Q.2 (iii) A] $y_0(x) = \frac{\beta}{2}(x^2+1)$

$$y_0(1) = \frac{\beta}{2}\cdot 2 = \beta$$
$$y_0(0) = \frac{\beta}{2} = \alpha$$

B] $y_{outer}(x;\varepsilon) = y_0(x) + \varepsilon^2 y_1(x) + \cdots$

$$\Rightarrow (x^2+1)\,y_1' - 2x\,y_1 = -y_0'' = -\beta, \quad y_1(1) = 0$$

$$\Rightarrow \left(\frac{y_1}{(x^2+1)}\right)' = \frac{-\beta}{(x^2+1)^2}.$$

Note: $\displaystyle\int \frac{dx}{(x^2+1)^2} = \int \cos^2\theta \, d\theta,$ where $x = \tan\theta$ since $x^2+1 = \sec^2\theta$ and $dx = \sec^2\theta\, d\theta$

$$= \frac{1}{2}\int (\cos(2\theta)+1)\, d\theta$$

$$= \frac{1}{4}\sin(2\theta) + \frac{\theta}{2}$$

$$= \frac{1}{2}\sin\theta\cos\theta + \frac{\theta}{2}$$
$$= \frac{1}{2}\tan\theta\cdot\cos^2\theta + \frac{\theta}{2}$$

$$= \frac{1}{2}x\left(\frac{1}{x^2+1}\right) + \tan^{-1}(x)$$

$$\therefore\ y_1(x) = (x^2+1)\left\{-\frac{\beta}{2}\frac{x}{(x^2+1)} - \beta\tan^{-1}(x) + c_1\right\}$$

$$\therefore\ y_1(1) = -\frac{\beta}{2} - 2\beta\cdot\frac{\pi}{4} + 2c_1 = -\frac{\beta}{2}(1+\pi) + 2c_1$$

$$\Rightarrow c_1 = \frac{\beta}{4}(1+\pi)$$

i.e. $y_1(0) = \frac{\beta}{4}(1+\pi) = \frac{\alpha}{2}(1+\pi)$ not necessarily 0.

Now consider the inner solution

$$y_{inner}(x;\varepsilon) = \hat{y}_0(\eta) + \varepsilon^2 \hat{y}_1(\eta) + \cdots$$

$$\Rightarrow \quad \hat{y}_{1,\eta\eta} + \hat{y}_{1,\eta} = 0, \qquad \hat{y}_1(0) = 0$$

$$\Rightarrow \quad \hat{y}_1(\eta) = a_1 + b_1 e^{-\eta}$$ as before but now the condition at $x=0$ is different.

$$\hat{y}_1(0) = a_1 + b_1 = 0 \Rightarrow b_1 = -a_1$$

$$\therefore \quad \hat{y}_1(\eta) = a_1 - a_1 e^{-\eta}$$

<u>Matching</u>

$$\xi = \frac{x}{\mu} \Rightarrow \eta = \frac{\mu}{\varepsilon^2}\xi$$

Require $\mu \to 0^+$, $\frac{\varepsilon^2}{\mu} \to 0^+$ as $\varepsilon \to 0^+$.

Note that for small x, we have

$$\tan^{-1} x = \frac{1}{2i}\log\left(\frac{1+ix}{1-ix}\right)$$

$$= \frac{1}{2i}\left\{ ix - \frac{(ix)^2}{2} + \frac{(ix)^3}{3} + \cdots - \left(-ix - \frac{(-ix)^2}{2} + \frac{1}{3}(-ix)^3 + \cdots\right)\right\}$$

$$= \frac{1}{2i}\left\{ 2ix + \frac{2}{3}ix^3 + \cdots \right\}$$

$$= x + \frac{x^3}{3} + \cdots$$

$$\therefore \quad y_0(x) = \frac{\beta}{2}\left(1 + \mu^2\xi^2\right)$$

$$y_1(x) = -\frac{\beta}{2}\mu\xi - \beta\left(1+\mu^2\xi^2\right)\left\{\mu\xi + \frac{\mu^3\xi^3}{3} + \cdots\right\}$$
$$+ \frac{\beta}{4}(1+\pi)\left(1 + \mu^2\xi^2\right)$$

while

$$\hat{y}_0(\eta) = a_0 + (\alpha - a_0)\, e^{-\frac{\mu}{\varepsilon^2}\xi}$$

$$\hat{y}_1(\eta) = a_1 - a_1\, e^{-\frac{\mu}{\varepsilon^2}\xi}$$

$$\therefore\ \frac{\beta}{2}\left(1 + \mu^2\xi^2\right) - \frac{\beta}{2}\mu\varepsilon^2\xi - \beta\varepsilon^2\mu\xi + O(\varepsilon^2\mu^3\xi^3)$$
$$+ \frac{\beta}{4}\varepsilon^2(1+\pi) + O(\varepsilon^2\mu^2\xi^2)$$
$$= a_0 + \varepsilon^2 a_1 + O\left(e^{-\frac{\mu}{\varepsilon^2}\xi}\right) + \varepsilon^4\hat{y}_2\left(\frac{\mu}{\varepsilon^2}\xi\right)$$

Assuming $\mu^2 \ll \varepsilon^2$ (while $\frac{\varepsilon^2}{\mu} \to 0$; e.g. this can happen if $\mu = \varepsilon^{3/2}$),

we have

$$a_0 = \frac{\beta}{2}$$

$$a_1 = \frac{\beta}{4}(1+\pi)$$

Note: the other terms can match with $\varepsilon^4\hat{y}_2(\eta)$ to leadin