Q2 (i) $a(x) = x^2+1 > 0$

$\therefore$ BL is at $x=0$

(ii) $\beta \neq 2\alpha$.

A] Let $y_{outer}(x;\varepsilon) = y_0(x) + \varepsilon^2 y_1(x) + \cdots$

$$\Rightarrow \quad (x^2+1)\,y_0' - 2x y_0 = 0\,, \quad y_0(1) = \beta$$

$$\Rightarrow \quad \frac{y_0'}{y_0} = \frac{2x}{x^2+1}$$

$$\Rightarrow \quad \log y_0 = \log(x^2+1) + \text{const}$$

$$\Rightarrow \quad y_0(x) = c_0\,(x^2+1)$$

$$y_0(1) = \beta = c_0(2) \quad \Rightarrow \quad c_0 = \frac{\beta}{2}$$

$$\therefore \quad y_0(x) = \frac{\beta}{2}(x^2+1)$$

$$\lim_{x\to 0^+} y_0(x) = \frac{\beta}{2} \neq \alpha$$

$\therefore$ BL is necessary at $x=0$.

B] $$\eta = \frac{x}{\delta(\varepsilon)} \quad \Rightarrow \quad \frac{\varepsilon^2}{\delta^2}\, y_{\eta\eta} + \frac{(1+\delta^2\eta^2)}{\delta}\, y_\eta - 2\delta\eta y = 0$$

Need $\dfrac{\varepsilon^2}{\delta^2} = \dfrac{1}{\delta} \Rightarrow \delta = \varepsilon^2$.

I.e., $$y_{\eta\eta} + (1+\varepsilon^4\eta^2)\,y_\eta - 2\varepsilon^4\eta y = 0$$

$$\therefore \quad y_{inner}(x;\varepsilon) = \hat{y}_0(\eta) + \varepsilon^2\hat{y}_1(\eta) + \cdots$$

$$\Rightarrow \quad \hat{y}_{0\,\eta\eta} + \hat{y}_{0\,\eta} = 0\,, \quad \hat{y}_0(0) = \alpha$$

$$\Rightarrow \quad \hat{y}_{0\eta} + \hat{y}_0 = a_0,$$

$$\Rightarrow \quad \hat{y}_0 e^{\eta} = a_0 e^{\eta} + b_0$$

$$\Leftrightarrow \quad \hat{y}_0(\eta) = a_0 + b_0 e^{-\eta}$$

$$\hat{y}_0(0) = \alpha = a_0 + b_0 \quad \Rightarrow \quad b_0 = \alpha - a_0$$

$$\therefore \quad \hat{y}_0(\eta) = a_0 + (\alpha - a_0) e^{-\eta}$$

Matching to leading order:

$\Rightarrow$

$$\frac{\beta}{2} = a_0$$

More precisely, define the matching variable $\xi = \dfrac{x}{\mu(\varepsilon)}$

$\Rightarrow \eta = \dfrac{\mu}{\varepsilon^2}\xi$. Need $\mu \to 0^+$ as $\varepsilon \to 0^+$

$\dfrac{\varepsilon^2}{\mu} \to 0^+$ as $\varepsilon \to 0^+$

$$\Rightarrow \quad y_{outer} \sim \frac{\beta}{2}\left(1 + \mu^2 \xi^2\right) \sim \frac{\beta}{2}$$

$$y_{inner} \sim a_0 + (\alpha - a_0) e^{-\frac{\mu}{\varepsilon^2}\xi} \sim a_0$$

$$\therefore \quad a_0 = \frac{\beta}{2}.$$