

Assignment 2

Lecturer: *Nalini Joshi*

Due: 2pm Thursday 18 May 2010

in Carlaw Room 830

1. Consider the ODE

$$(1.1) \quad \frac{d^4 y}{dx^4} = x y,$$

(i) Use a generalized Fourier transform

$$(1.2a) \quad \widehat{y}(k) = \int_{\gamma} e^{-ikx} y(x) dx$$

$$(1.2b) \quad y(x) = \frac{1}{2\pi} \int_{\gamma} e^{ikx} \widehat{y}(k) dk$$

to deduce an integral representation of the solutions for appropriate paths γ .

- (ii) Suppose the integrand in Equation (1.2b) is written as $\exp(ix^\mu \rho(\lambda))$, where $\lambda = x^\nu k$, for appropriate μ and ν . For $x \in \mathbb{R}$, $x > 0$, draw the sectors S_j , $j = 1, \dots, 5$, of convergence of (1.2b) in the λ -plane. Hence find the pairs of sectors where the ends of the paths γ can lie in order for the integral representation to be convergent.
- (iii) For $x \in \mathbb{R}$, $x > 0$, let $Y_0(x)$ be a solution of Equation (1.1) obtained by taking an integral representation with path γ_0 whose ends lie in the interior of the upper half plane. By using the method of steepest descents, find the unique leading-order asymptotic behaviour of $Y_0(x)$ as $x \rightarrow +\infty$. Why does this define a unique solution?
- (iv) Let $x = r e^{i\theta}$, where $\theta > 0$. Draw the sectors S_j , $j = 1, \dots, 5$ again (in the λ -plane) for (A) $\theta = 4\pi/5$ and (B) $\theta = 8\pi/5$. In case (B), what is the asymptotic behaviour of $Y_0(x)$ as $r \rightarrow +\infty$?

2. The following boundary value problem arises from plasma physics.

$$(2.1a) \quad \frac{d^2 \psi}{dx^2} + \left(\frac{2}{x} - \frac{\epsilon}{x^2} \right) \frac{d\psi}{dx} - \psi = 0$$

$$(2.1b) \quad \psi(0) = -1, \quad \lim_{x \rightarrow +\infty} \psi(x) = 0, \quad \epsilon \rightarrow 0^+$$

- (i) Where do you expect to see a boundary layer?
- (ii) Find the leading-order term $y_0(x)$ in the outer expansion of $y(x)$. Show that the approximate solution fails to satisfy the boundary condition at the point that is contained in the boundary layer.
- (iii) Find the first term $\widehat{y}_0(\eta)$ in the inner expansion of $y(x)$, where $\eta = x/\delta(\epsilon)$ for an appropriate δ .
- (iv) Find the values of any free constants in your results for $y_0(x)$ and $\widehat{y}_0(\eta)$.