$= \frac{i x^{1/3}}{2\pi} e^{\frac{3}{4}x^{4/3}} \int_{-\infty}^{\infty} \exp\left[-\frac{3}{2} x^{4/3} (\lambda+1)^2\right] d\lambda$$

let $\lambda + 1 = \sqrt{\frac{2}{3}}\, x^{-2/3} t$

$$= \frac{i x^{1/3}}{2\pi} \cdot \sqrt{\frac{2}{3}}\, x^{-2/3} e^{\frac{3}{4}x^{4/3}} \int_{-\infty}^{\infty} e^{-t^2} dt$$

$$= \frac{i x^{-1/3}}{\sqrt{6}\,\pi} e^{\frac{3}{4}x^{4/3}} \cdot \sqrt{\pi}$$

$$= \frac{i}{\sqrt{6\pi}} x^{-1/3} e^{\frac{3}{4}x^{4/3}} \qquad \text{as } x \to +\infty .$$