

**Assignment 2**

Lecturer: *Nalini Joshi*

**Due: 4pm Thursday 08 October 2009**

*in Carlaw Room 520*

1. Consider the ODE

$$(1.1) \quad y''' = xy,$$

(i) Use a generalized Fourier transform

$$\hat{y}(k) = \int_{\tilde{\gamma}} e^{-ikx} y(x) dx$$
$$y(x) = \frac{1}{2\pi} \int_{\gamma} e^{ikx} \hat{y}(k) dk$$

to deduce an integral representation of the solutions for appropriate paths  $\gamma$ . Find the pairs of sectors in the complex  $k$ -plane where the ends of the paths  $\gamma$  can lie in order for the integral representation to be convergent.

(ii) Let  $Y_0(x)$  be a solution of Equation (1.1) obtained by taking an integral representation with path  $\gamma$  that lies along the imaginary axis, where  $\gamma$  is directed from the negative imaginary axis to the positive imaginary axis. By using this integral representation, show that

$$Y_0(x) \sim \frac{i}{\sqrt{6\pi}} x^{-1/3} \exp(3x^{4/3}/4) \quad \text{as } x \rightarrow +\infty.$$

2. Suppose  $0 < \epsilon \ll 1$  is given. Consider the ODE

$$(2.2) \quad \epsilon^2 y'' + (x^2 + 1)y' - 2xy = 0,$$

governing  $y(x)$  for  $x \in [0, 1]$ , where the solution satisfies the boundary conditions

$$y(0) = \alpha, y(1) = \beta.$$

(i) Where do you expect to see a boundary layer?

(ii) Consider the case  $\beta \neq 2\alpha$ .

*A:* Find the leading-order term  $y_0(x)$  in the outer expansion of  $y(x)$ . Show that for this case, the approximate solution fails to satisfy the boundary condition at the point that is contained in the boundary layer.

*B:* Find the first term  $\hat{y}_0(\eta)$  in the inner expansion of  $y(x)$ , where  $\eta = x/\delta(\epsilon)$  for an appropriate  $\delta$ .

*C:* Find the values of any free constants in your results for parts *A* and *B*.

(iii) Consider the case  $\beta = 2\alpha$ .

*A:* Show that  $y_0(x)$  satisfies both boundary conditions, i.e., the one at  $x = 0$  and  $x = 1$ .

*B:* Show by deducing the next (non-zero) term in each of the outer and inner expansions that a boundary layer is still needed.

*C:* Find the values of any free constants in your results for part *B* by carrying out a process of asymptotic matching.