When Applied Mathematics Collided with Algebra

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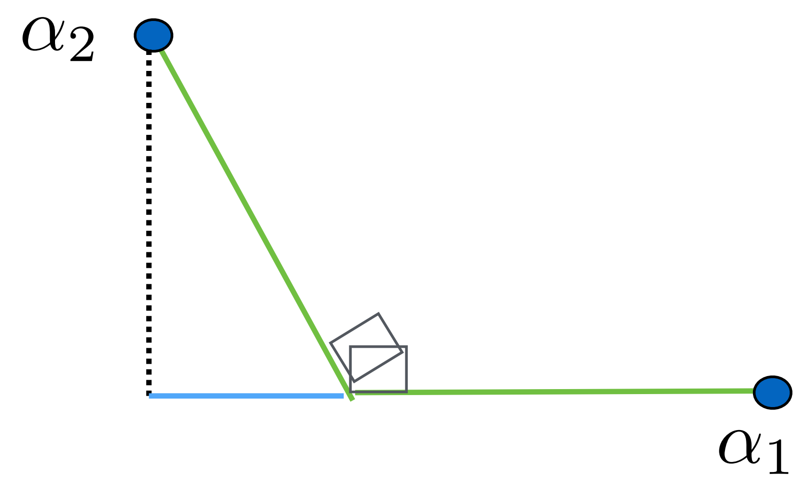
Supported by the Australian Research Council

A Reflection



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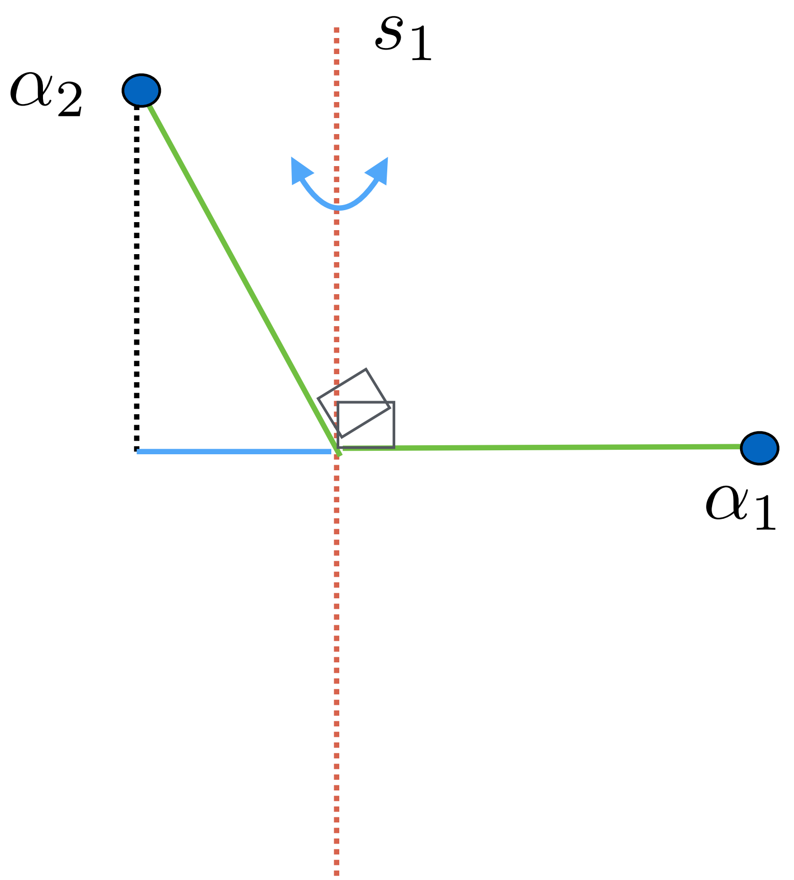

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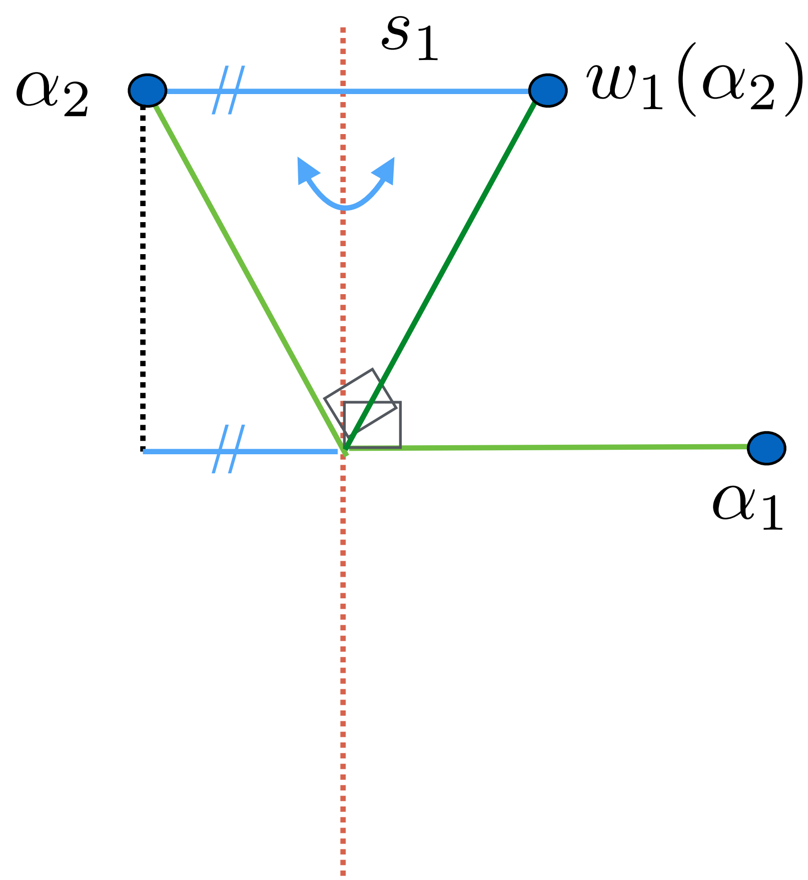

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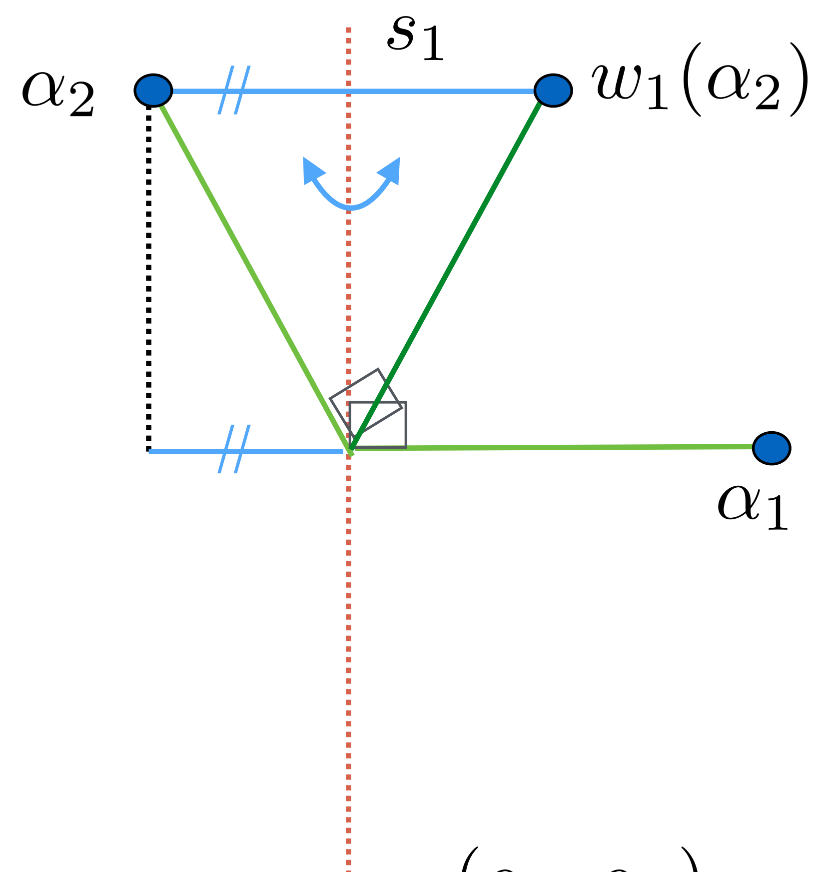
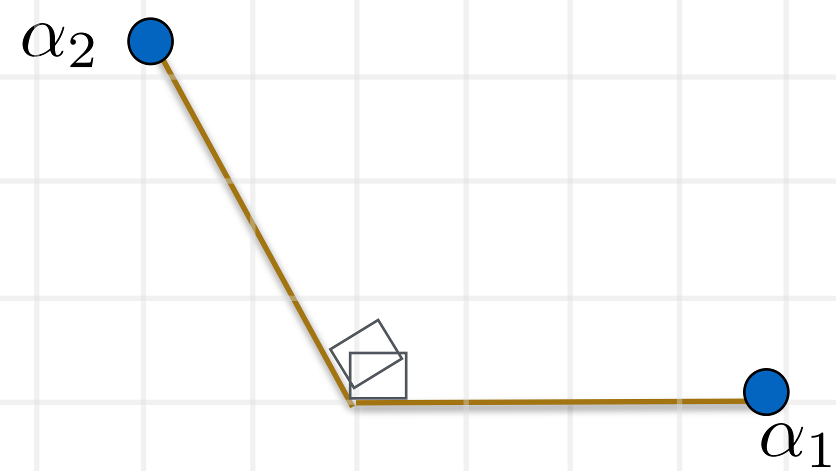


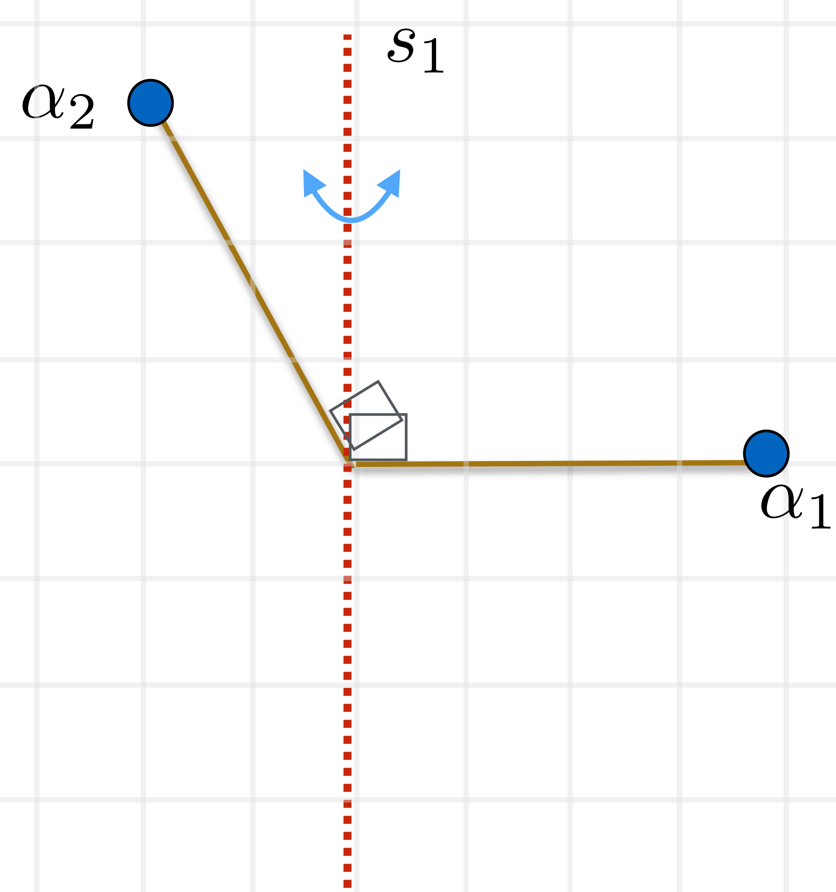
$$
\begin{aligned}
w_1(\alpha_2) &= \alpha_2 - 2\,\frac{(\alpha_1, \alpha_2)}{(\alpha_1, \alpha_1)}\,\alpha_1 \\
&= (-1, \sqrt{3}) + (2, 0) \\
&= (1, \sqrt{3})
\end{aligned}
$$

Root System



α_1 and α_2 are "simple" roots

Root System



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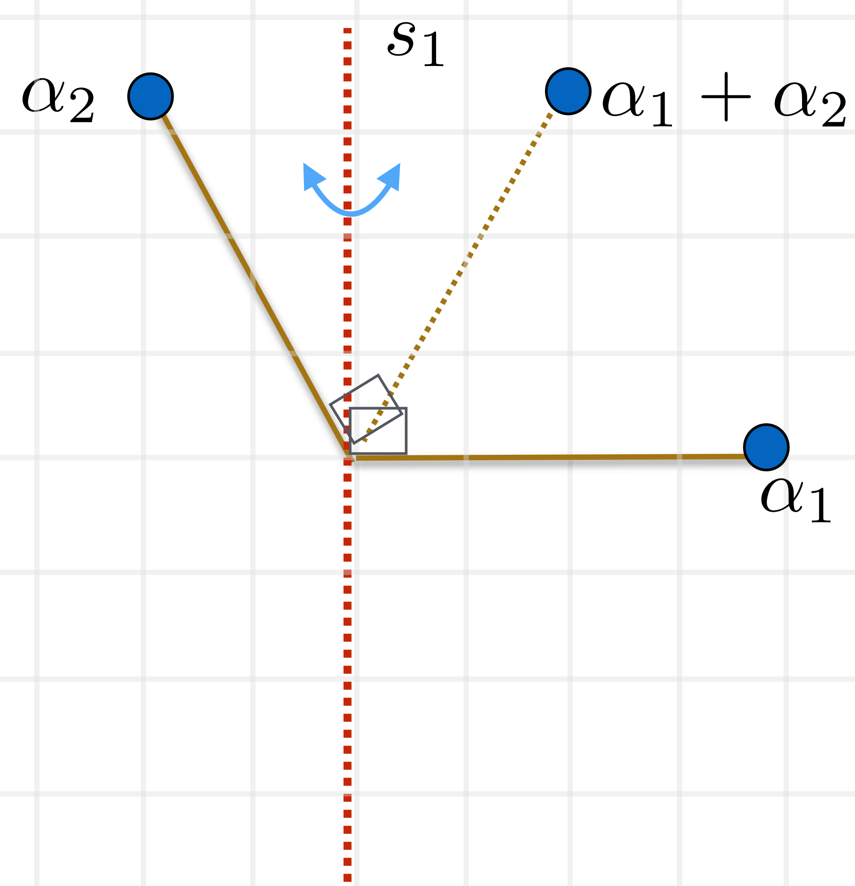

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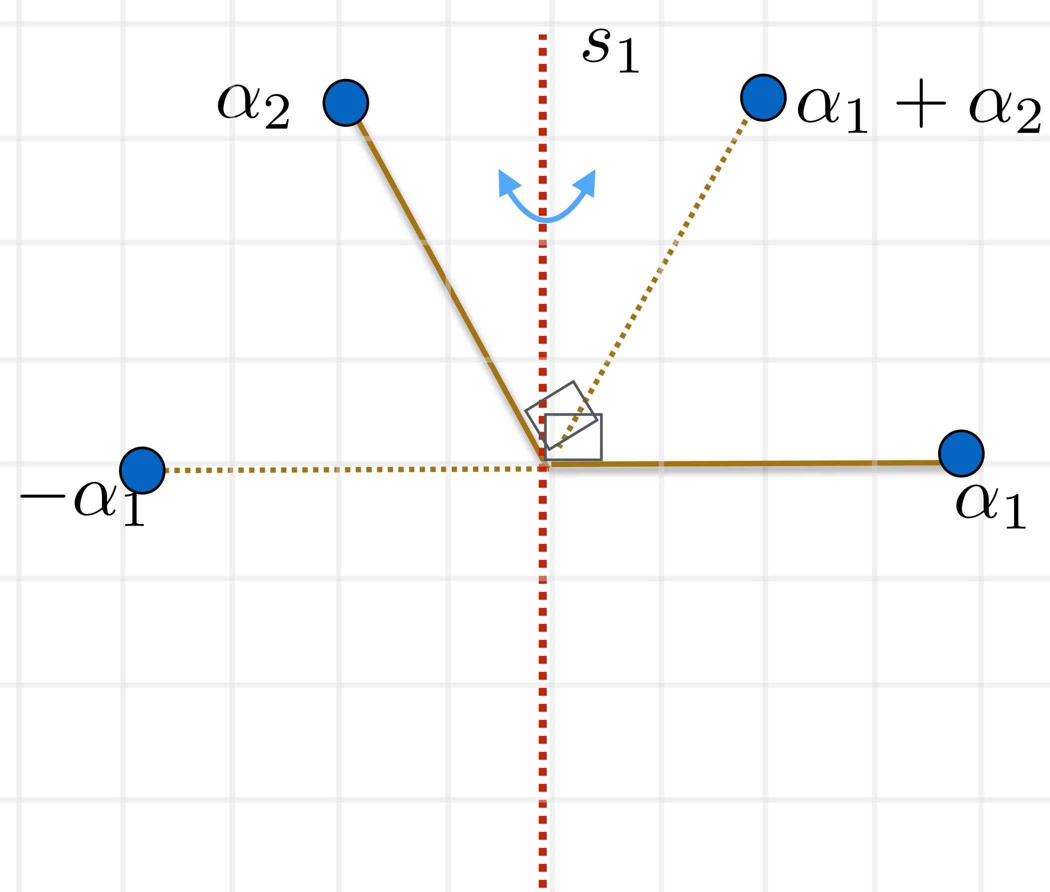

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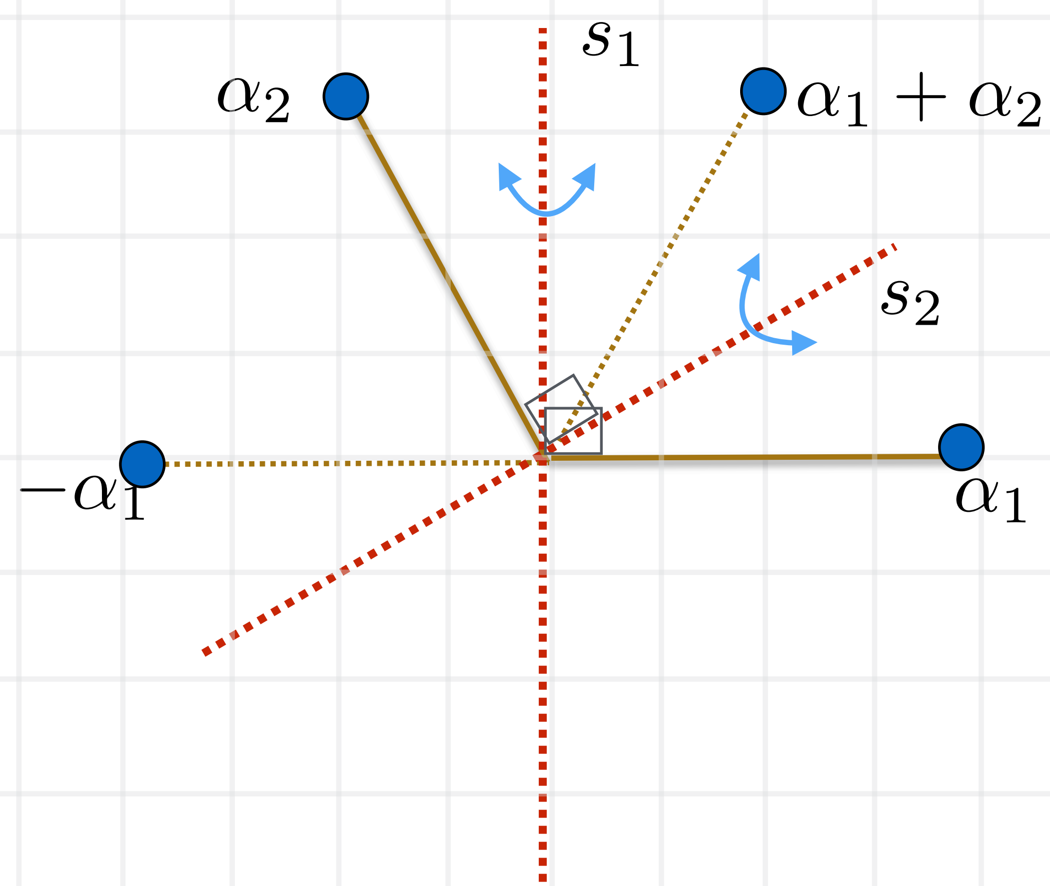

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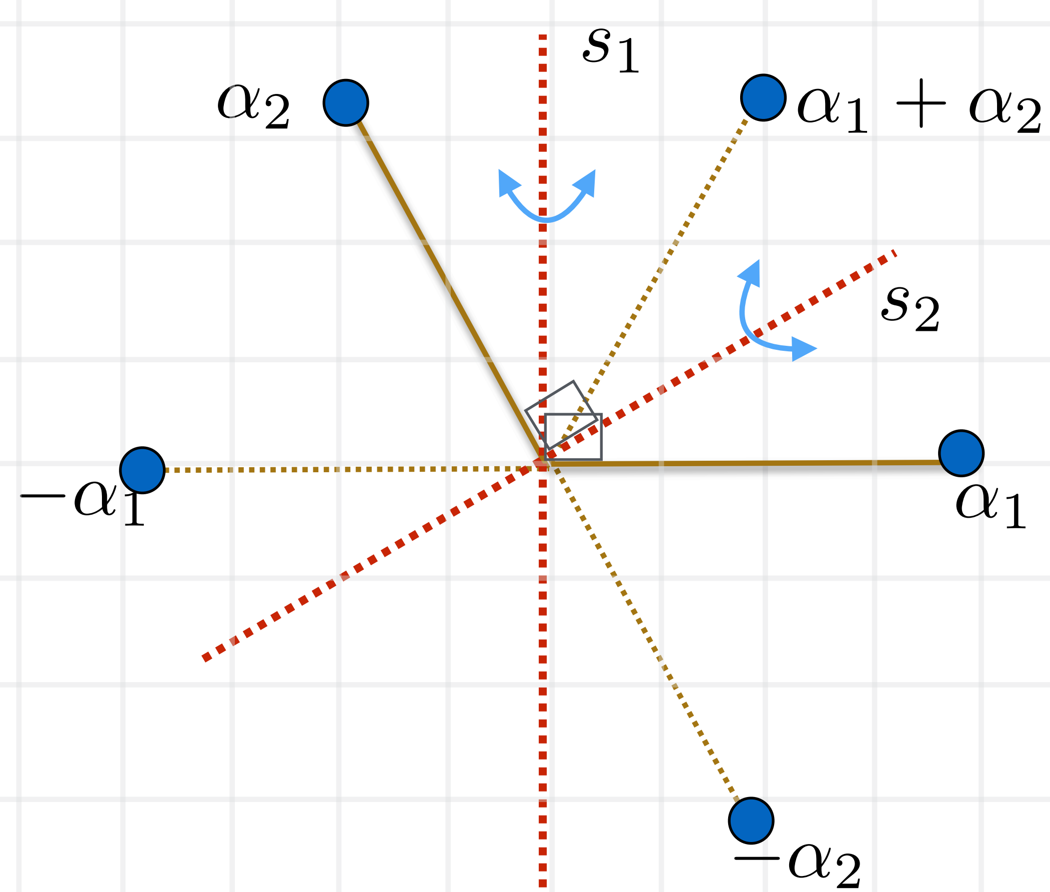

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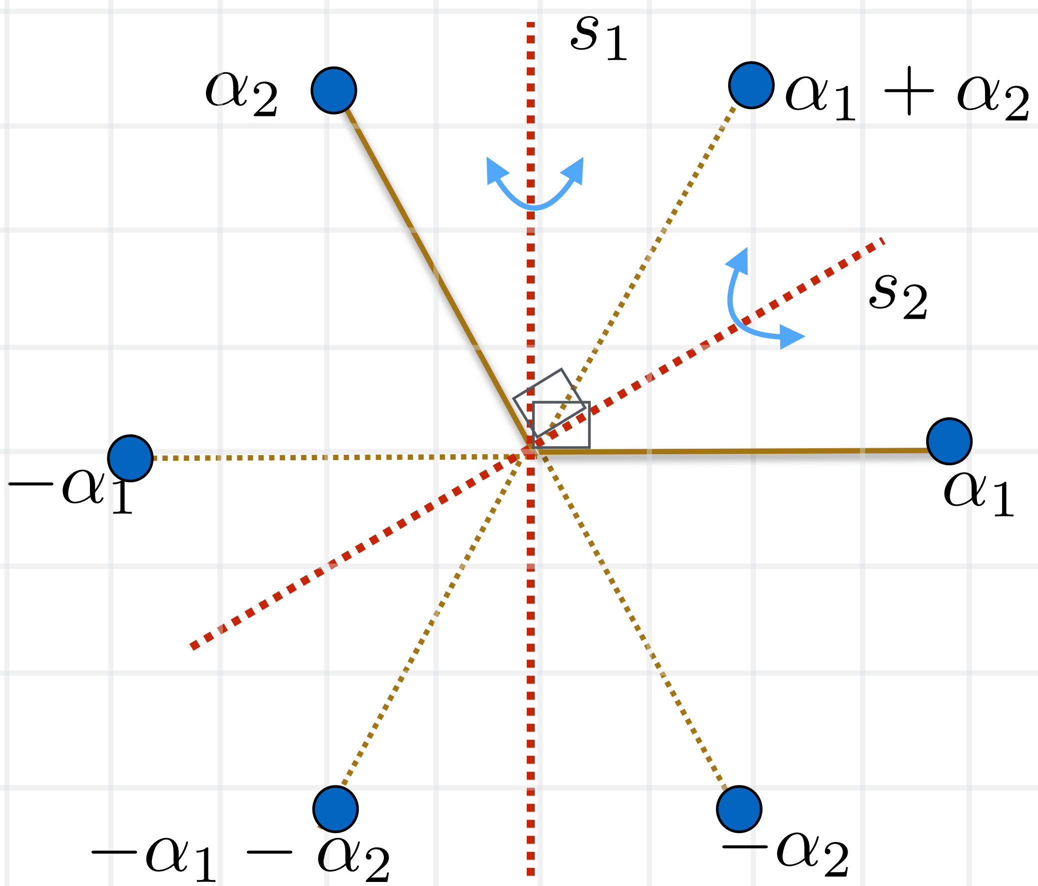

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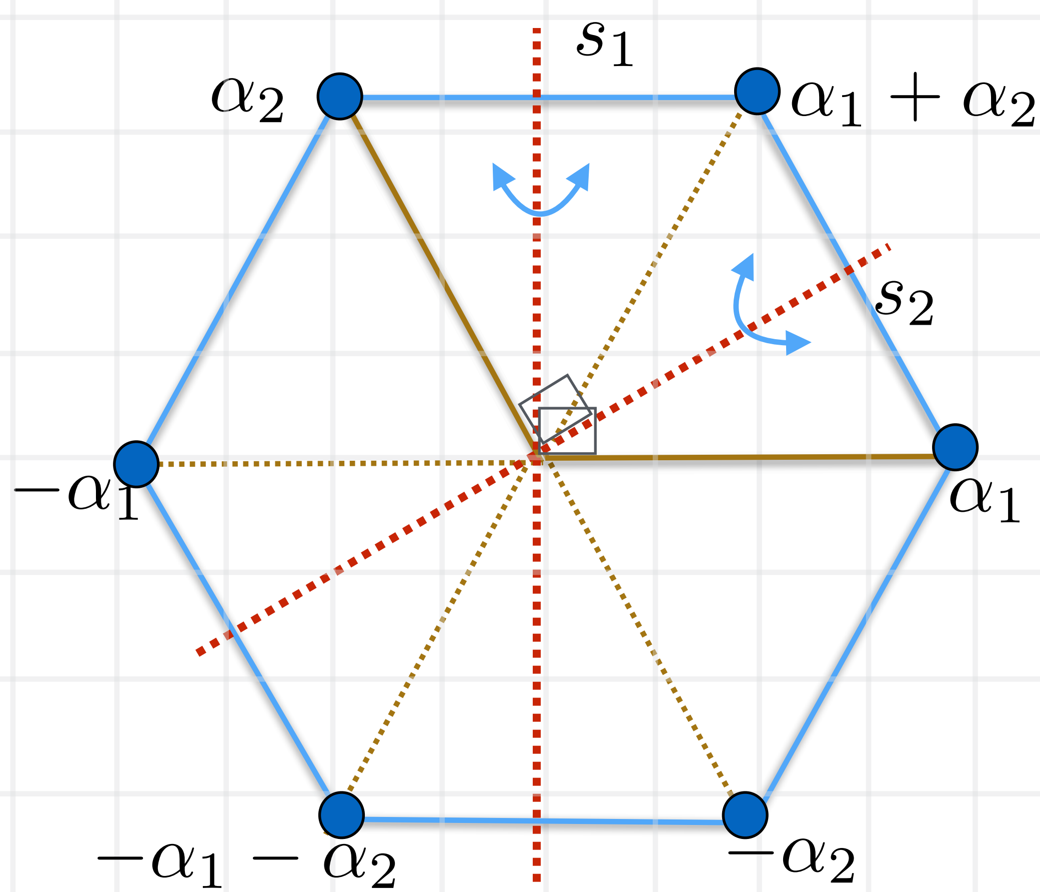
Root System



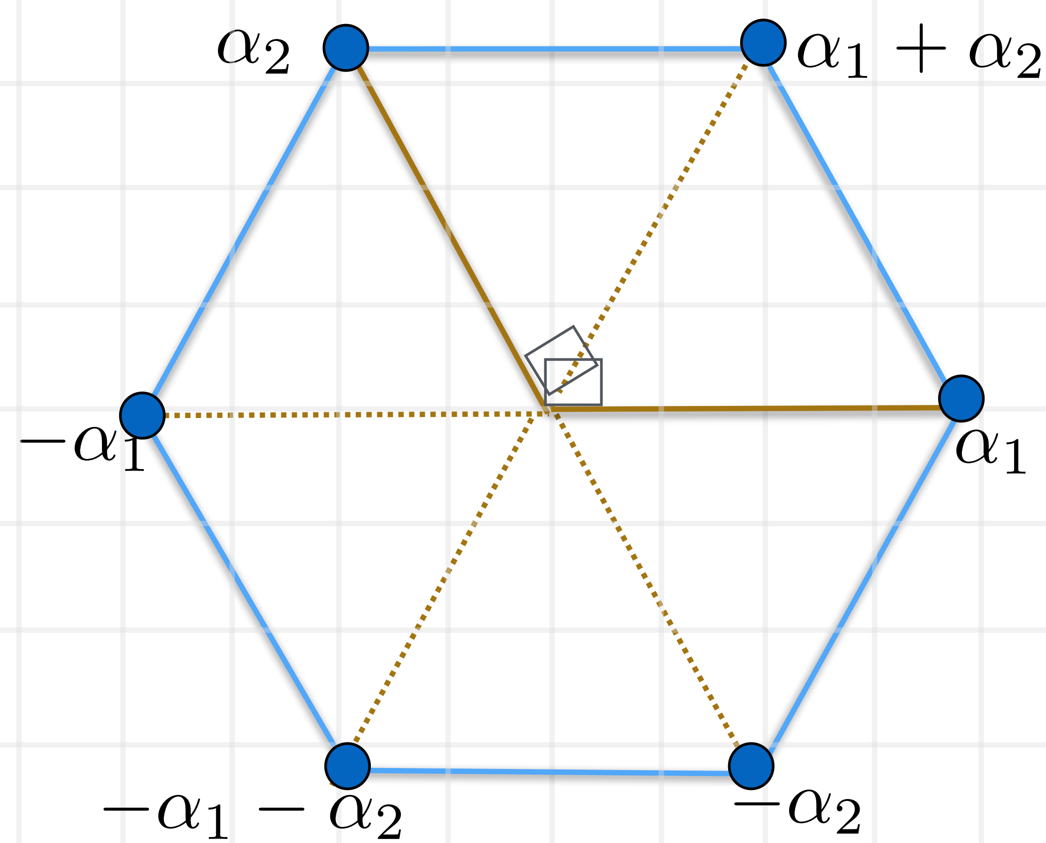
α_1 and α_2 are "simple" roots

Reflection Groups

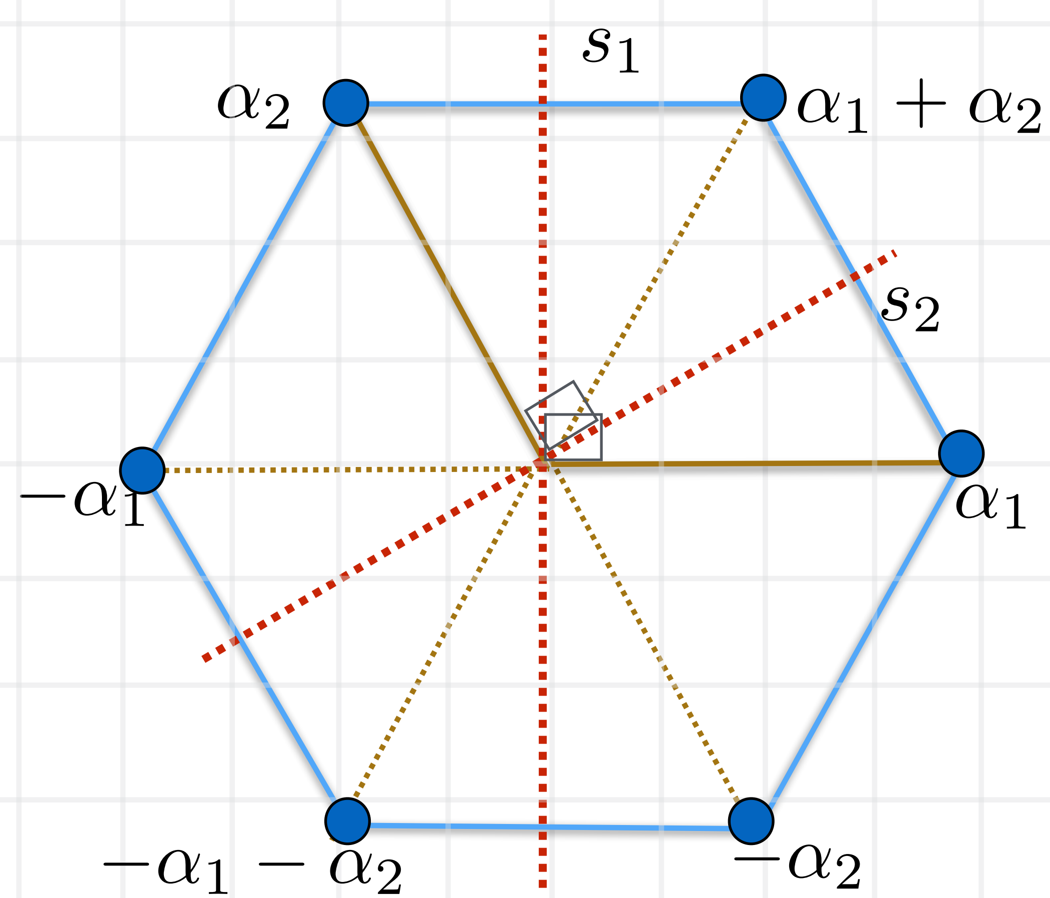
- Roots: $\alpha_1, \alpha_2, \ldots, \alpha_n$
- Reflections: $w_i(\alpha_j) = \alpha_j - \, 2 \, \dfrac{(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \, \alpha_i$
- Co-roots: $\check{\alpha}_i = \, 2 \, \dfrac{\alpha_i}{(\alpha_i, \alpha_i)}$
- Weights: $h_1, h_2, \ldots, h_n$

$$(h_i, \check{\alpha}_i) = \delta_{ij}$$

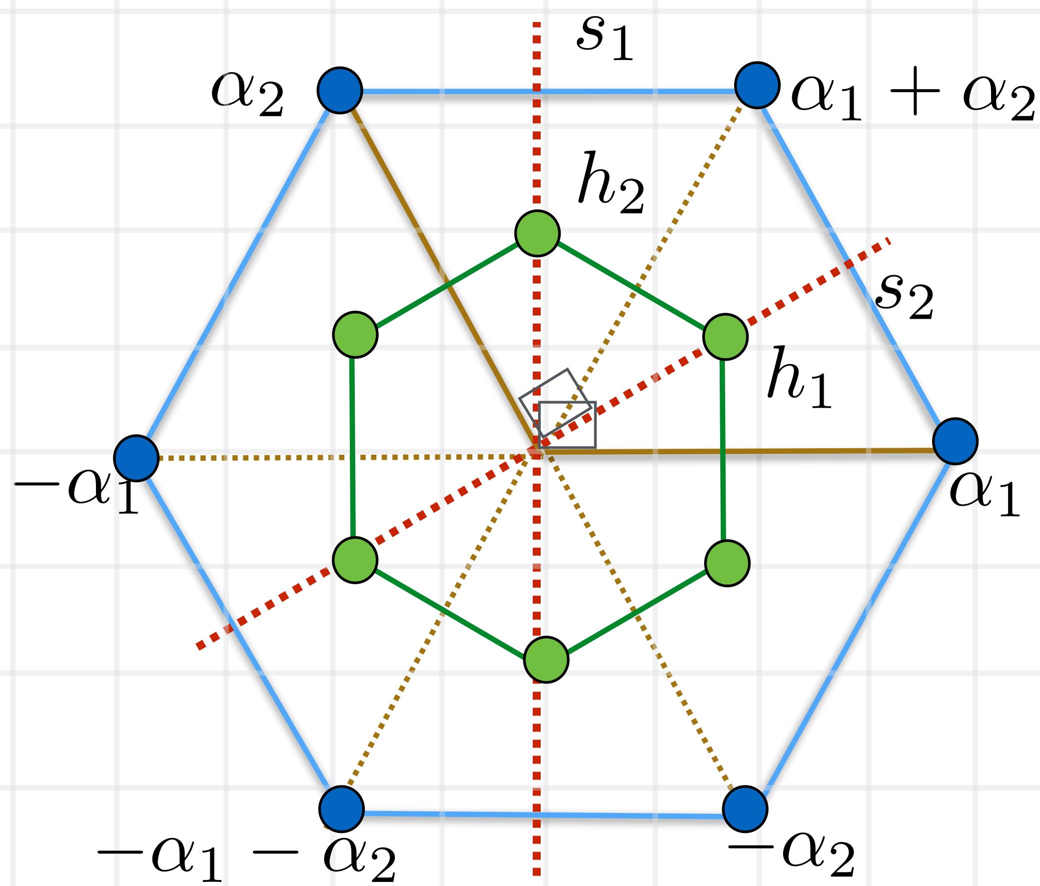
A_2



A_2

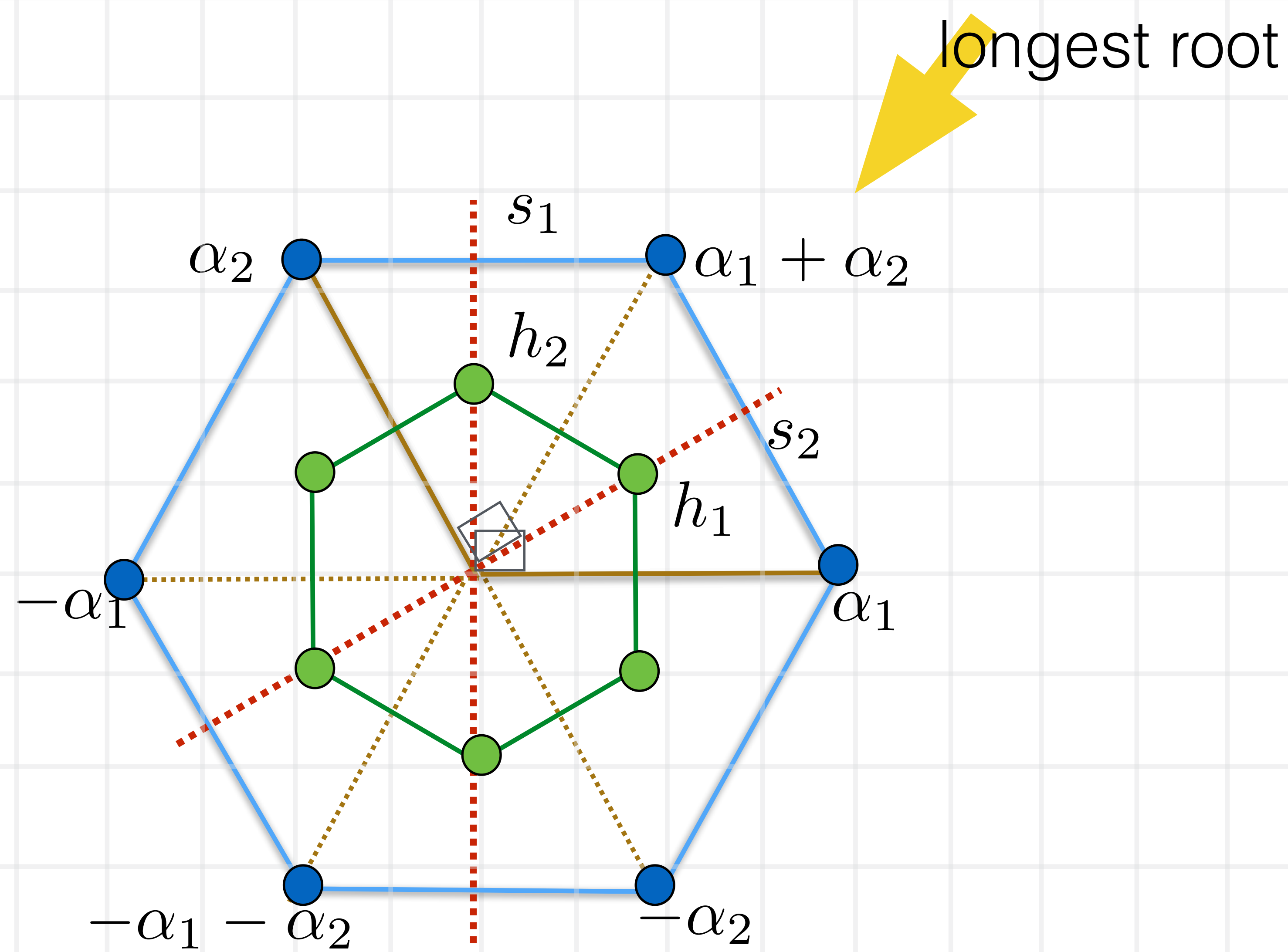

s_1
α_2
$\alpha_1+\alpha_2$
s_2
$-\alpha_1$
α_1
$-\alpha_1-\alpha_2$
$-\alpha_2$

A$_2$

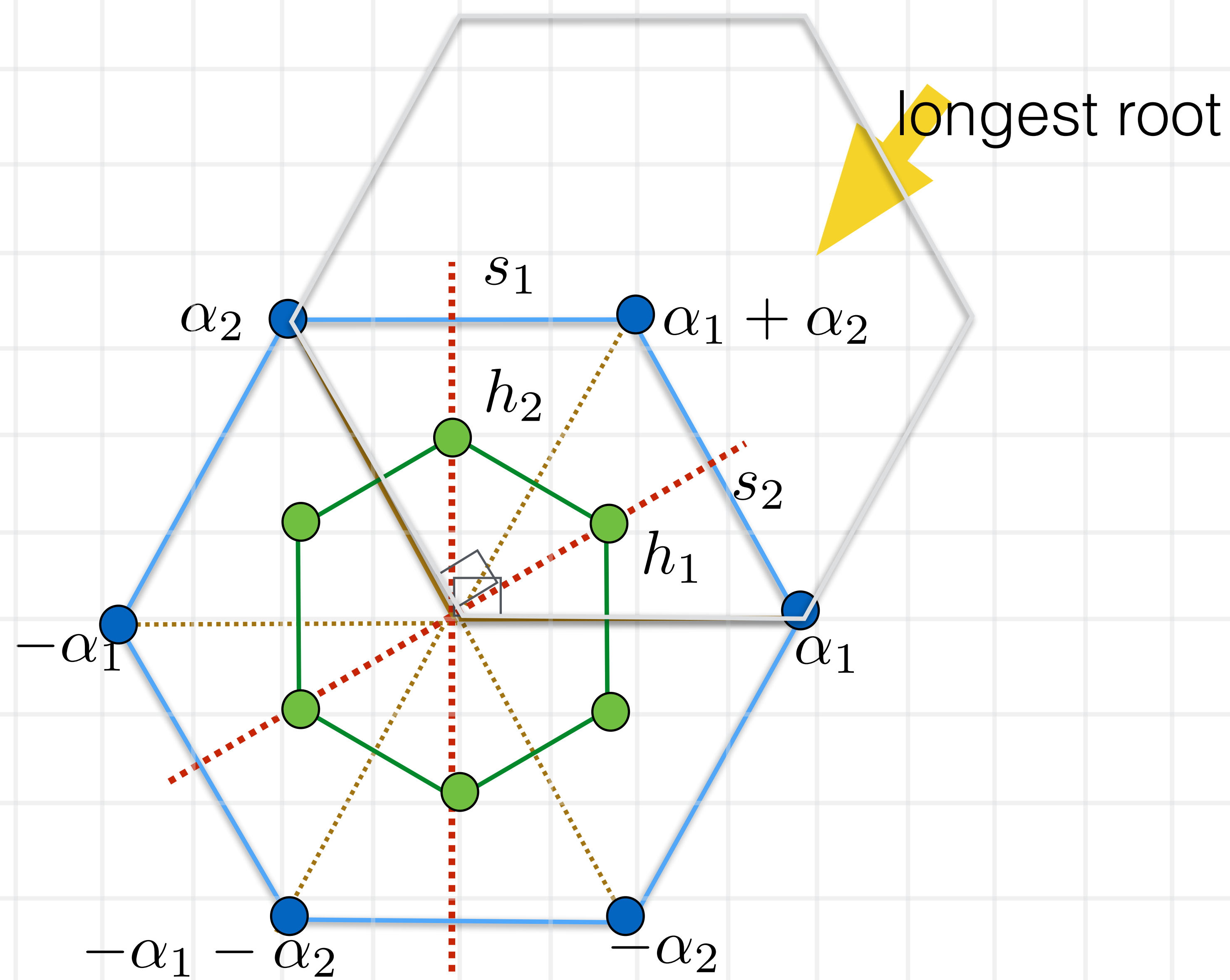


s_1
α_2
$\alpha_1 + \alpha_2$
h_2
s_2
h_1
$-\alpha_1$
α_1
$-\alpha_1 - \alpha_2$
$-\alpha_2$

A_2


longest root
s_1
s_2
h_1
h_2
α_1
α_2
$\alpha_1 + \alpha_2$
$-\alpha_1$
$-\alpha_2$
$-\alpha_1 - \alpha_2$

A$_2$


longest root
s_1
s_2
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α_1
α_2
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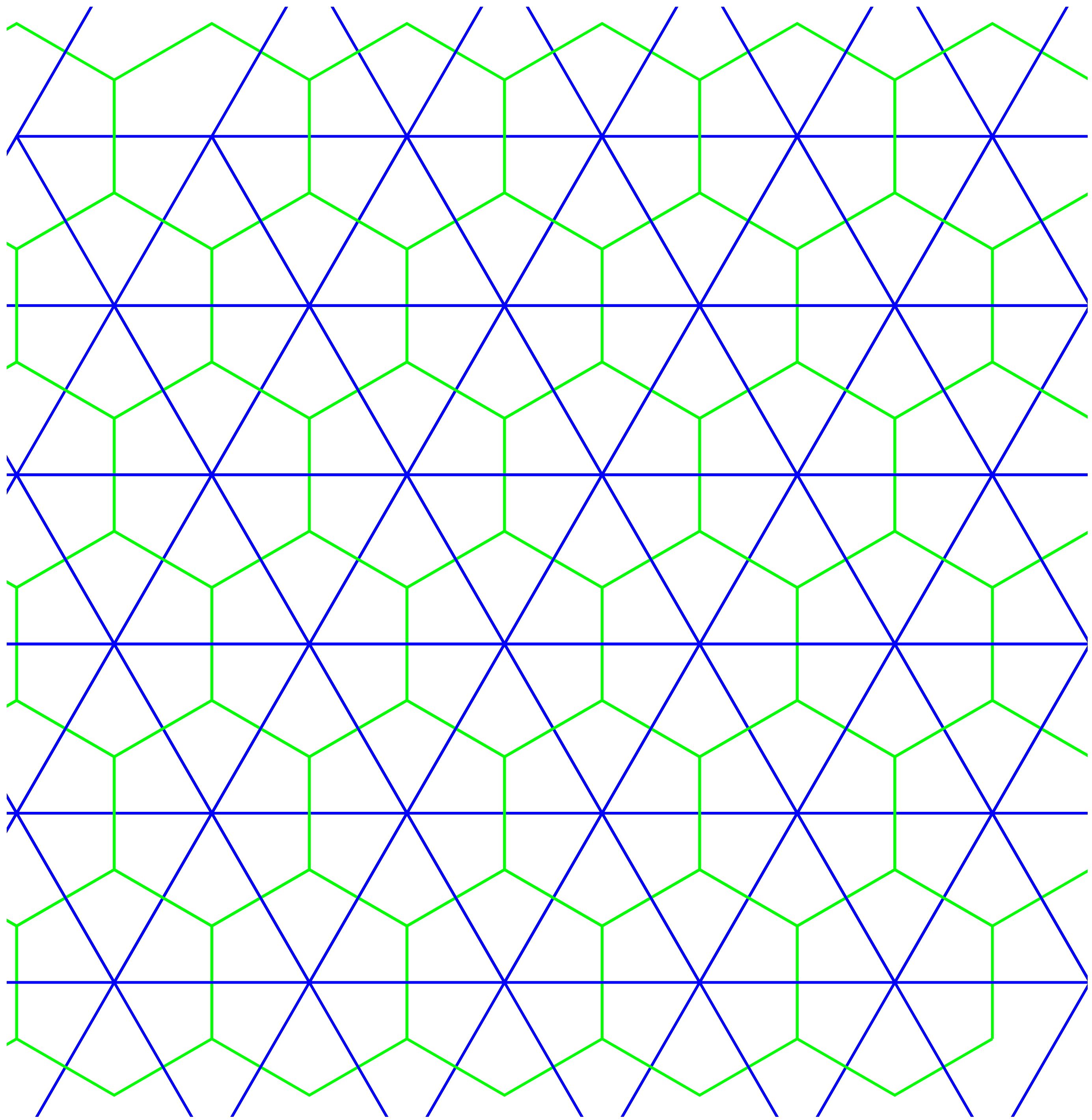
$-\alpha_1$
$-\alpha_2$
$-\alpha_1 - \alpha_2$

Translation by longest root

$A_2^{(1)}$

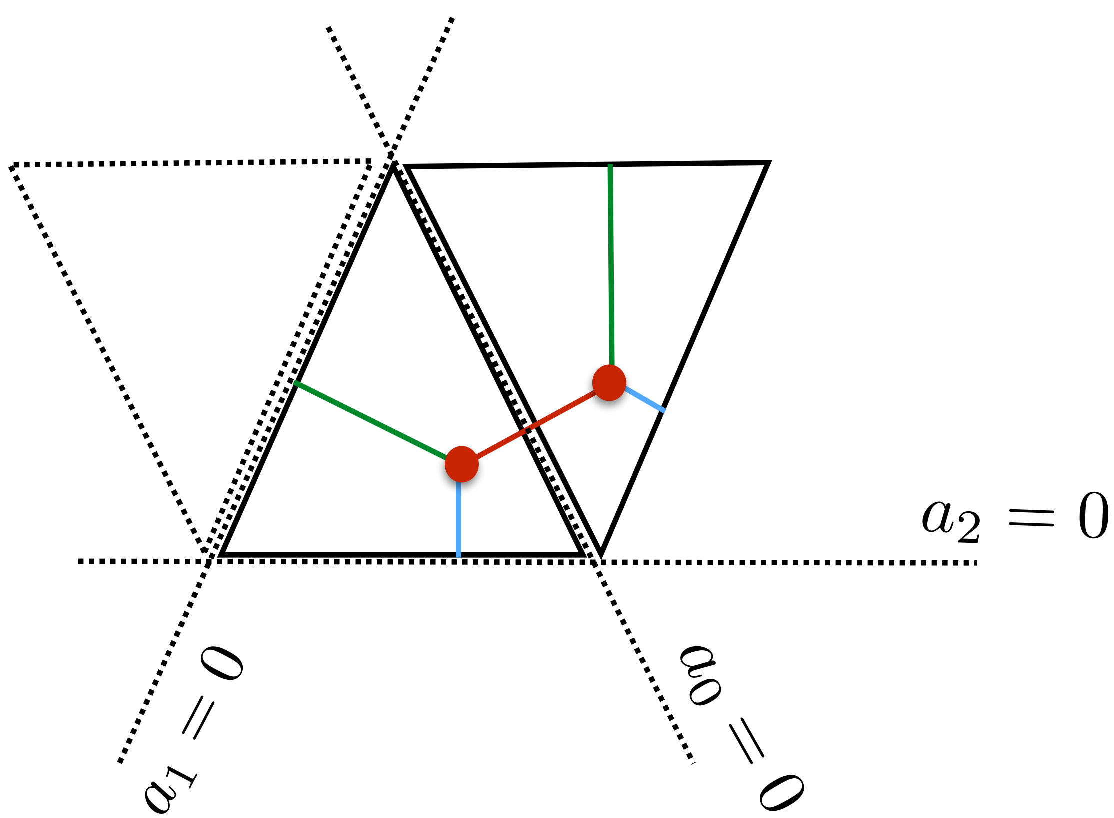


On the Lattice

$$\widetilde{\mathcal{W}}(A_2^{(1)}) = \langle s_0, s_1, s_2, \pi \rangle$$

$$s_j^2 = 1,\ \ (s_j s_{j+1})^3 = 1,\ \ (j = 0, 1, 2)$$

$$\pi^3 = 1,\ \ \pi s_j = s_{j+1}\pi$$



$$s_0(a_0, a_1, a_2) = (-a_0, a_1 + a_0, a_2 + a_0)$$

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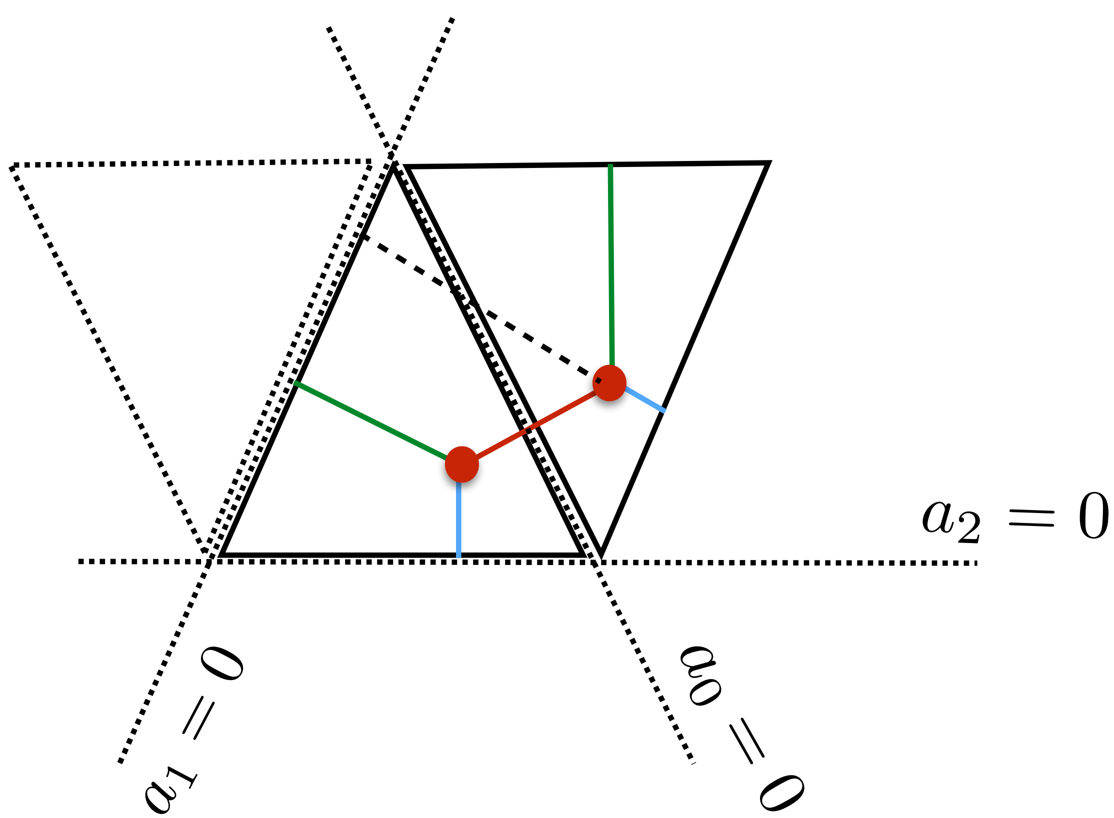

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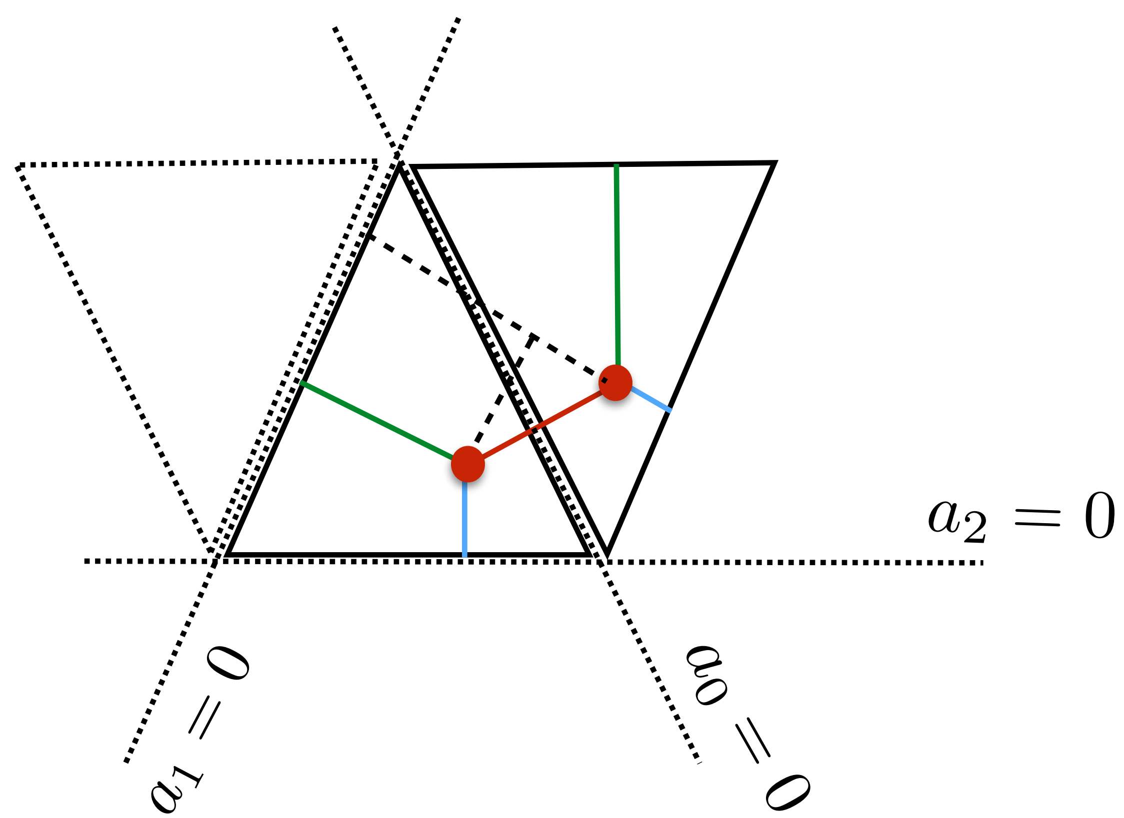

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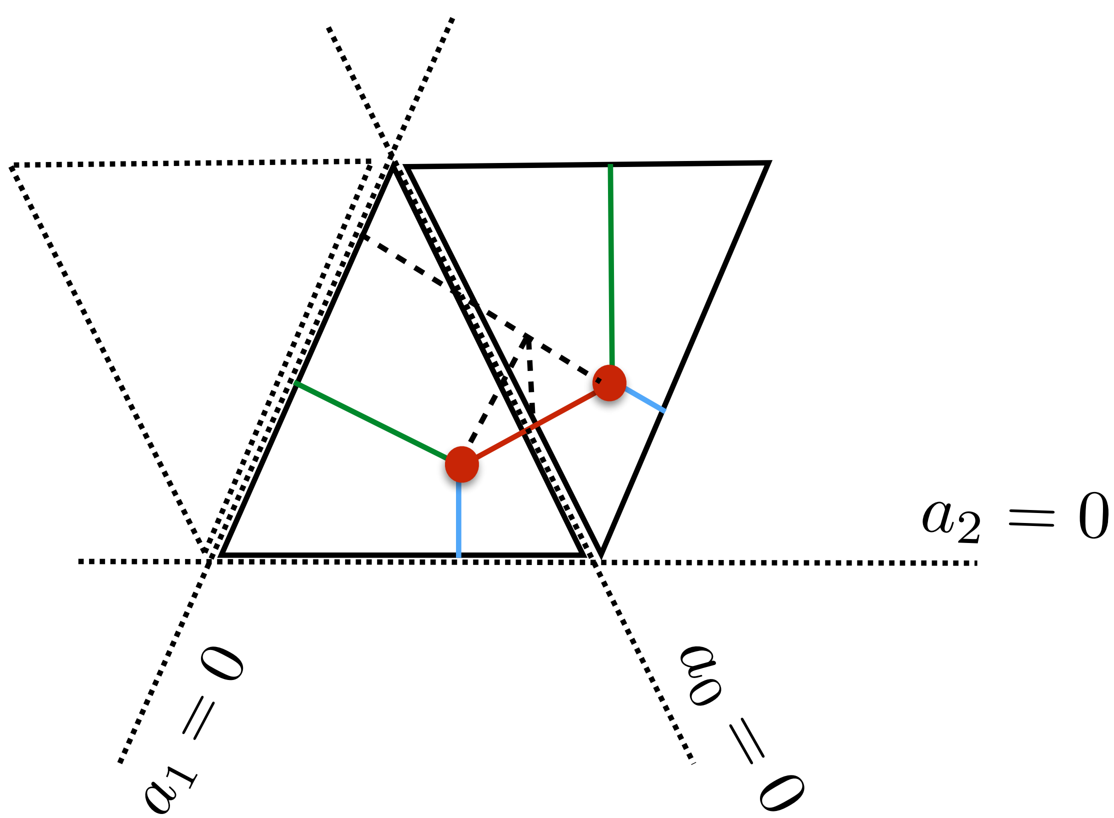

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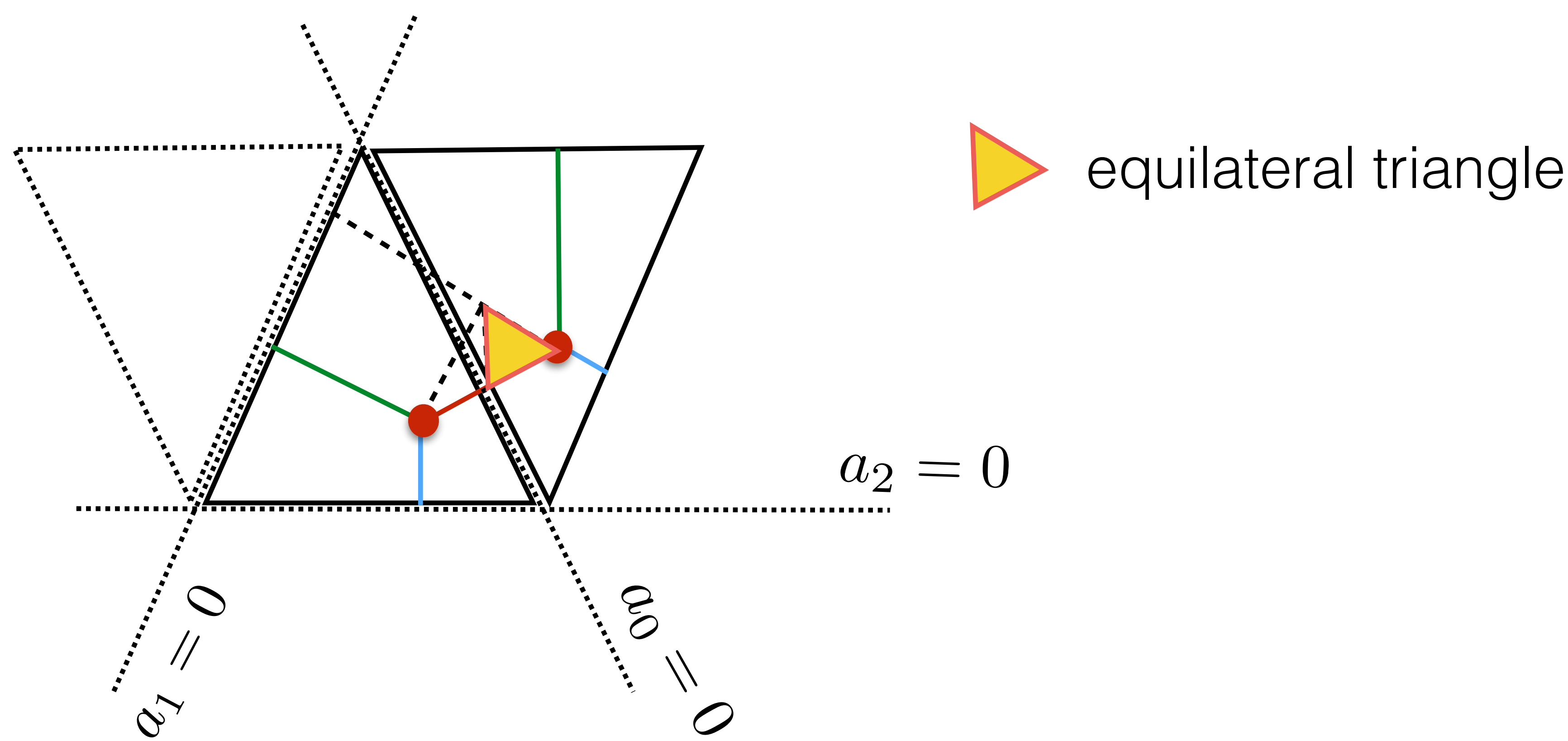

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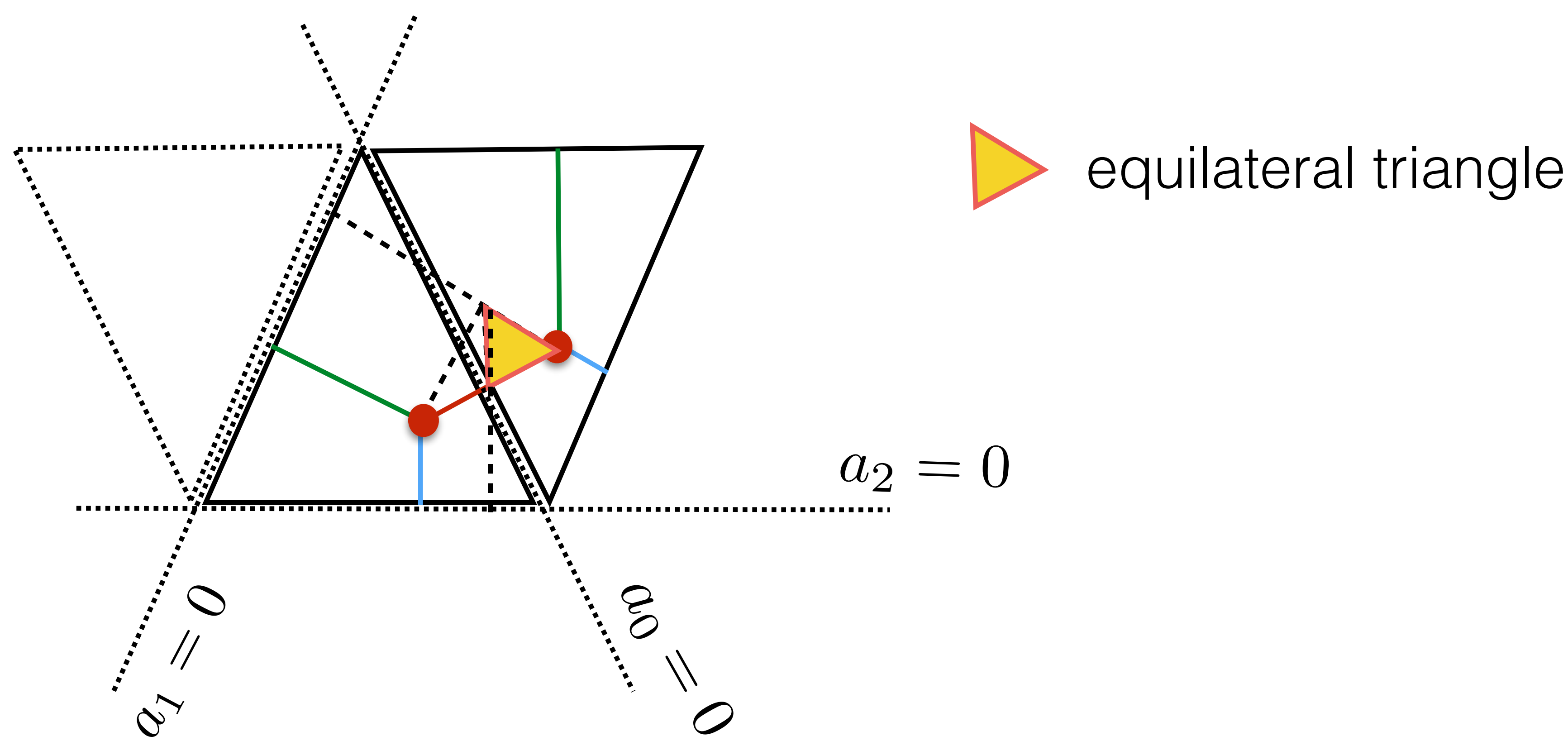


$$s_0(a_0, a_1, a_2) = (-a_0, a_1 + a_0, a_2 + a_0)$$

Cremona Isometries

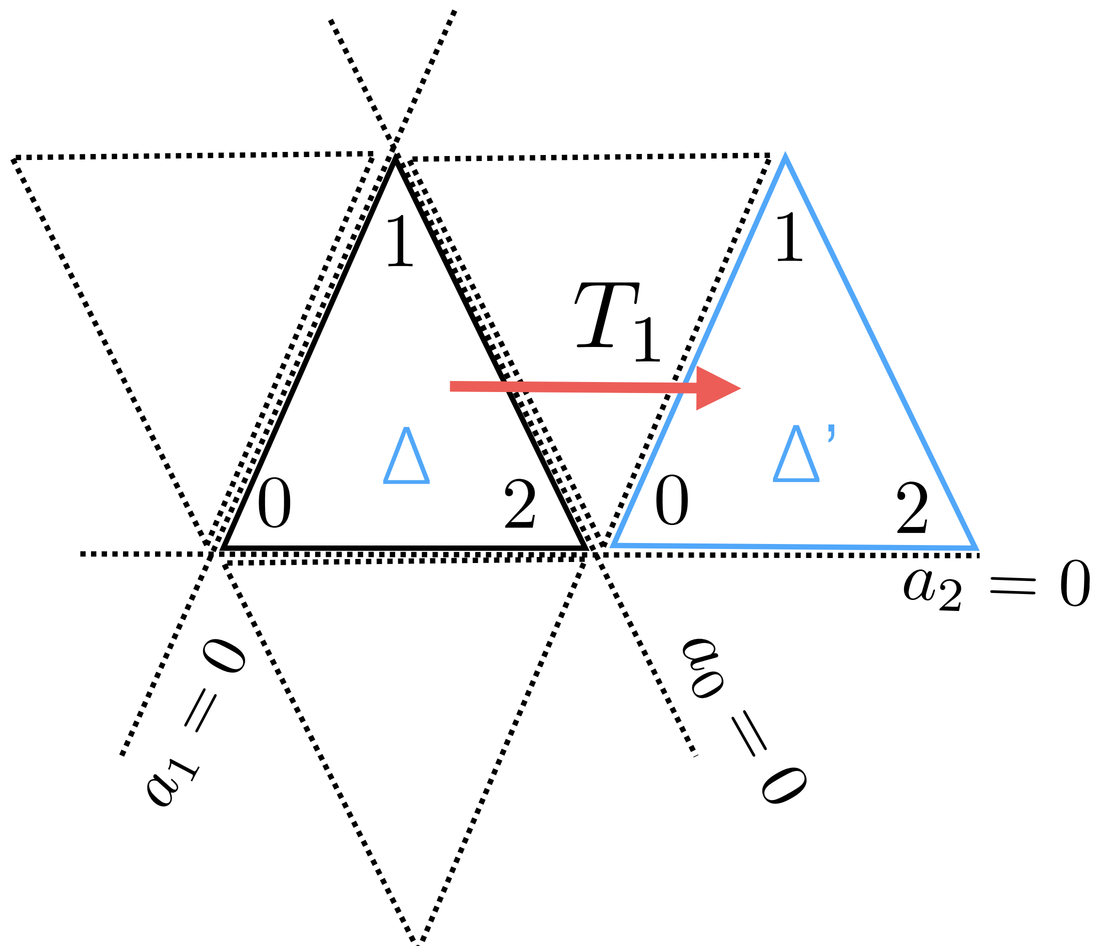
	a_0	a_1	a_2	f_0	f_1	f_2
s_0	$-a_0$	$a_1 + a_0$	$a_2 + a_0$	f_0	$f_1 + \dfrac{a_0}{f_0}$	$f_2 - \dfrac{a_0}{f_0}$
s_1	$a_0 + a_1$	$-a_1$	$a_2 + a_1$	$f_0 - \dfrac{a_1}{f_1}$	f_1	$f_2 - \dfrac{a_1}{f_1}$
s_2	$a_0 + a_2$	$a_1 + a_2$	$-a_2$	$f_0 + \dfrac{a_2}{f_2}$	$f_1 - \dfrac{a_2}{f_1}$	f_2

Cremona Isometries

	a_0	a_1	a_2	f_0	f_1	f_2
s_0	$-a_0$	$a_1 + a_0$	$a_2 + a_0$	f_0	$f_1 + \frac{a_0}{f_0}$	$f_2 - \frac{a_0}{f_0}$
s_1	$a_0 + a_1$	$-a_1$	$a_2 + a_1$	$f_0 - \frac{a_1}{f_1}$	f_1	$f_2 - \frac{a_1}{f_1}$
s_2	$a_0 + a_2$	$a_1 + a_2$	$-a_2$	$f_0 + \frac{a_2}{f_2}$	$f_1 - \frac{a_2}{f_1}$	f_2

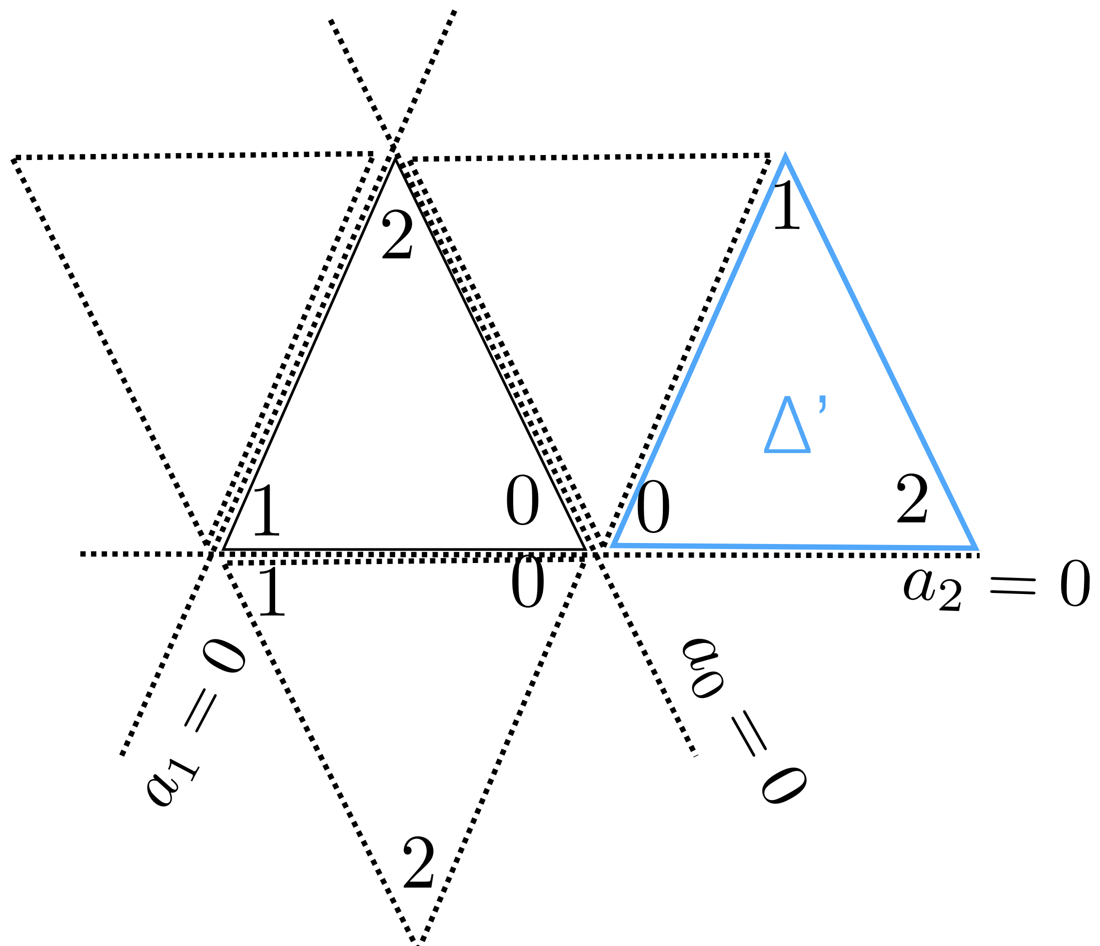
Discrete Dynamics I

- Translation



Discrete Dynamics II

- Translation as reflections ?



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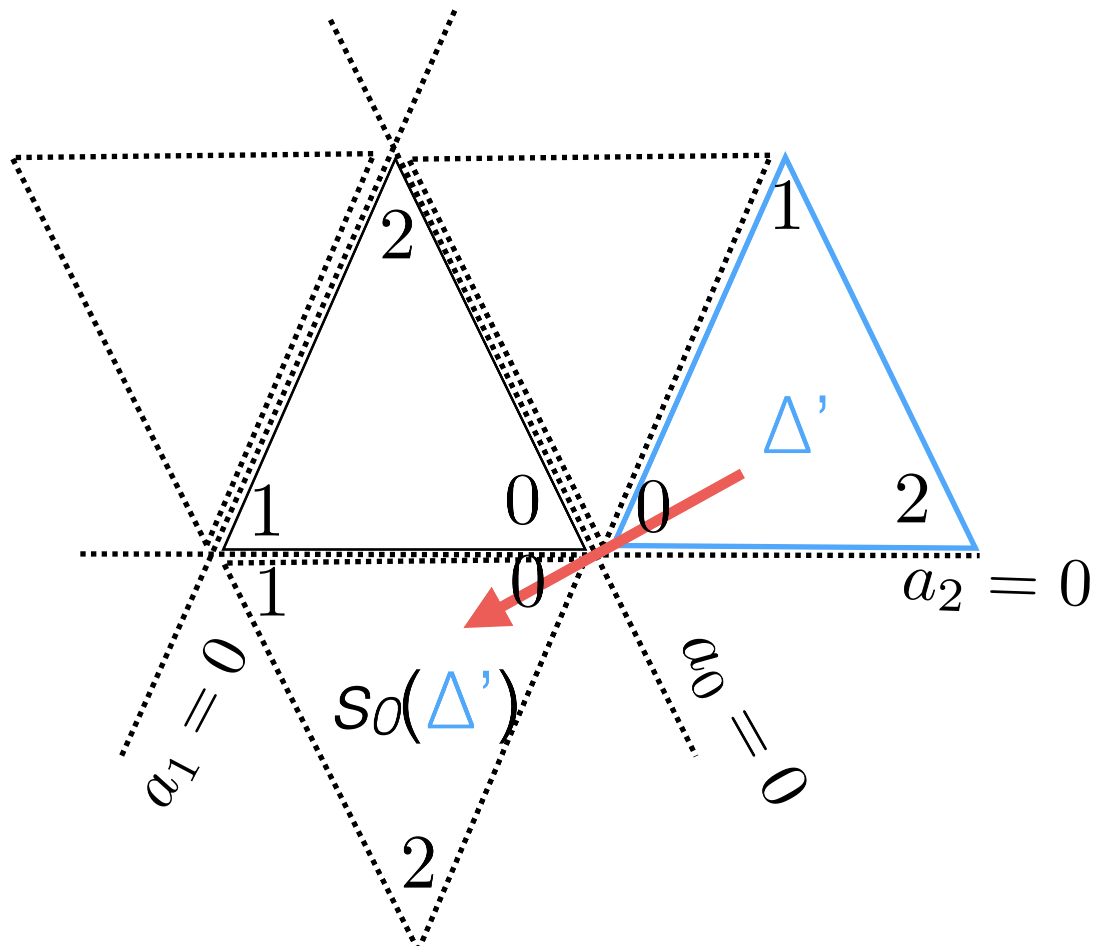
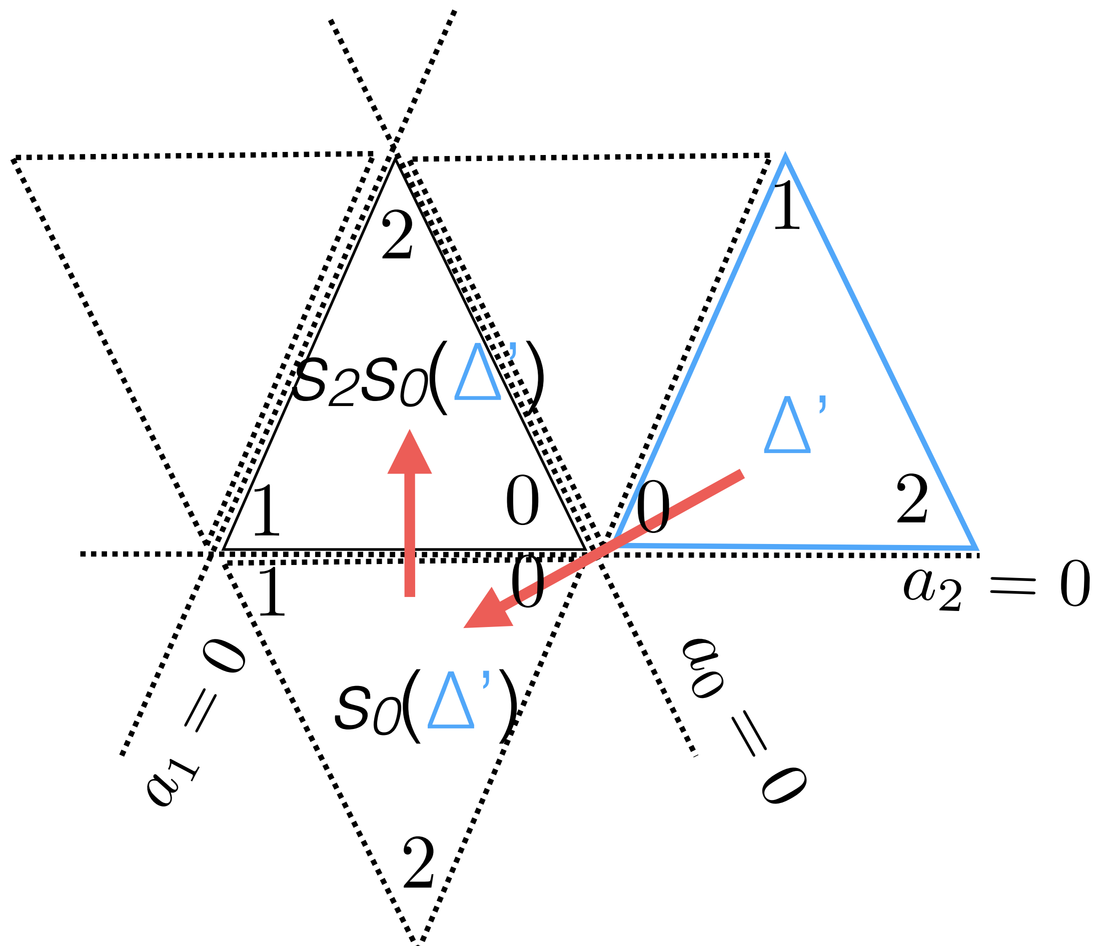


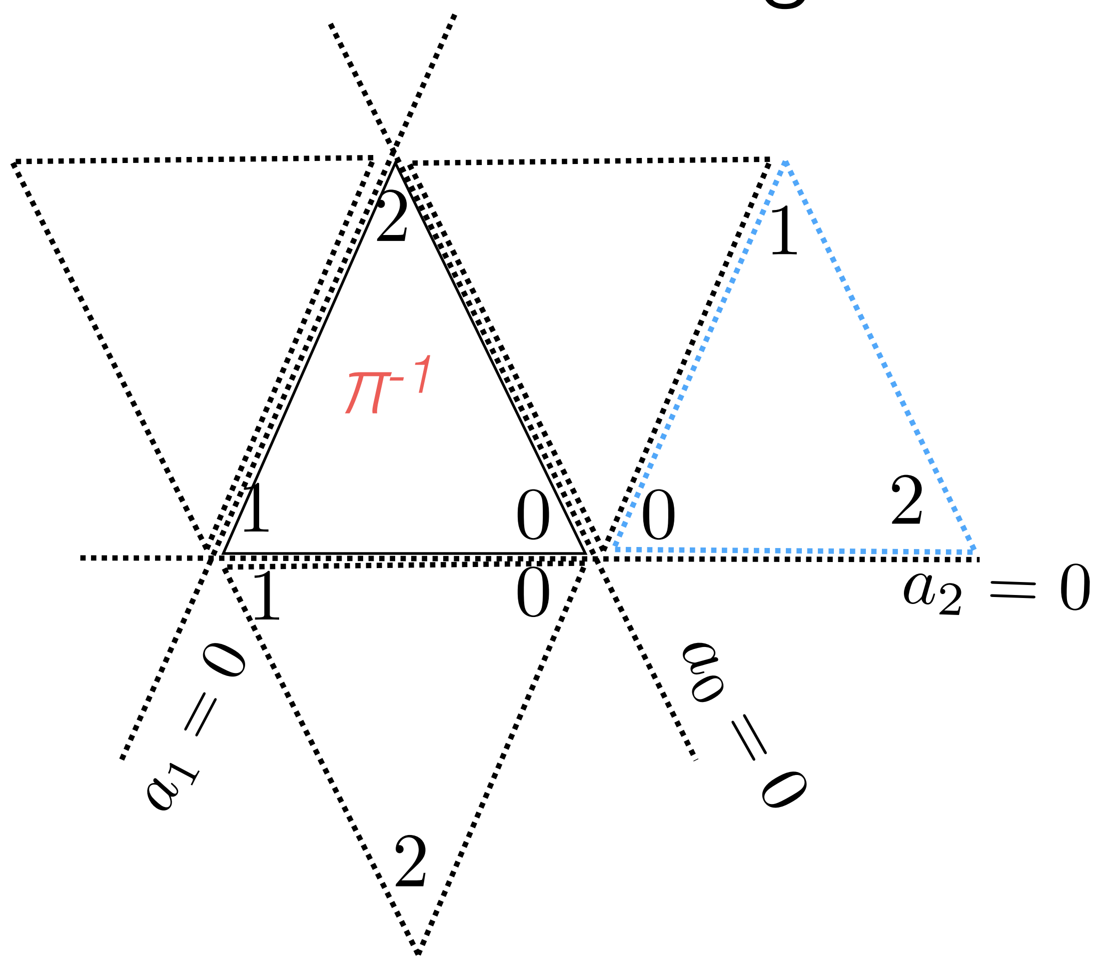
Discrete Dynamics II

- Translation as reflections ?



Discrete Dynamics III

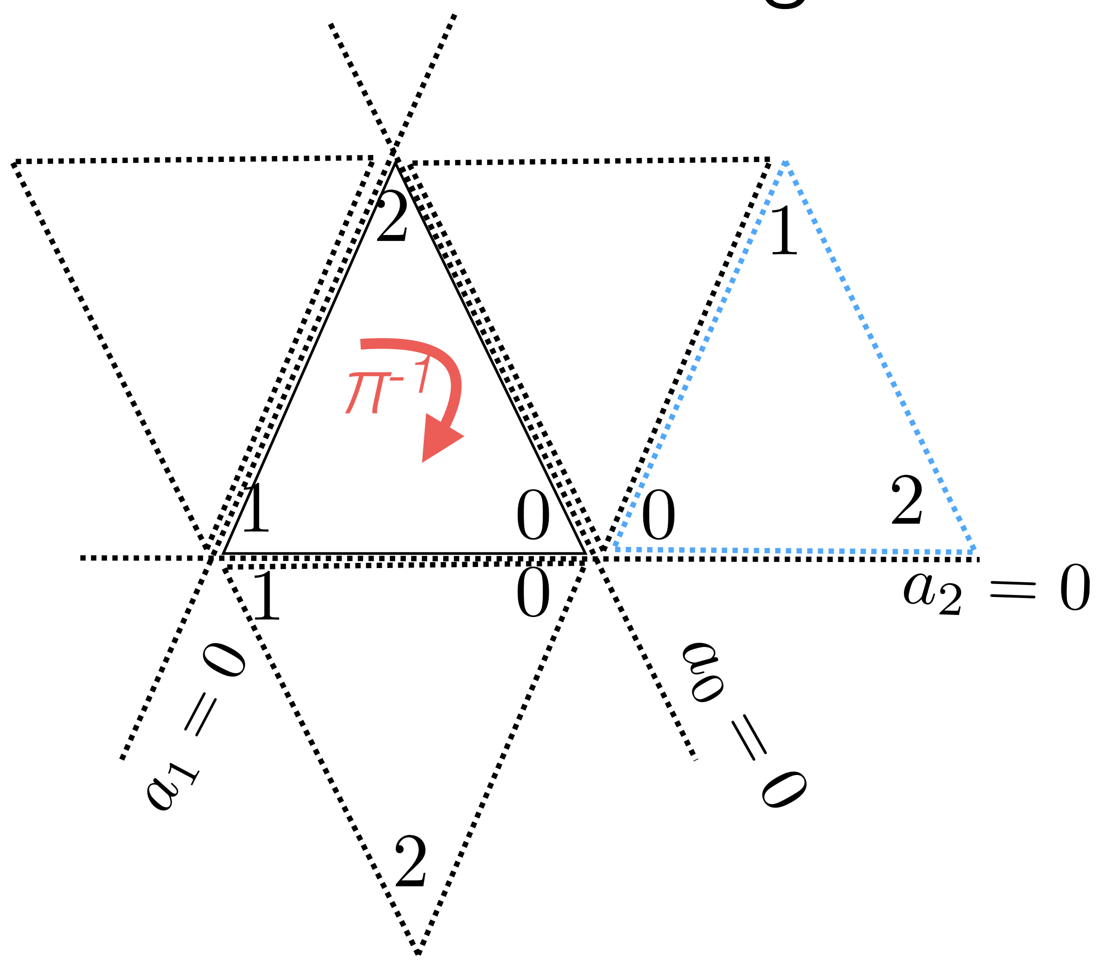
- Translation as reflection + diagram automorphism



$$I = \pi^{-1}\, s_2\, s_0\, T_1 \text{ where } \pi^3 = 1 \Rightarrow \;\; T_1 = \pi\, s_2\, s_1$$

Discrete Dynamics III

- Translation as reflection + diagram automorphism



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Discrete Dynamics III

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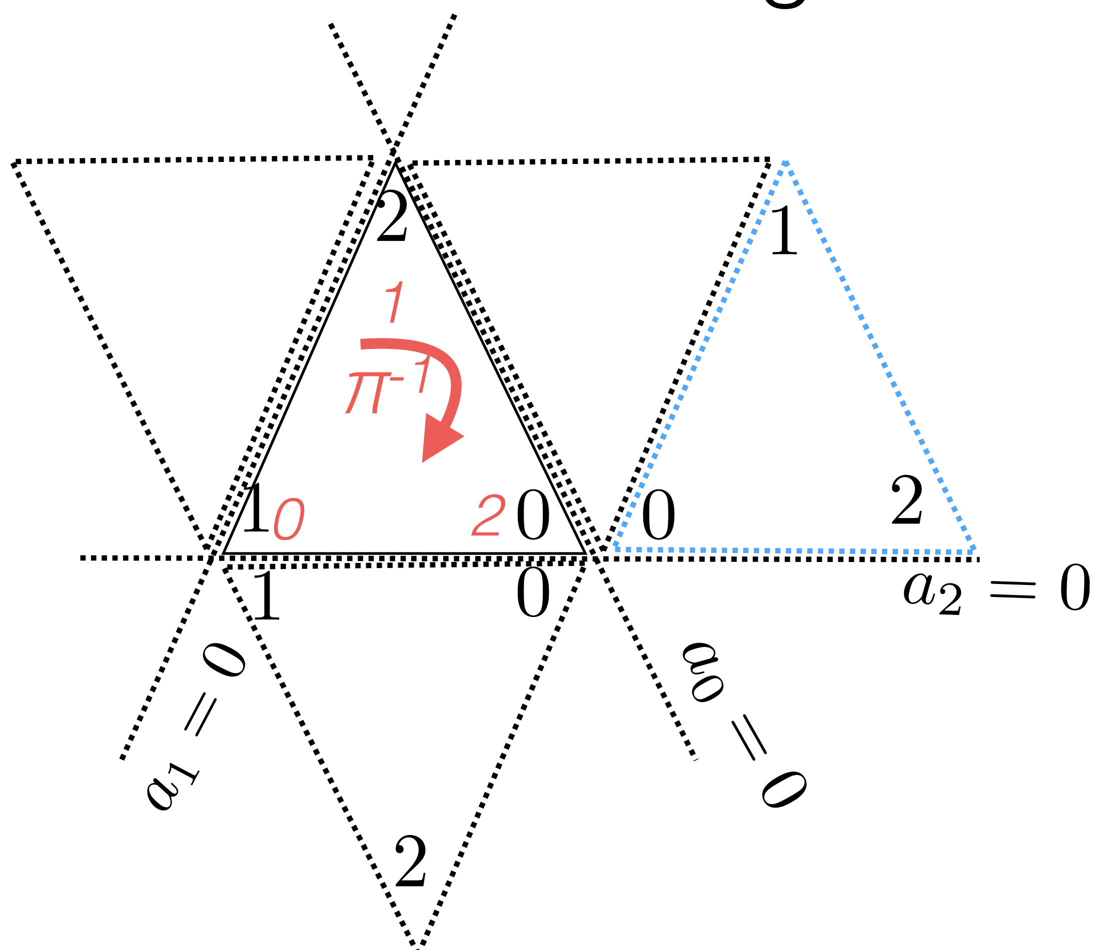

$$I = \pi^{-1}\, s_2\, s_0\, T_1 \text{ where } \pi^3 = 1 \Rightarrow \quad T_1 = \pi\, s_2\, s_1$$

Discrete Dynamics IV

Noting that

$$T_1(a_0) = a_0 + 1, T_1(a_1) = a_1 - 1, T_1(a_2) = a_2$$

Define

$$u_n = T_1^n(f_1), v_n = T_1^n(f_0)$$

$$\Rightarrow \qquad \begin{cases} u_n + u_{n+1} & = t - v_n - \frac{a_0+n}{v_n} \\ v_n + v_{n-1} & = t - u_n + \frac{a_1-n}{u_n} \end{cases}$$

Discrete Dynamics IV

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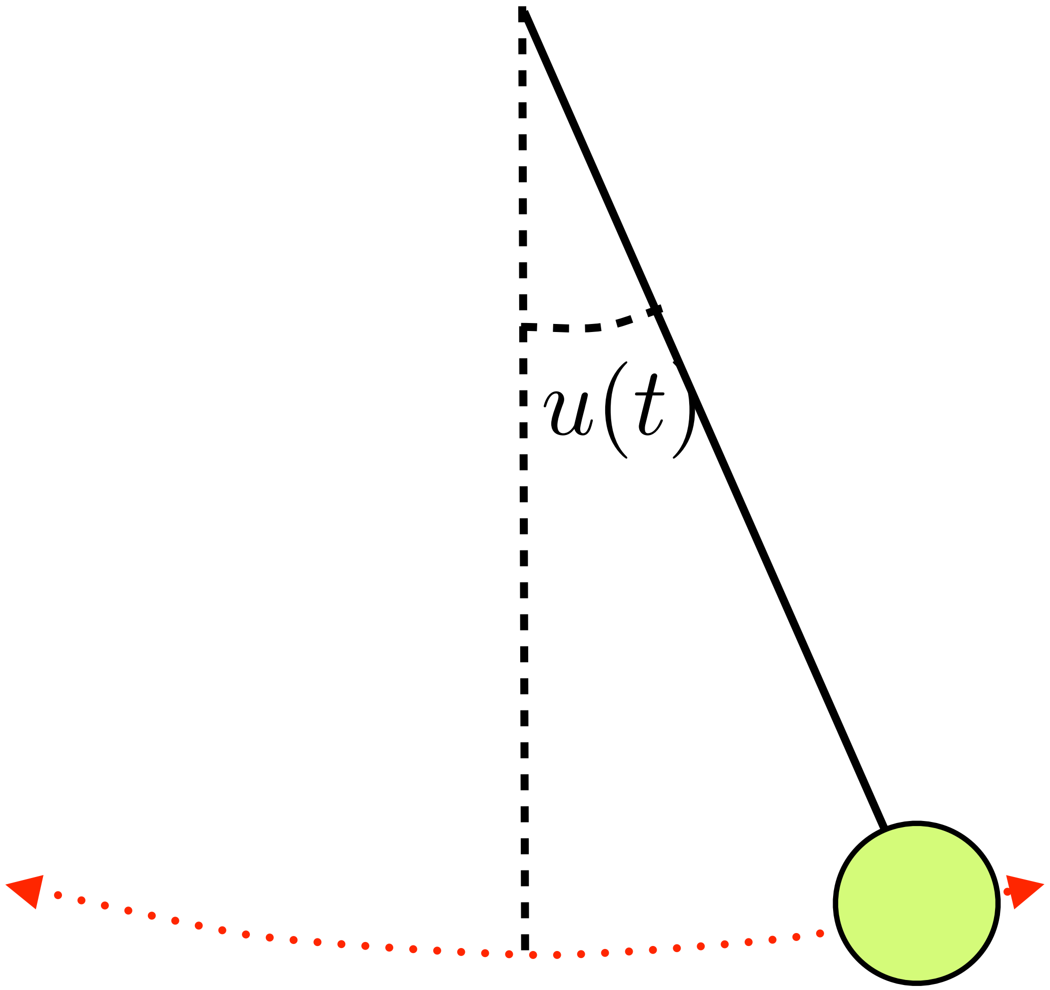
These are discrete Painlevé equations.

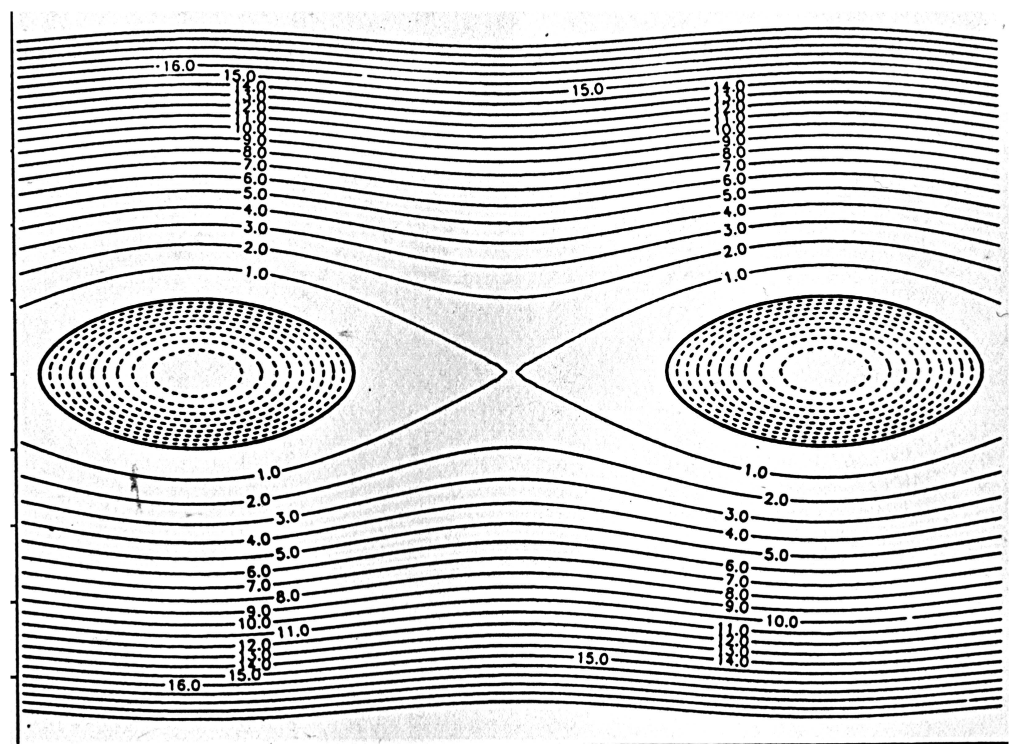
What does this have to do with applied mathematics?

Dynamical Systems



$$\begin{cases} \dot{u}(t) = v(t) \\ \dot{v}(t) = -\sin\big(u(t)\big) \end{cases}$$

Phase Space



$$H(u, v) = \frac{v^2}{2} - \cos\big(u(t)\big)$$

Another View

$$f(t) = e^{iu(t)}$$

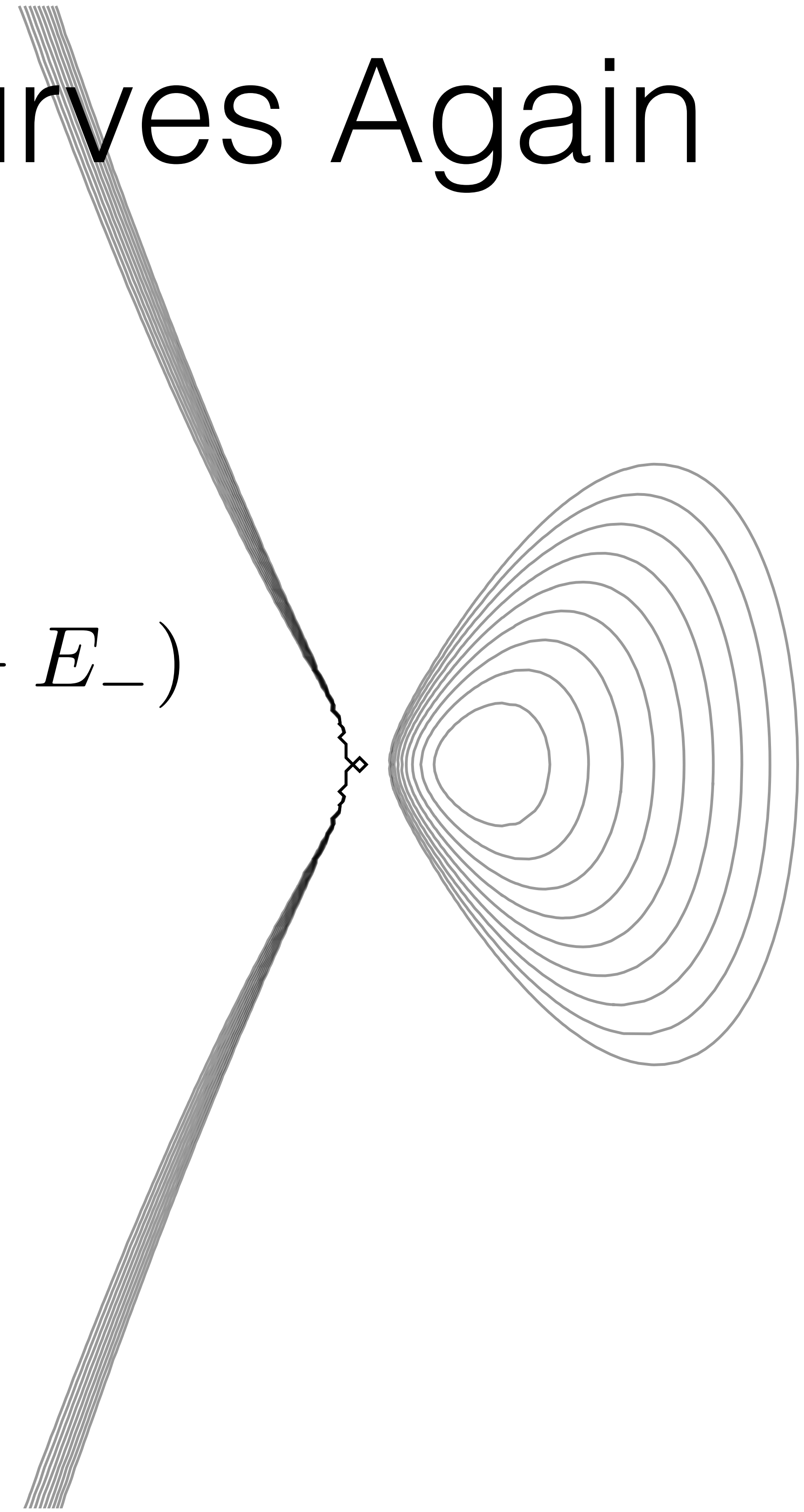
$$\Rightarrow \begin{cases} \ddot{f} & = \frac{\dot{f}^2}{f} - \frac{1}{2}\left(f^2 - 1\right) \\ E & = \frac{\dot{f}^2}{2f^2} + \frac{1}{2}\left(f + \frac{1}{f}\right) \end{cases}$$

$$\Rightarrow \dot{f}^2 = -f^3 - 2\,E\,f^2 - 1$$

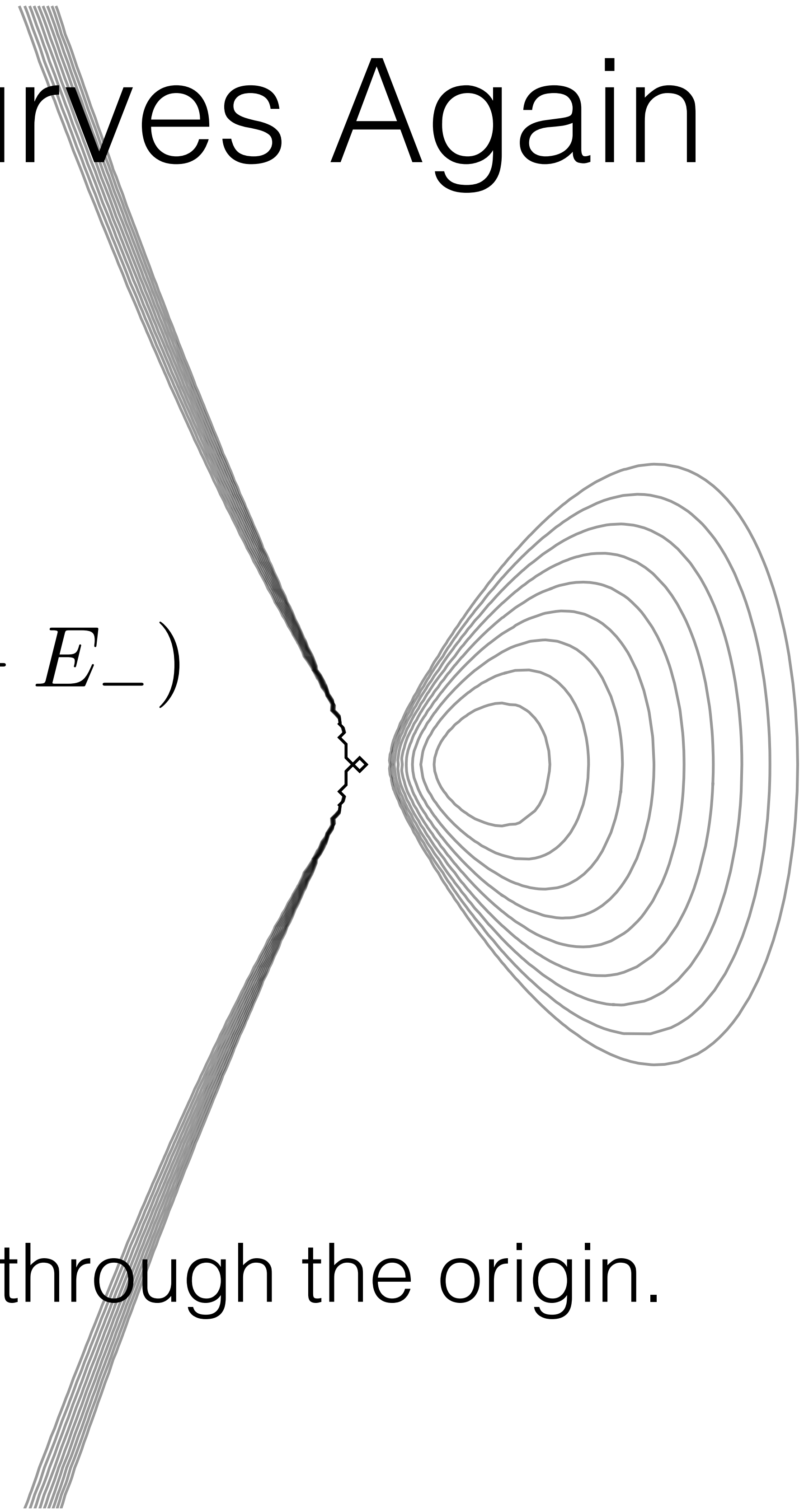
Phase Curves Again

$$y^2 = -x\,(x - E_+)\,(x - E_-)$$

$$E_\pm = E \pm \sqrt{E^2 - 1}$$



Phase Curves Again



$$y^2 = -x\,(x - E_+)\,(x - E_-)$$

$$E_\pm = E \pm \sqrt{E^2 - 1}$$

The trajectories all go through the origin.

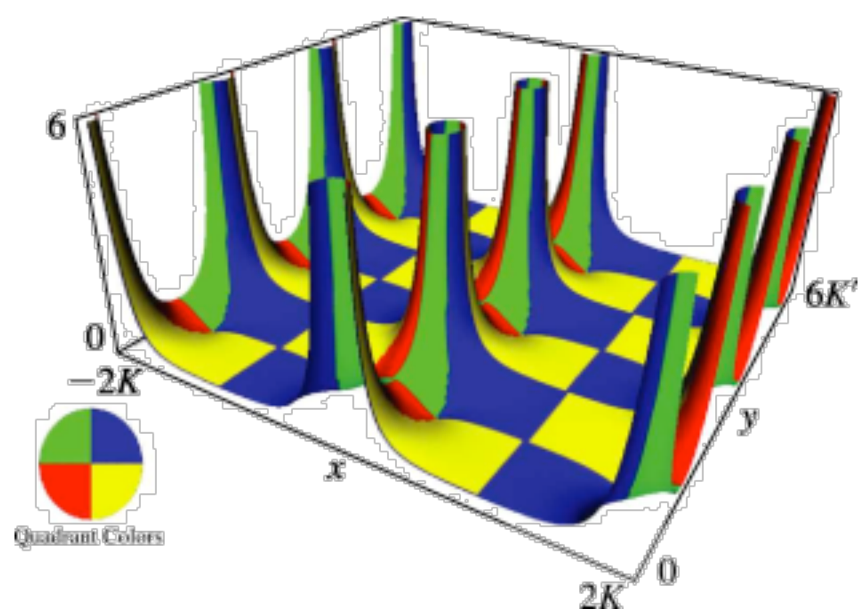
Two Problems

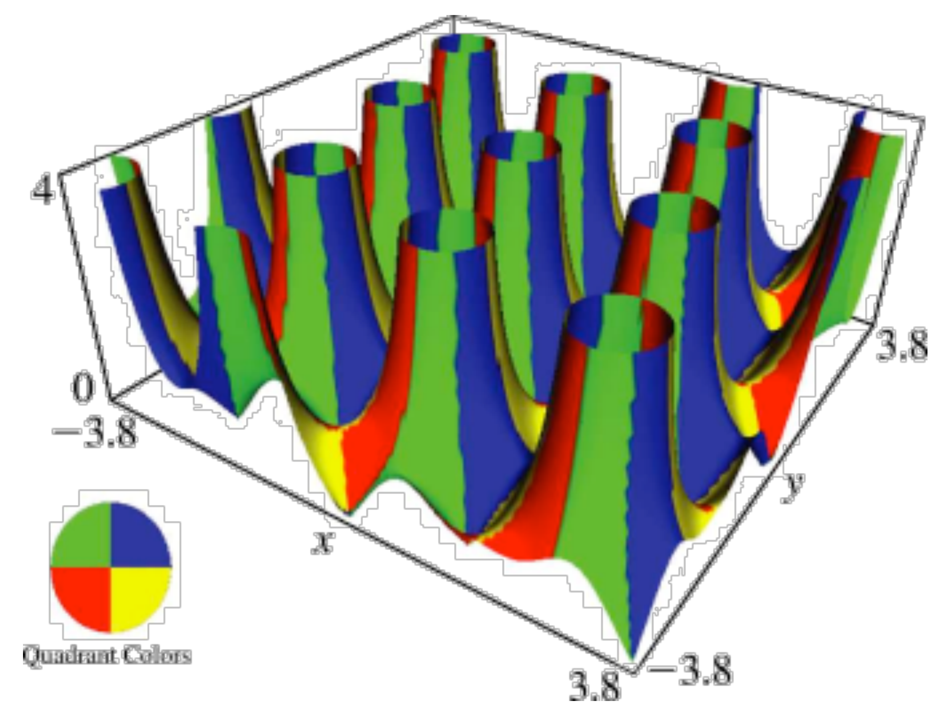
- The trajectories are indistinguishable as they pass through the origin.
- The phase space is no longer compact; Liouville's theorem* does not necessarily hold.
- These properties are shared by many nonlinear mathematical models.

* Liouville's thm gives the solution by quadratures.

Elliptic Functions

- Doubly-periodic, meromorphic functions





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Elliptic Functions in phase space

$$\ddot{w} = 6\,w^2 - \frac{g_2}{2}$$

$$\Rightarrow \qquad \frac{\dot{w}^2}{2} = 2\,w^3 - \frac{g_2}{2}\,w - \frac{g_3}{2}$$

$$\Rightarrow \qquad w(t) = \wp\big(t - t_0; g_2, g_3\big)$$

The phase space coordinatised by $(w, \dot{w})$ is not compact, due to poles.

Elliptic Functions parametrize curves

- In phase space, $\dot{w} = y, w = x,$ the conserved quantity becomes

$$y^2 = 4\,x^3 - g_2\,x - g_3$$

- Initial values determine g_3

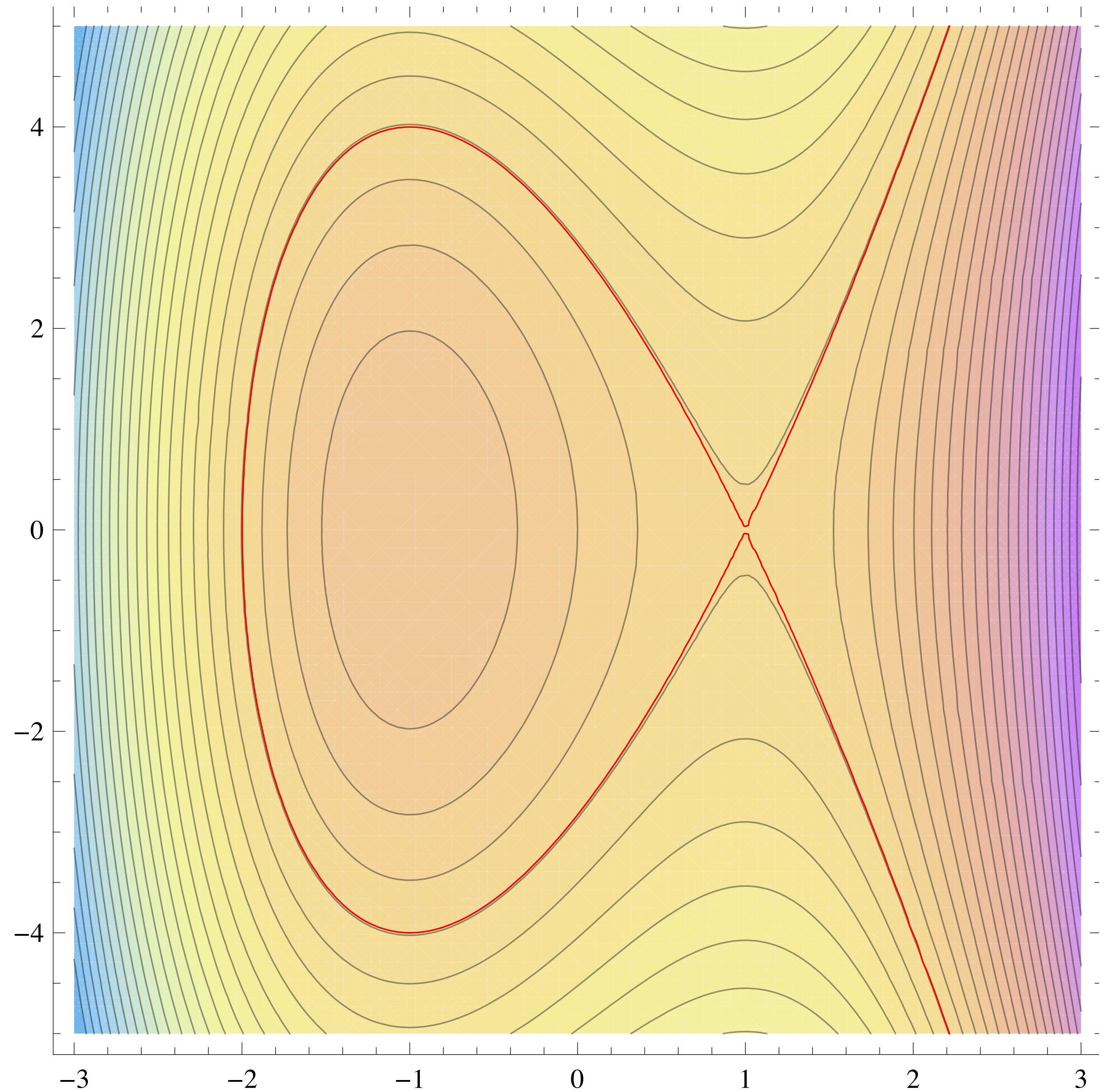
- Each value of g_3 defines a level curve of

$$f(x, y) = y^2 - 4\,x^3 + g_2\,x$$

Cubic Pencil

A Weierstrass cubic pencil:

$$y^2 - 4x^3 + g_2 x + g_3 = 0, \quad g_2 = 2, g_3 = -E$$



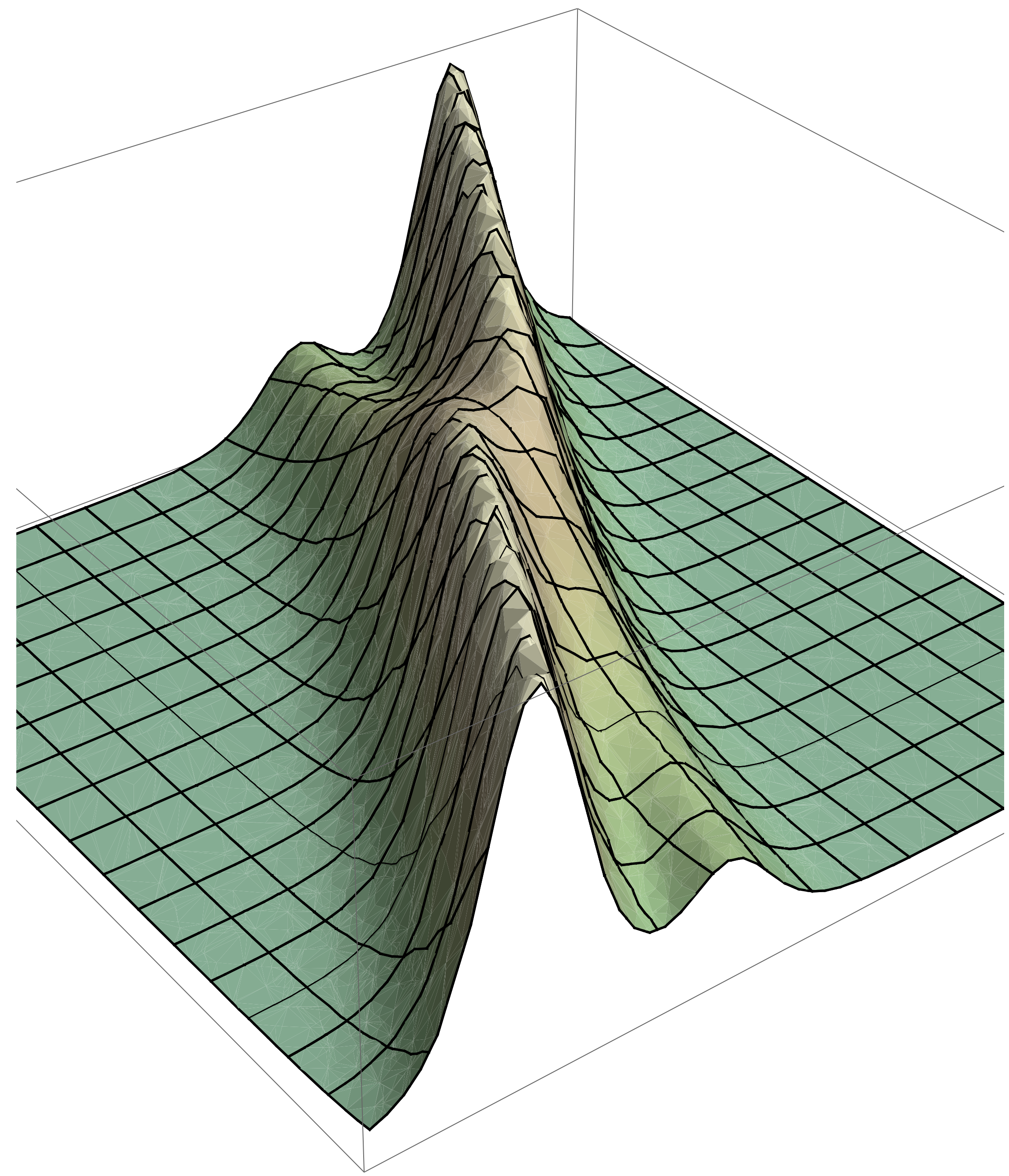
Motivation

- Korteweg-de Vries equation

$$w_\tau + 6\,w\,w_\xi + w_{\xi\xi\xi} = 0$$

$$\begin{cases} w = -2\,y(x) - 2\,\tau \\ x = \xi + 6\,\tau^2 \end{cases}$$

$$\Rightarrow \begin{cases} w_\tau & = -24\,\tau\,y_x - 2 \\ w_\xi & = -2\,y_x \\ w_{\xi\xi\xi} & = -2\,y_{xxx} \end{cases}$$



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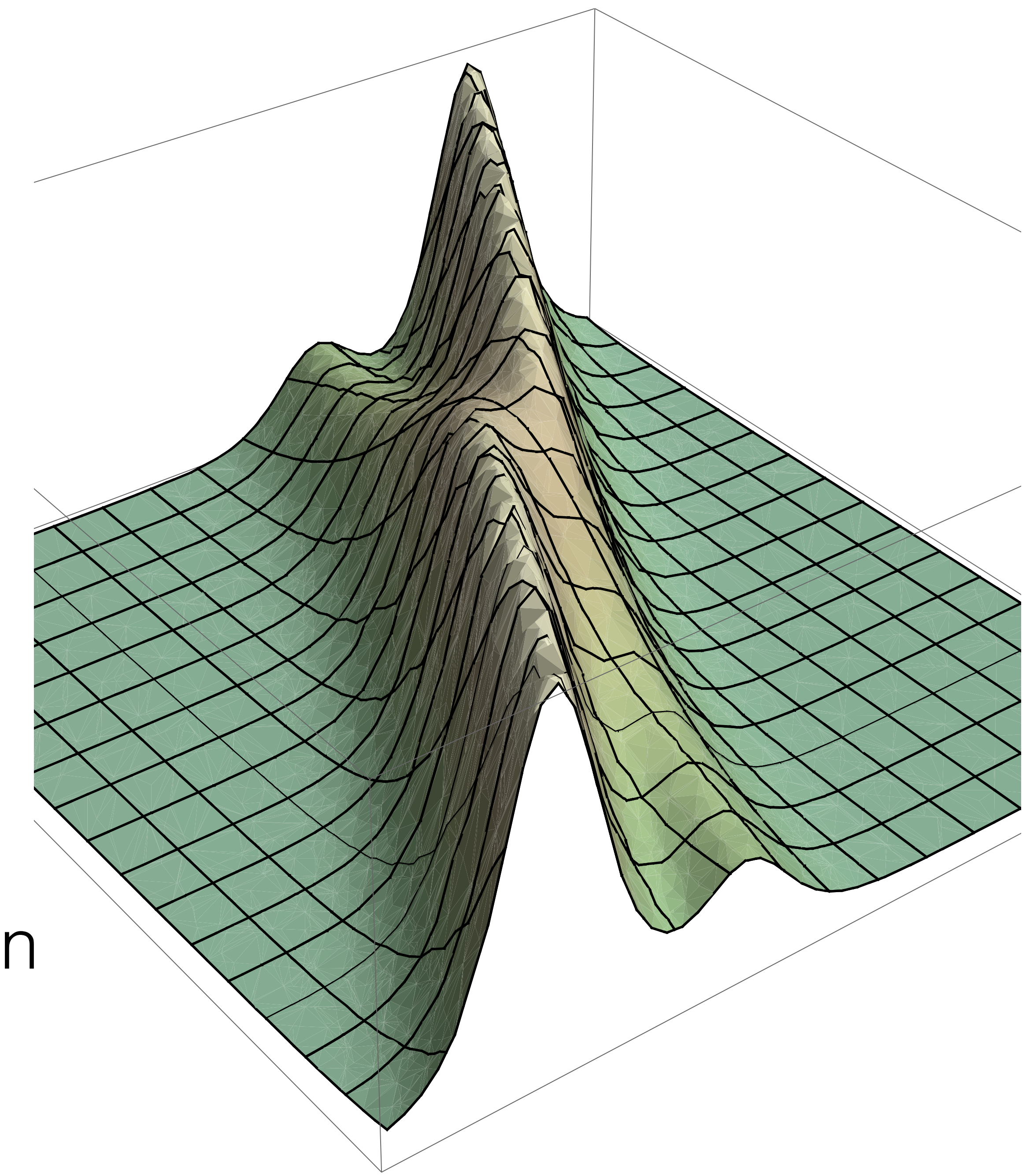
The first Painlevé equation

$$y'' = 6\,y^2 - x$$



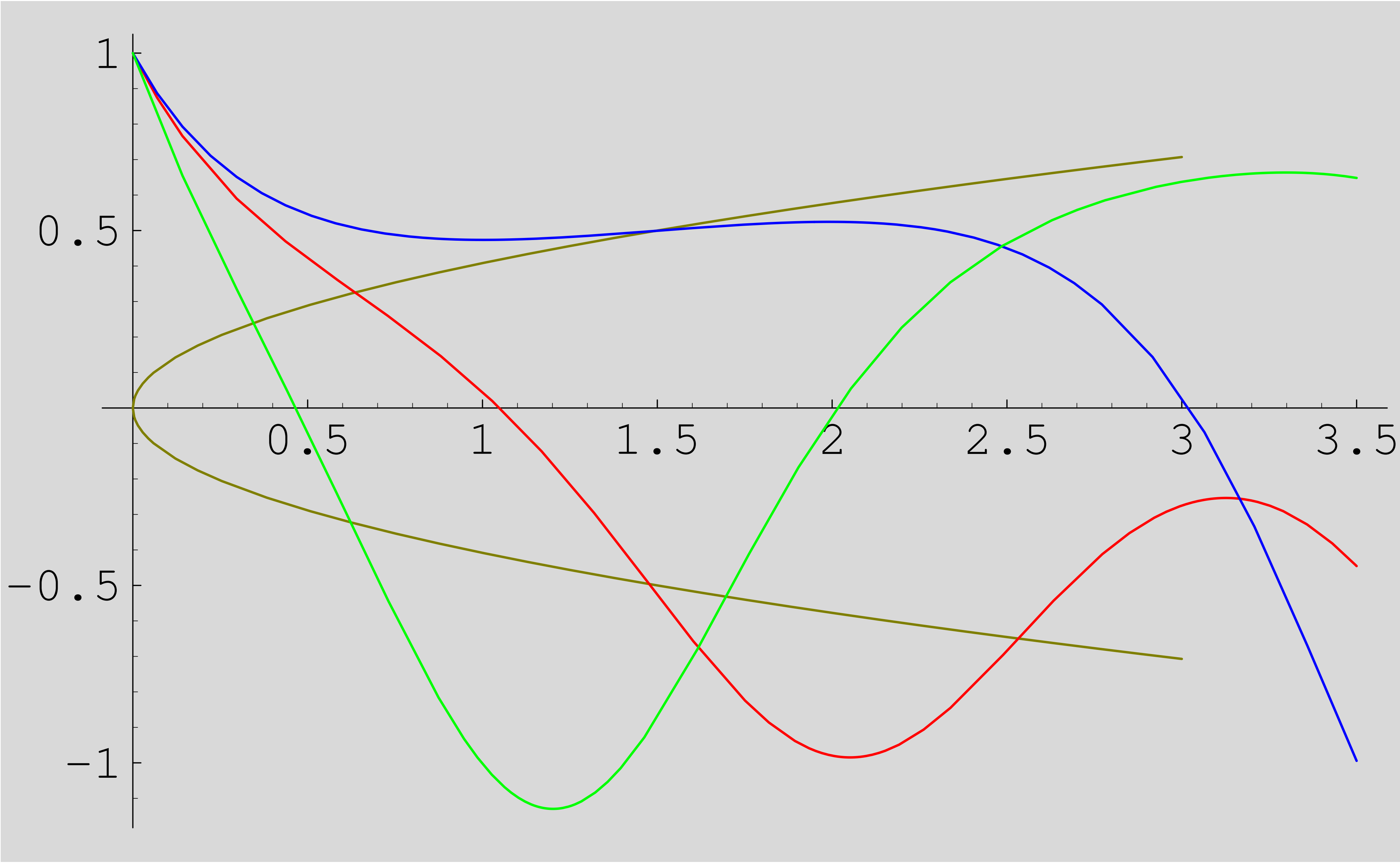
Applications

- Electrical structures of interfaces in steady electrolysis *L. Bass, Trans Faraday Soc 60 (1964)1656–1663*
- Spin-spin correlation functions for the 2D Ising model *TT Wu, BM McCoy, CA Tracy, E Barouch Phys Rev B13 (1976) 316–374*
- Spherical electric probe in a continuum gas *PCT de Boer, GSS Ludford, Plasma Phys 17 (1975) 29–41*
- Cylindrical Waves in General Relativity *S Chandrashekar, Proc. R. Soc. Lond. A 408 (1986) 209–232*
- Non-perturbative 2D quantum gravity *Gross & Migdal PRL 64(1990) 127-130*
- Orthogonal polynomials with non-classical weight function *AP Magnus J. Comput Appl. Anal. 57 (1995) 215–237*
- Level spacing distributions and the Airy kernel *CA Tracy, H Widom CMP 159 (1994) 151–174*
- Spatially dependent ecological models: *J & Morrison Anal Appl 6 (2008) 371-381*
- Gradient catastrophe in fluids: *Dubrovin, Grava & Klein J. Nonlin. Sci 19 (2009) 57-94*

What do we know about the solutions of these equations?

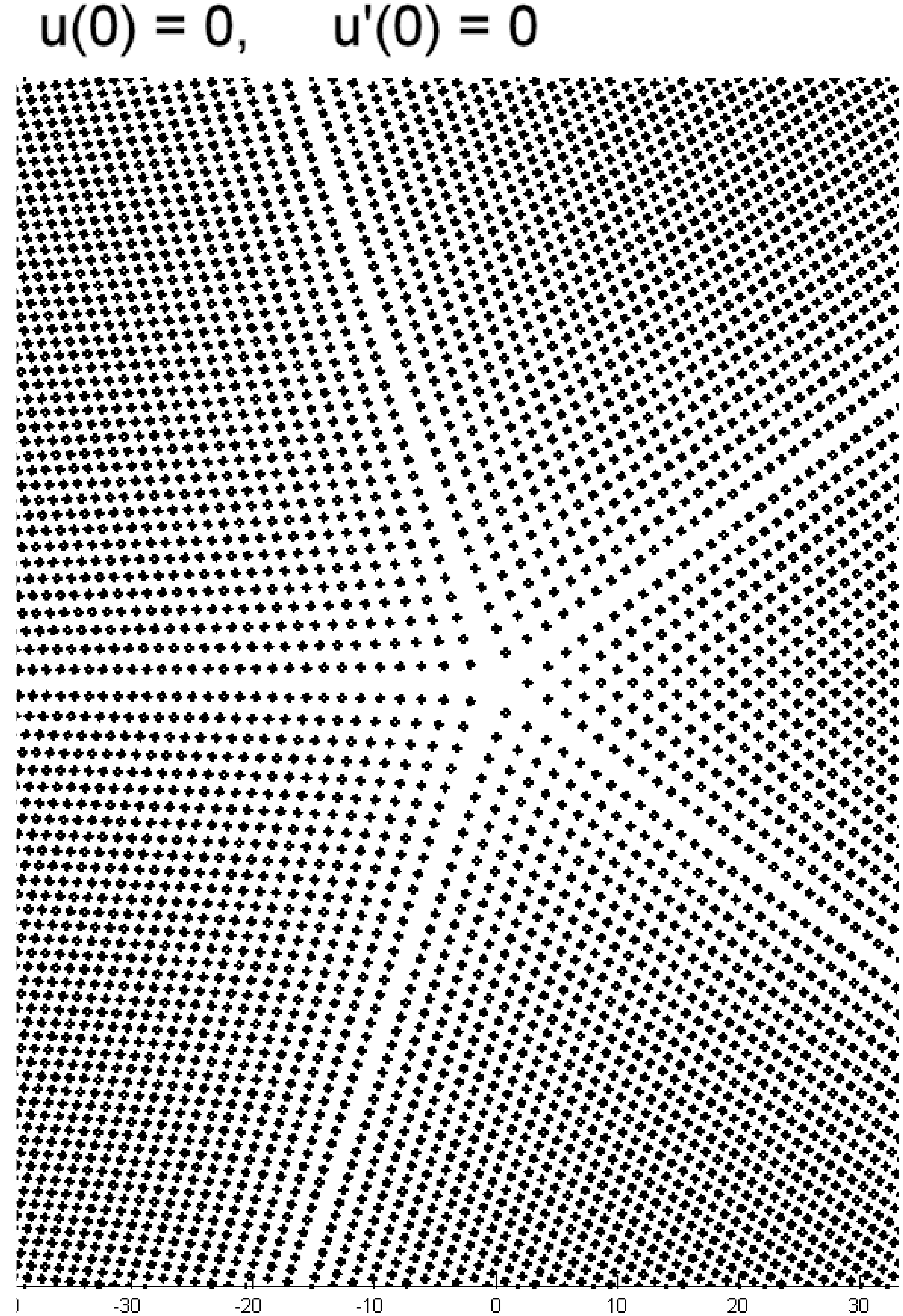
Real Solutions

Consider P_I $\quad w_{tt} = 6w^2 - t$ for $w(t), t \in \mathbb{R}$



Complex Solutions

- Movable poles
- Transcendentality of general solutions
- Special solutions
- Asymptotic behaviours



Fornberg & Weideman 2009

$\mathrm{P_I}$

General Solutions

- P_I: $w_{tt} = 6w^2 - t$
- in system form

$$\frac{d}{dt}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} w_2 \\ 6\,w_1^2 - t \end{pmatrix}$$

- has t-dependent Hamiltonian

$$H = \frac{w_2^2}{2} - 2\,w_1^3 + t\,w_1$$

Perturbed Form

- Or, in Boutroux's coordinates:

$$w_1 = t^{1/2}\, u_1(z),\ \ w_2 = t^{3/4} u_2(z),\ \ z = \frac{4}{5} t^{5/4}$$

$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix} = \begin{pmatrix} u_2 \\ 6u_1^2 - 1 \end{pmatrix} - \frac{1}{5z} \begin{pmatrix} 2u_1 \\ 3u_2 \end{pmatrix}$$

- a perturbation of an elliptic curve as $|z| \to \infty$

$$E = \frac{u_2^2}{2} - 2u_1^3 + u_1 \ \Rightarrow\ \frac{dE}{dz} = \frac{1}{5z}\,(6E + 4u_1)$$

Perturbed Form

- Or, in Boutroux's coordinates:

$$w_1 = t^{1/2} u_1(z), \;\; w_2 = t^{3/4} u_2(z), \;\; z = \frac{4}{5} t^{5/4}$$

$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix} = \begin{pmatrix} u_2 \\ 6u_1^2 - 1 \end{pmatrix} - \frac{1}{5z} \begin{pmatrix} 2u_1 \\ 3u_2 \end{pmatrix}$$

- a perturbation of an elliptic curve as $|z| \to \infty$

$$E = \frac{u_2^2}{2} - 2u_1^3 + u_1 \;\; \Rightarrow \;\; \frac{dE}{dz} = \frac{1}{5z} \, (6E + 4u_1)$$

Similarly

- $\mathrm{P_{II}}$: $$w_{tt} = 2w^3 + tw + \alpha$$
- $\mathrm{P_{IV}}$: $$w_{tt} = \frac{{w_t}^2}{2w} + \frac{3w^3}{2} + 4tw^2 + 2(t^2 - 1 + \alpha_1 + 2\,\alpha_2)w - \frac{2\alpha_1^2}{w}$$

have system forms that are perturbations of autonomous systems in the limit $|t| \to \infty$

Projective Space

- What if *x*, *y* become unbounded?
- Use projective geometry: $x = \dfrac{u}{w}, y = \dfrac{v}{w}$

$$[x, y, 1] = [u, v, w] \;\; \in \mathbb{CP}^2$$

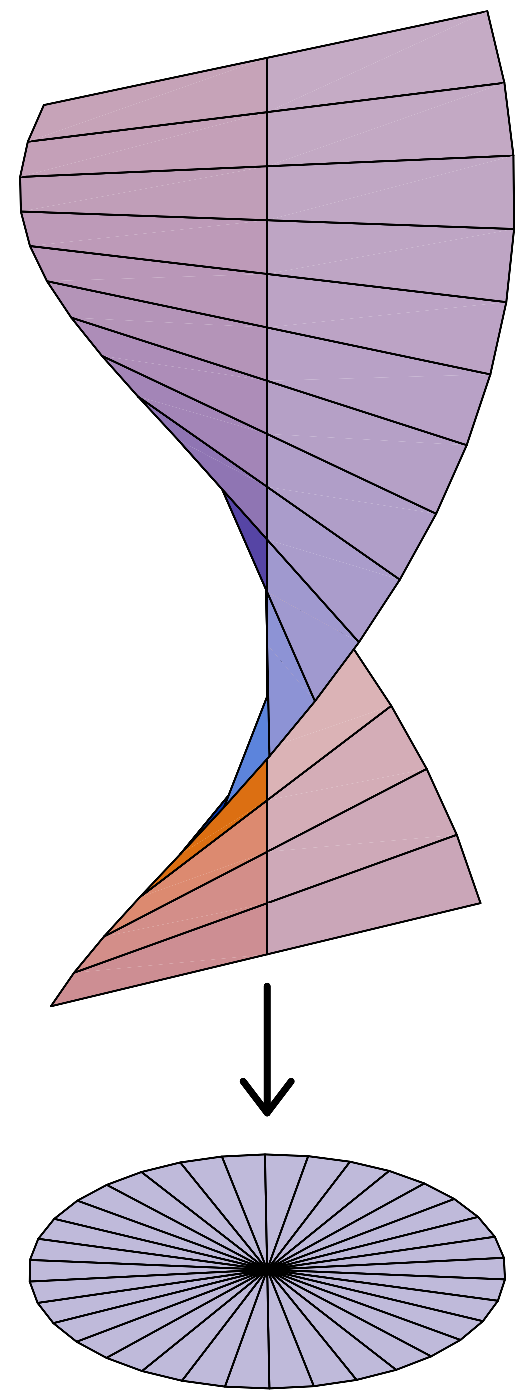
- The level curves of P_{I} are now

$$F_{\mathrm{I}} = wv^2 - 4u^3 + g_2uw^2 + g_3w^3$$

all intersecting at the base point [0, 1, 0].

$\Rightarrow$ To describe solutions, resolve the flow through this point

Resolving a base pt



From JJ Duistermaat, QRT Maps and Elliptic Surfaces, Springer Verlag, 2010

Resolution

- "Blow up" the singularity or base point:

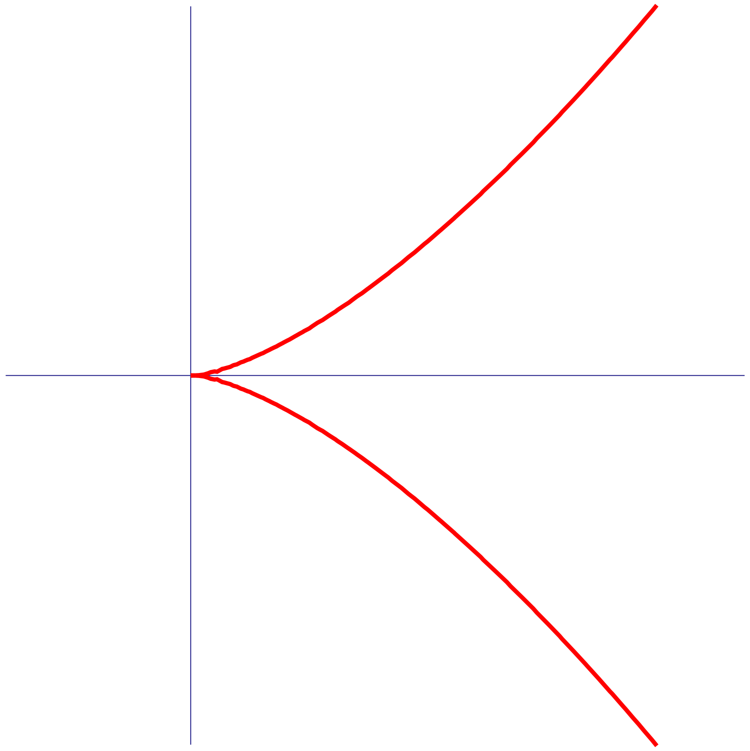
$$\begin{aligned} f(x,y) &= y^2 - x^3 \\ (x,y) &= (x_1, x_1\, y_1) \\ \Rightarrow & x_1^2\, y_1^2 - x_1^3 = 0 \\ \Leftrightarrow & x_1^2\, (y_1^2 - x_1) = 0 \end{aligned}$$

- Note that

$$x_1 = x, y_1 = y/x$$

Method

$$y^2 = x^3$$



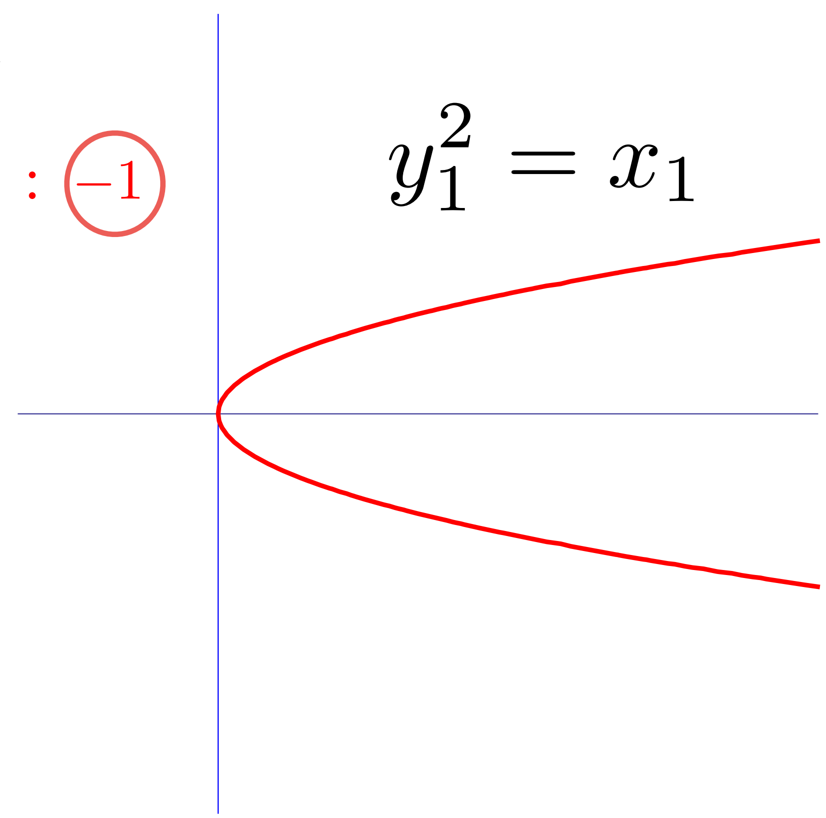
$$f(x, y) = y^2 - x^3$$

$$(x, y) = (x_1, x_1\, y_1)$$

$$f(x_1, x_1\, y_1) = x_1^2(y_1^2 - x_1)$$

$L_1 : -1$

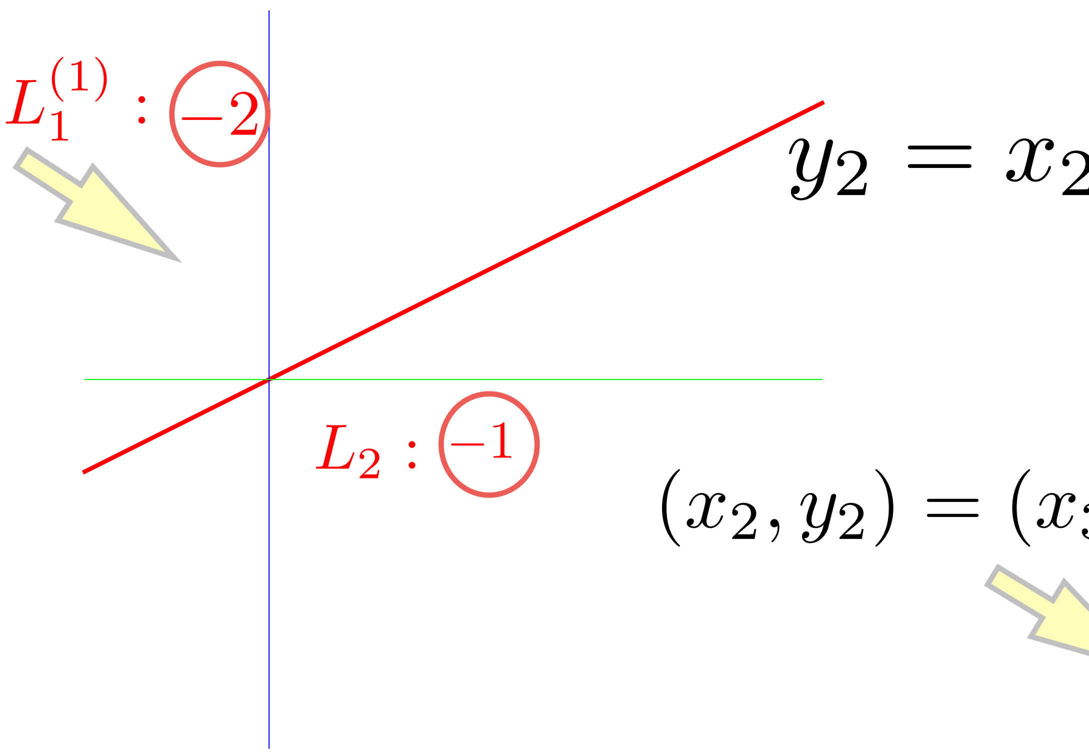
$$y_1^2 = x_1$$



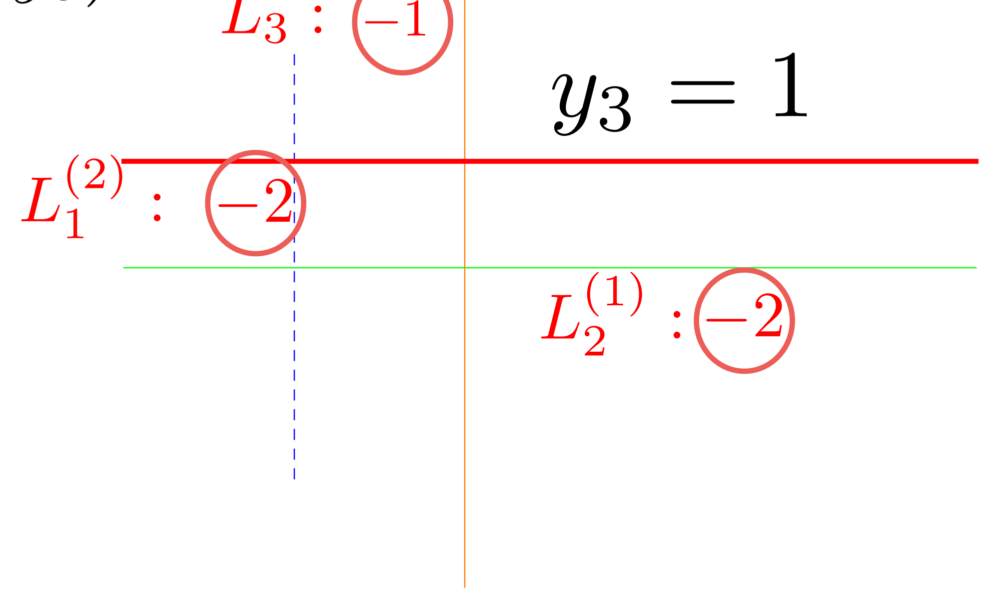
$$f_1(x_1, y_1) = y_1^2 - x_1$$

$$f_1(x_2\,y_2, y_2) = y_2(y_2 - x_2)$$

$$(x_1, y_1) = (x_2\,y_2, y_2)$$



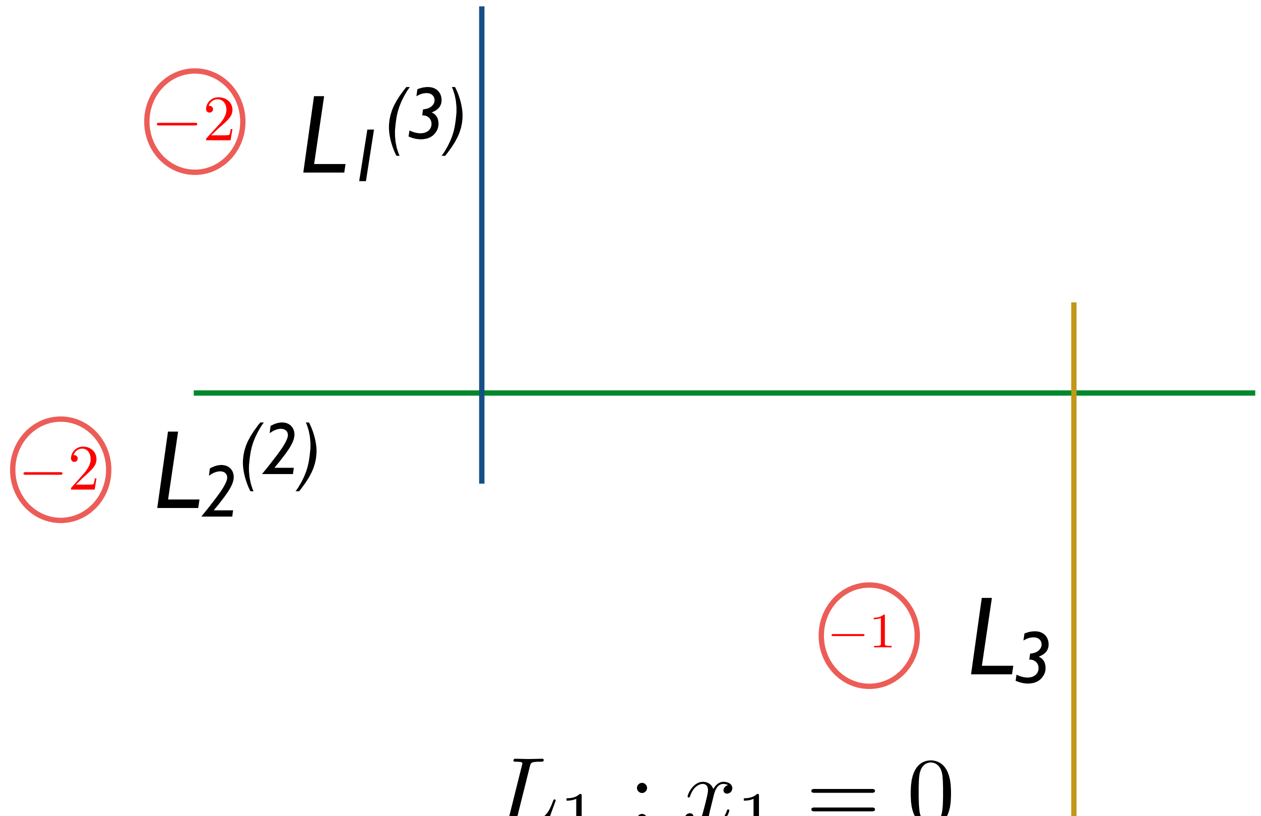
$$(x_2, y_2) = (x_3, x_3\,y_3)$$



$$f_2(x_2, y_2) = y_2 - x_2$$

$$f_2(x_3, x_3\,y_3) = x_3(y_3 - 1)$$

Initial-Value Space



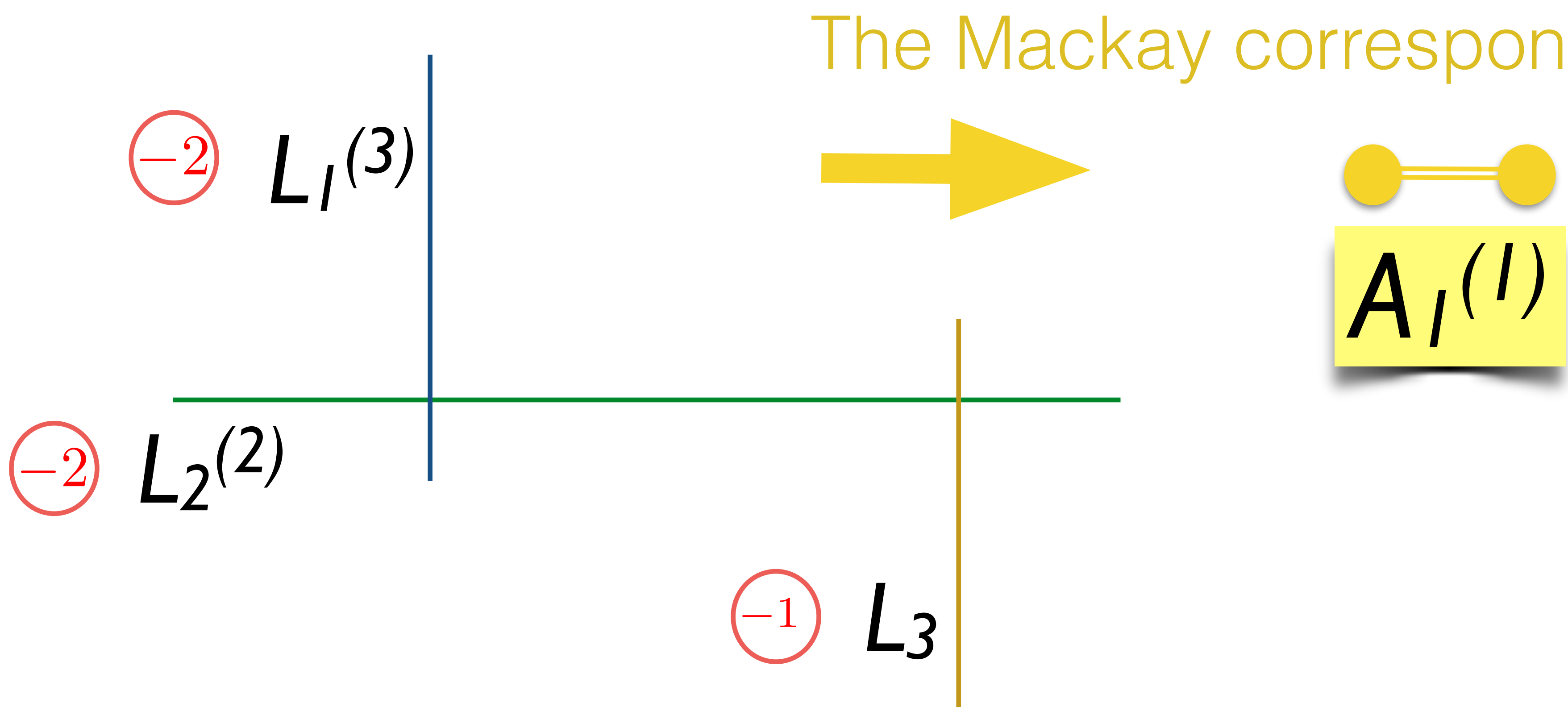
$$L_1 : x_1 = 0$$

$$L_2 : y_2 = 0$$

$$L_3 : x_3 = 0$$

Now the space is compactified and regularised.

Initial-Value Space



$$L_1 : x_1 = 0$$

$$L_2 : y_2 = 0$$

$$L_3 : x_3 = 0$$

Now the space is compactified and regularised.

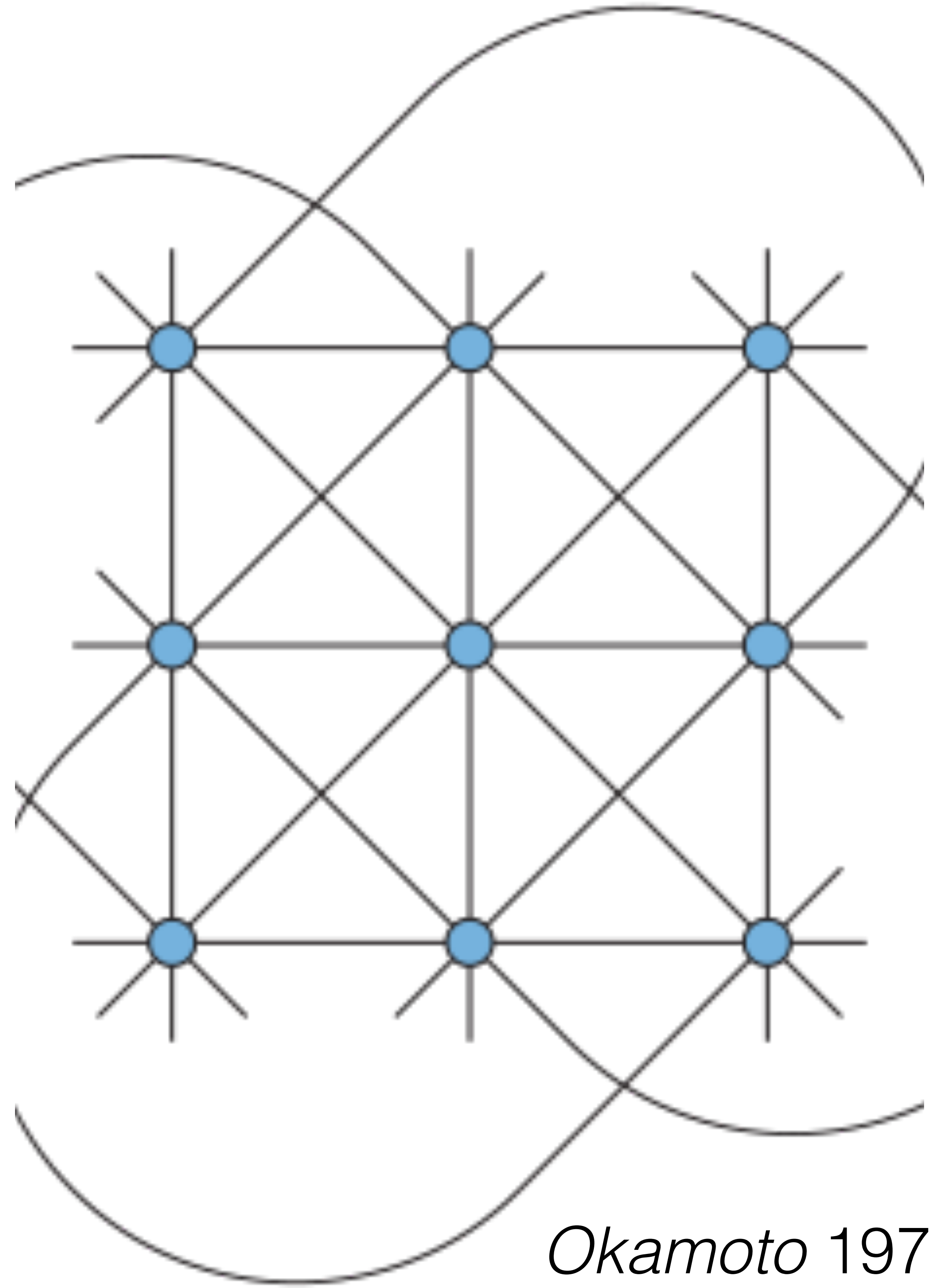
Good Resolution

- When all curves intersect each other transversally at distinct points, the result is called a "good resolution".
- Hironaka's theorem guarantees this in complex projective space.
- Note: each transformation had the form

$$x_1 = x, y_1 = y/x$$

Unifying Property

The space of initial values of a Painlevé system is resolved by “blowing up” 9 points in $\mathbb{CP}^2$
(or 8 points in $\mathbb{P}^1$x$\mathbb{P}^1$)



Okamoto 1979

Sakai 2001

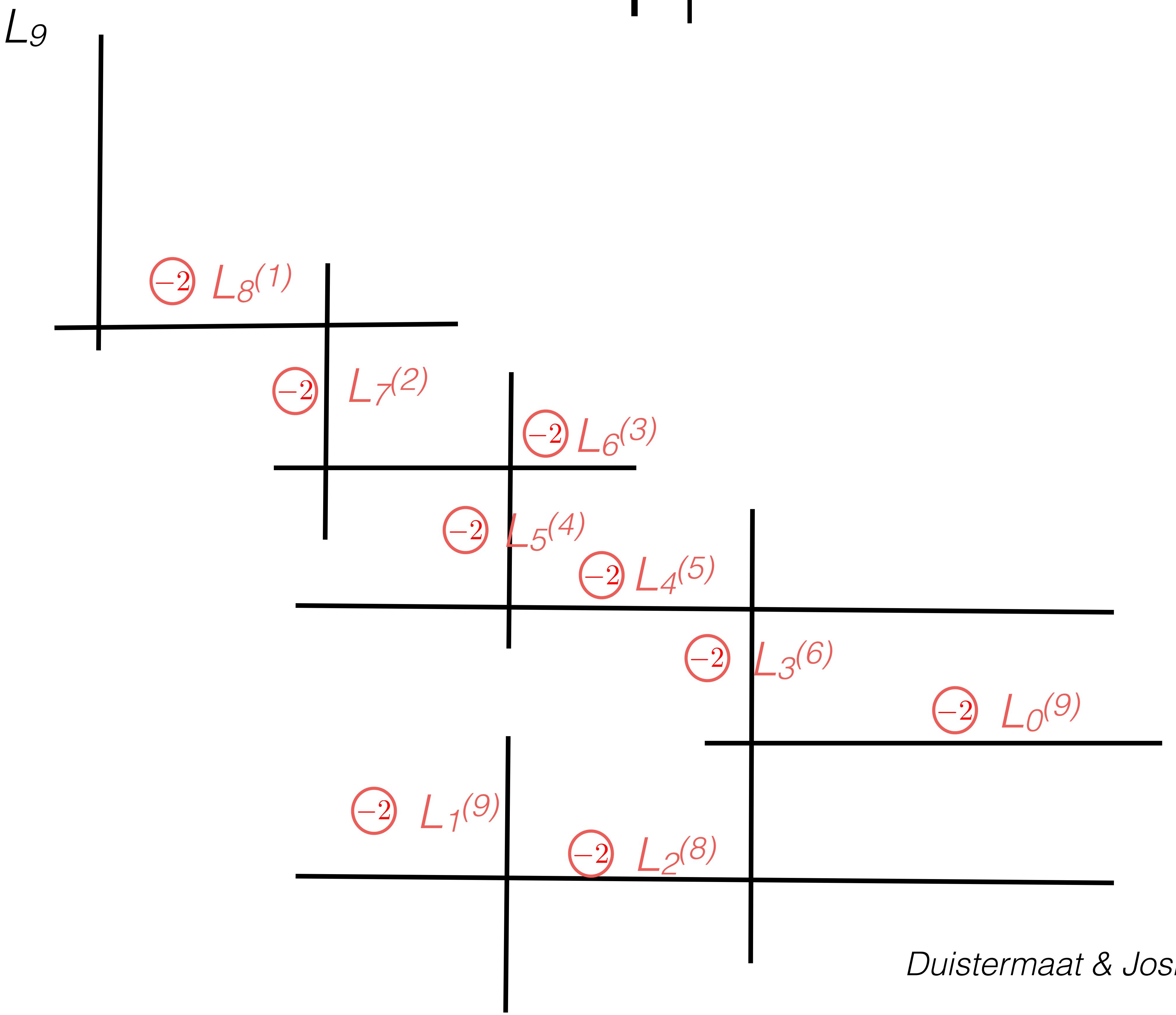
Initial-Value Space of P_I

- There are nine base points:

$$b_0 : u_{031} = 0, u_{032} = 0$$
$$b_1 : u_{111} = 0, u_{112} = 0$$
$$b_2 : u_{211} = 0, u_{212} = 0$$
$$b_3 : u_{311} = 4, u_{312} = 0$$
$$b_4 : u_{411} = 4, u_{412} = 0$$
$$b_5 : u_{511} = 0, u_{512} = 0$$
$$b_6 : u_{611} = 0, u_{612} = 0$$
$$b_7 : u_{711} = 32, u_{712} = 0$$
$$b_8 : u_{811} = -\frac{2^8}{(5\,z)}, u_{812} = 0$$

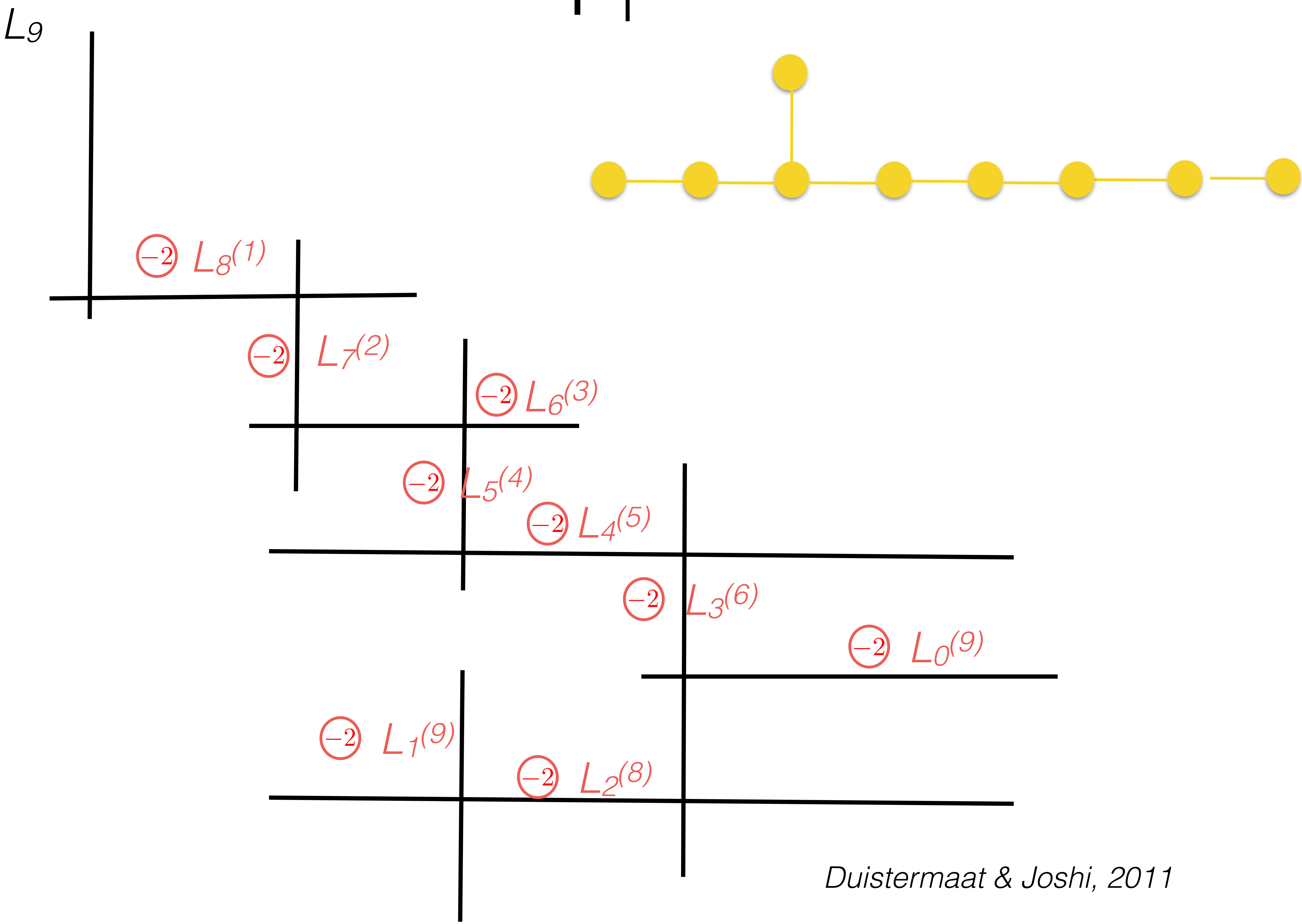
- Only the last one differs from the elliptic case.

P_I



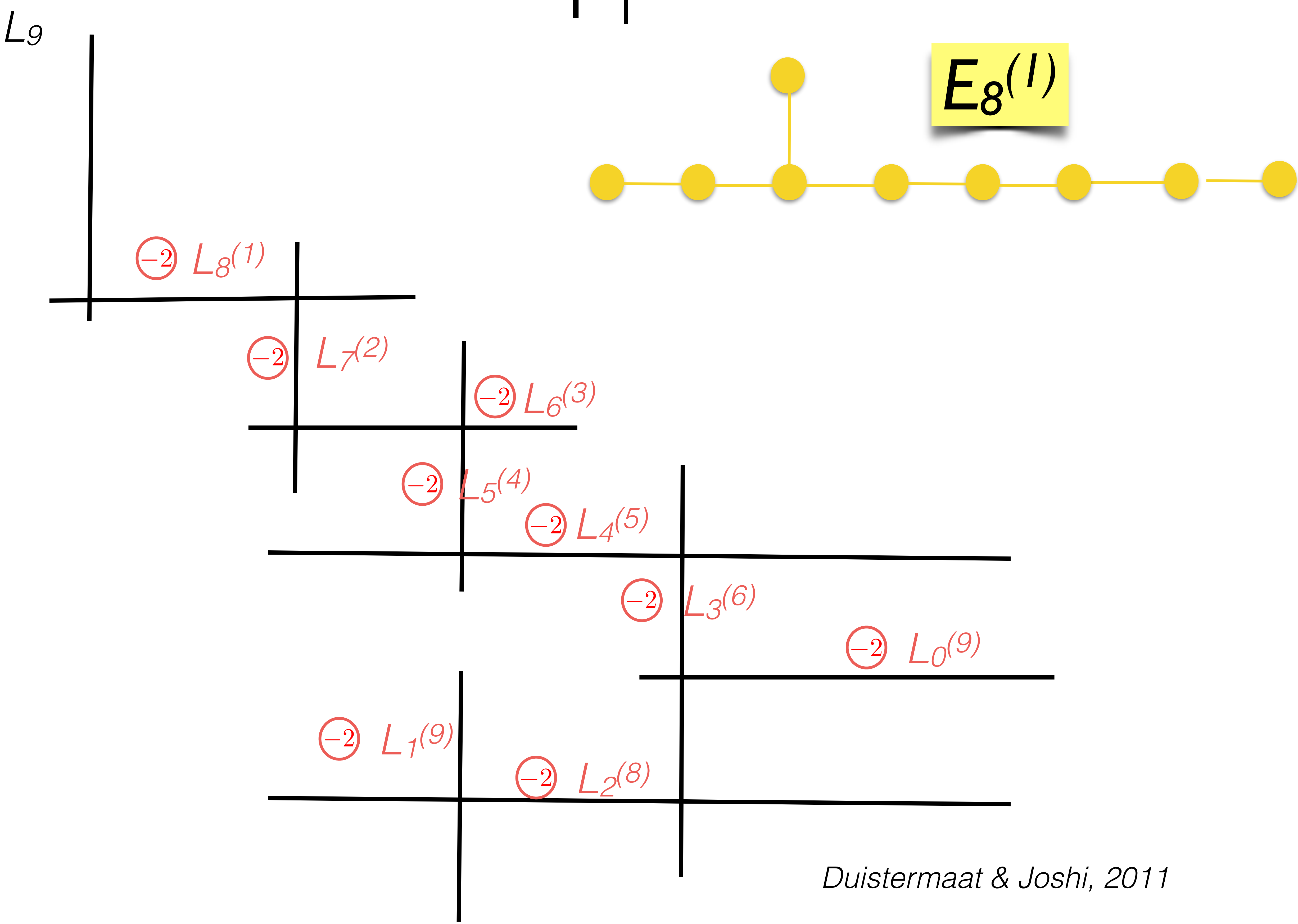
Duistermaat & Joshi, 2011

P_I



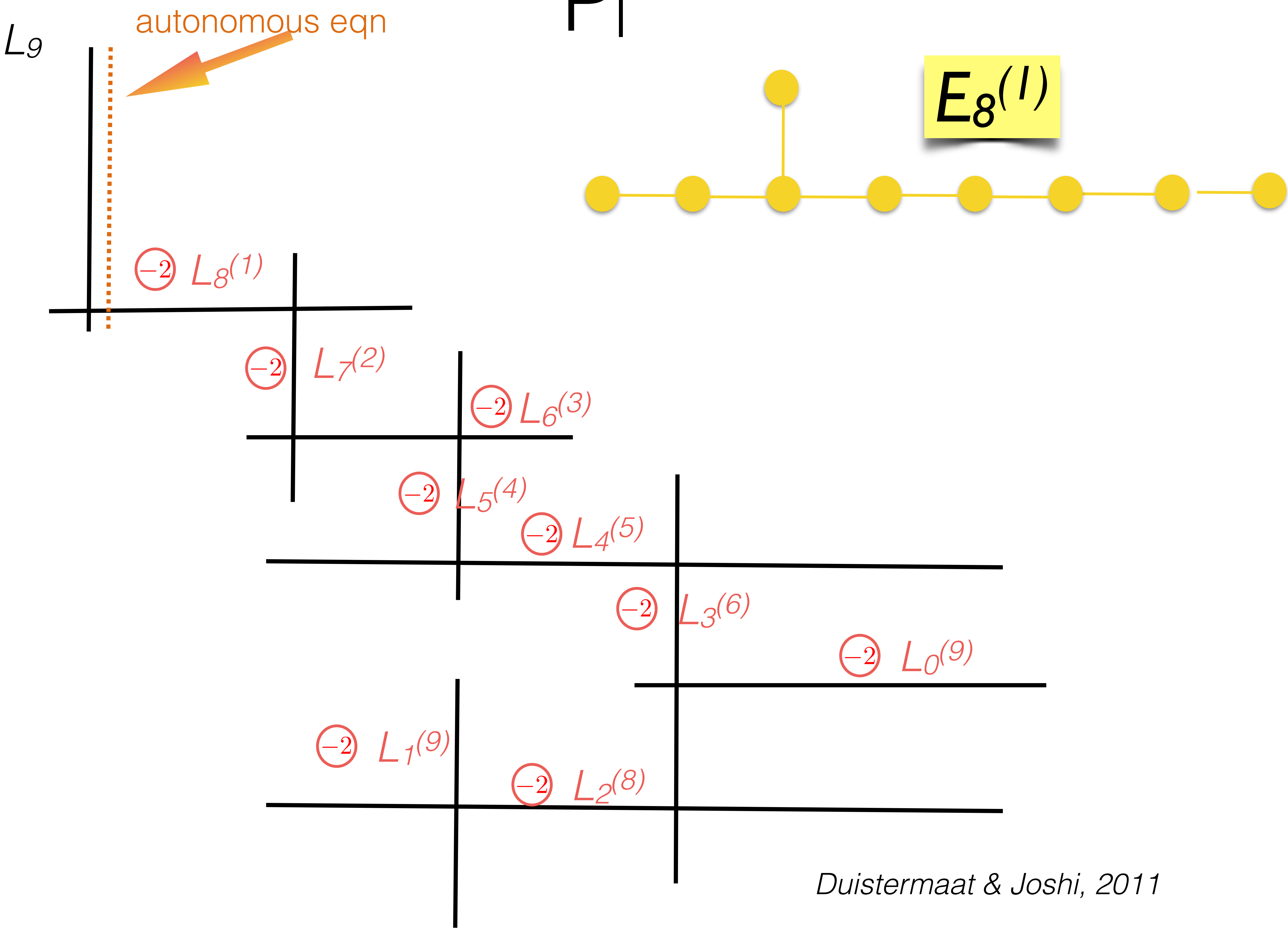
Duistermaat & Joshi, 2011

P_I



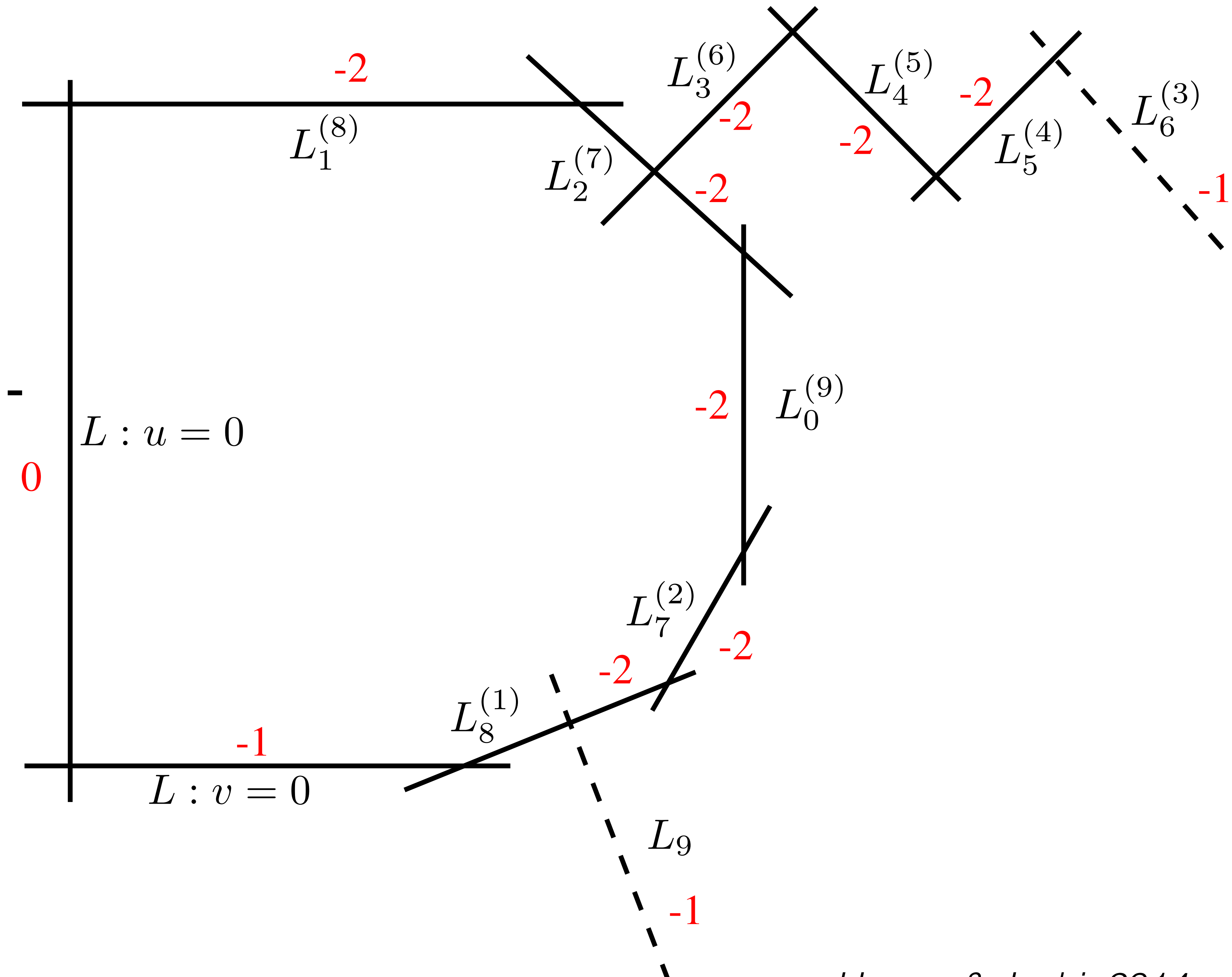
Duistermaat & Joshi, 2011

P_I



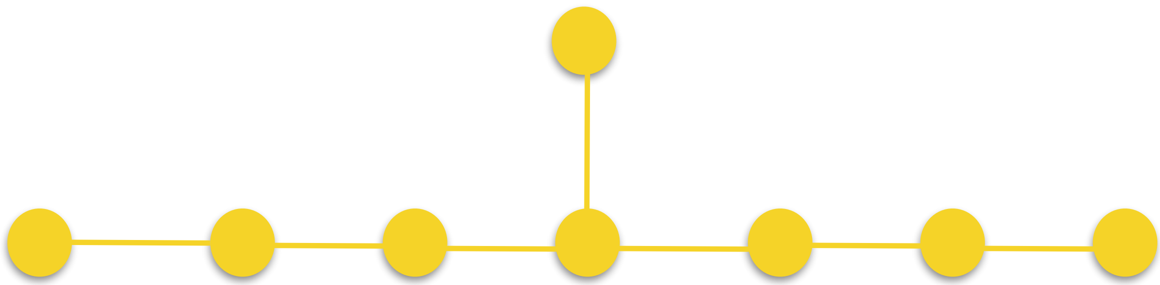
Duistermaat & Joshi, 2011

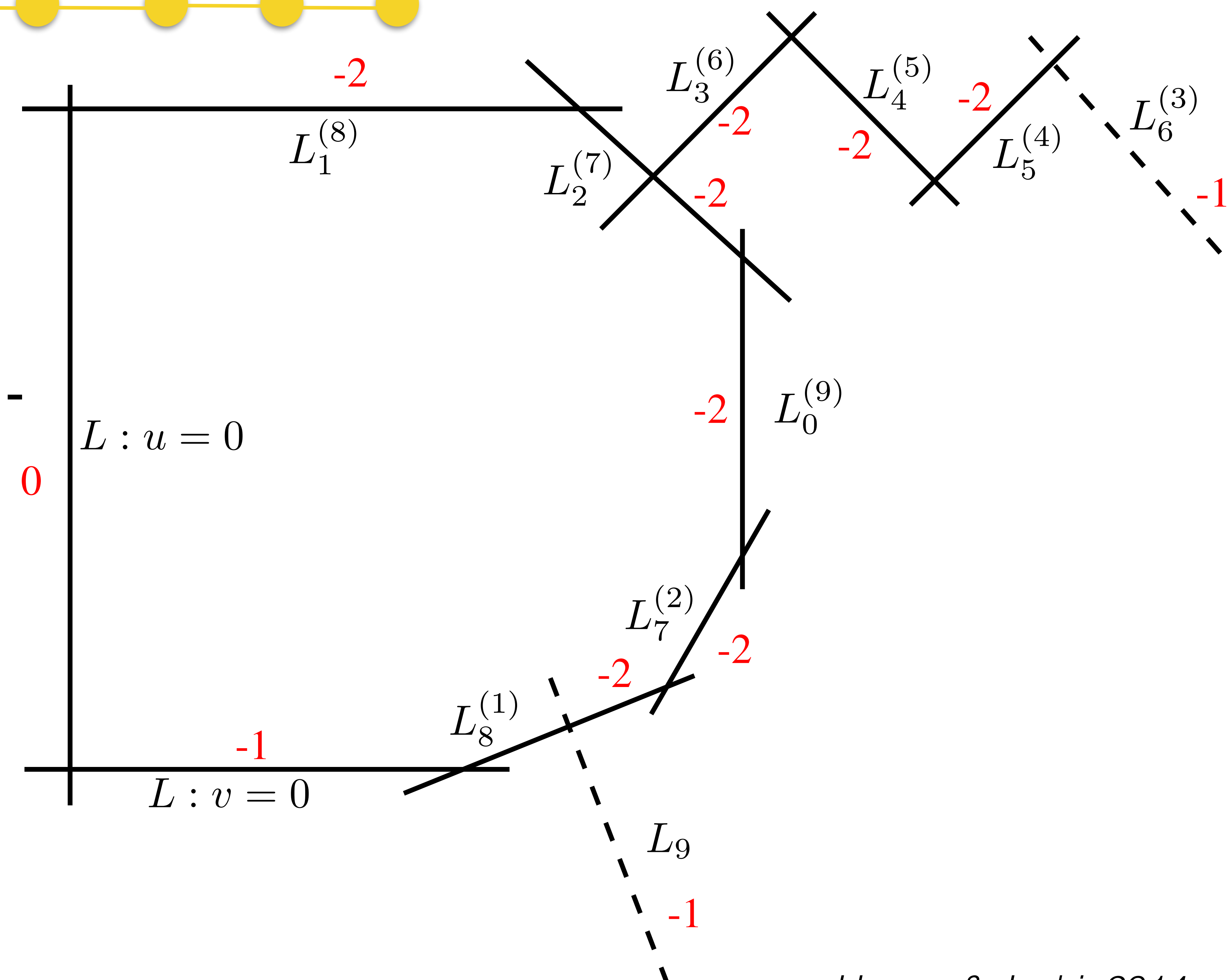
P_{II}



Howes & Joshi, 2014

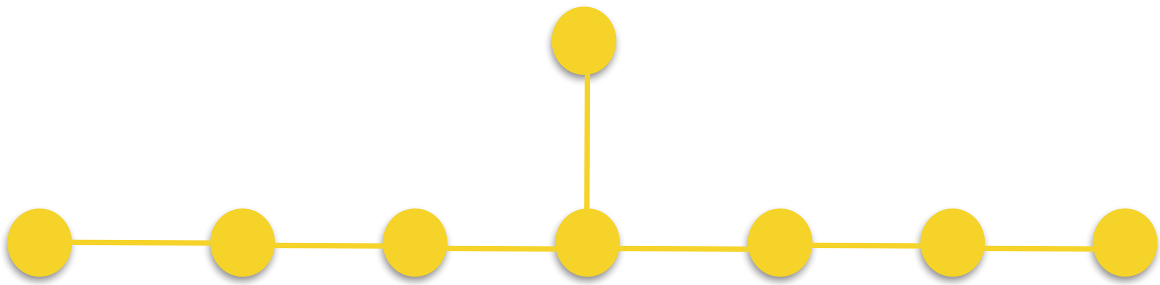
$\mathrm{P_{II}}$



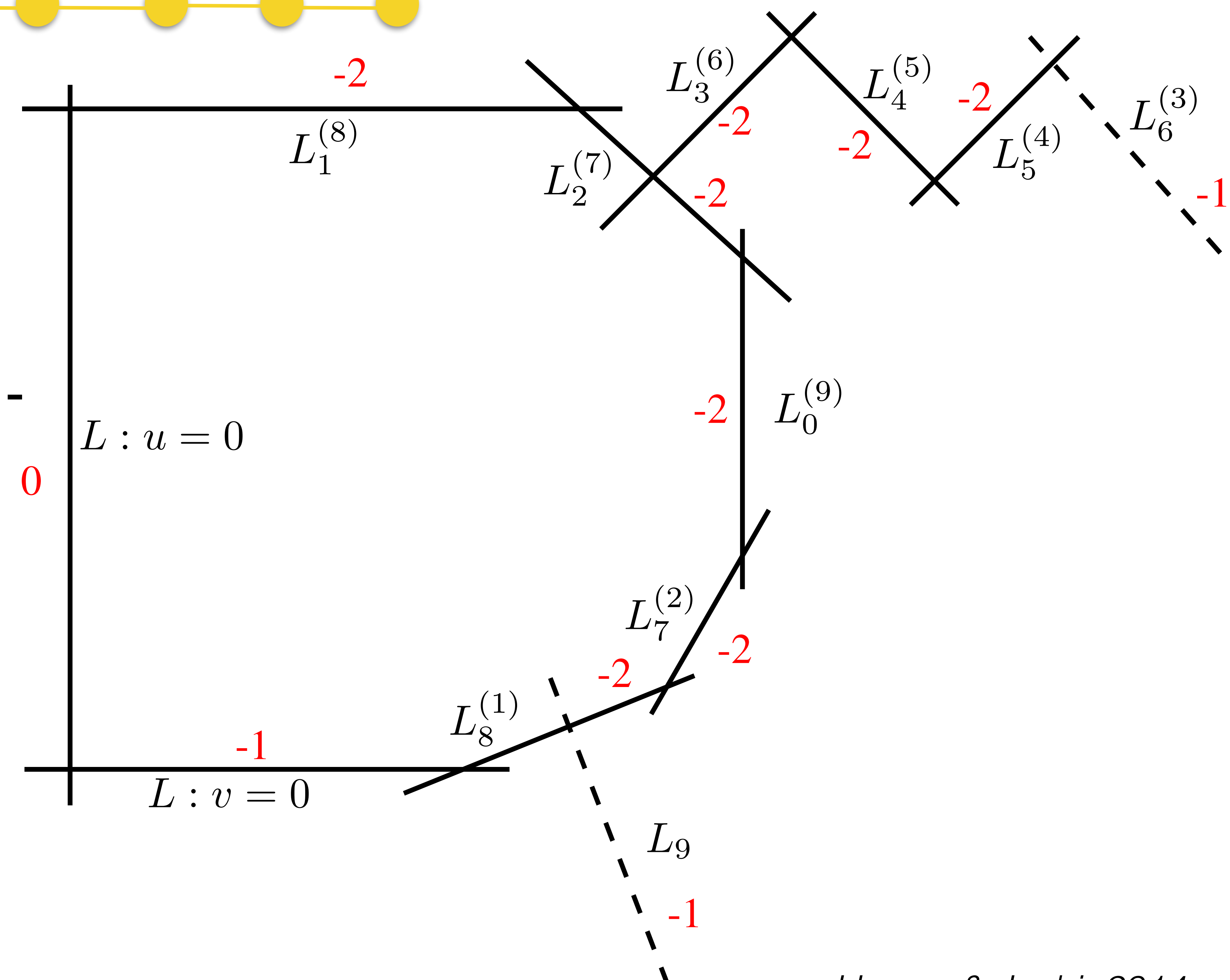


Howes & Joshi, 2014

P_{II}

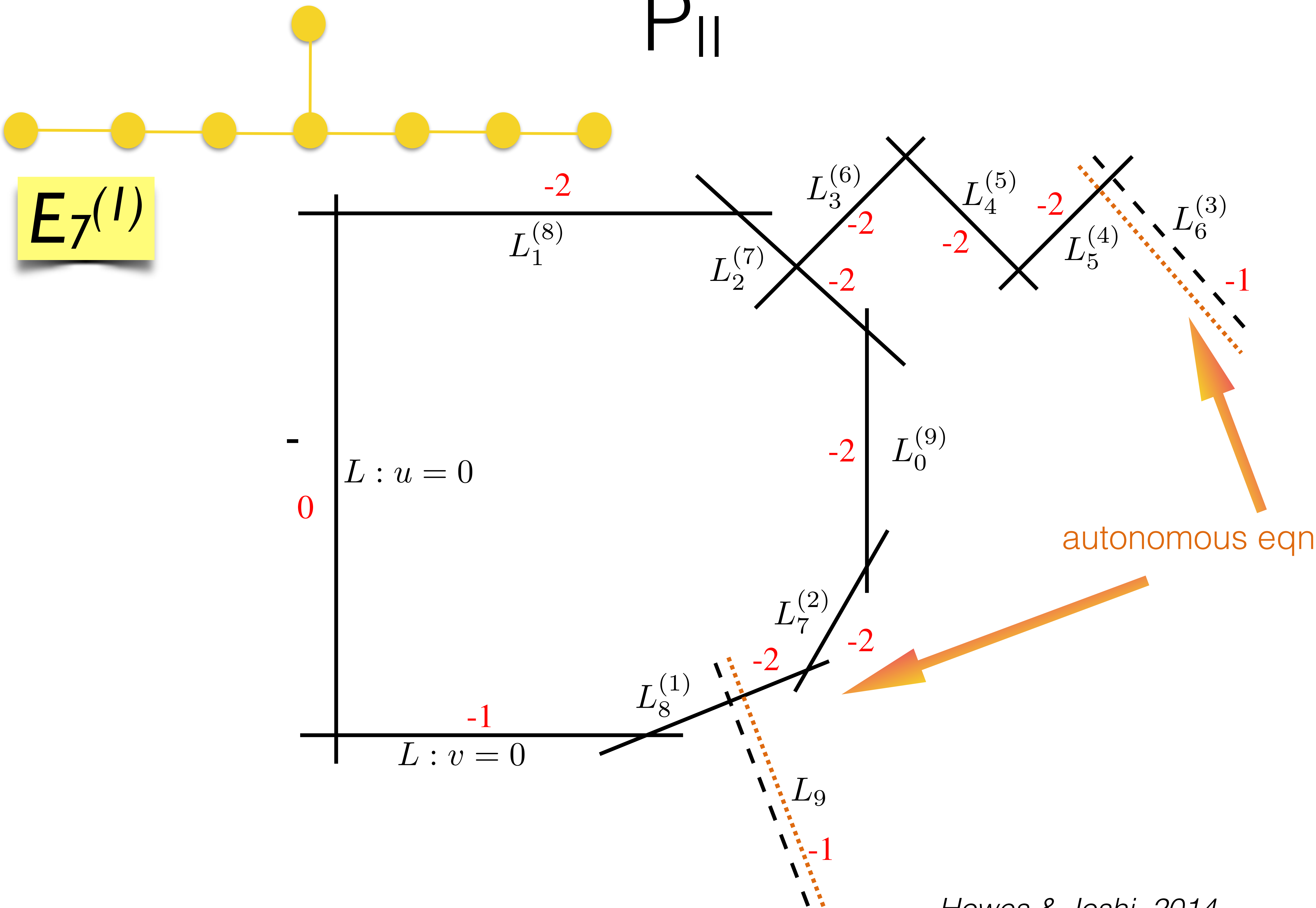


$E_7^{(1)}$



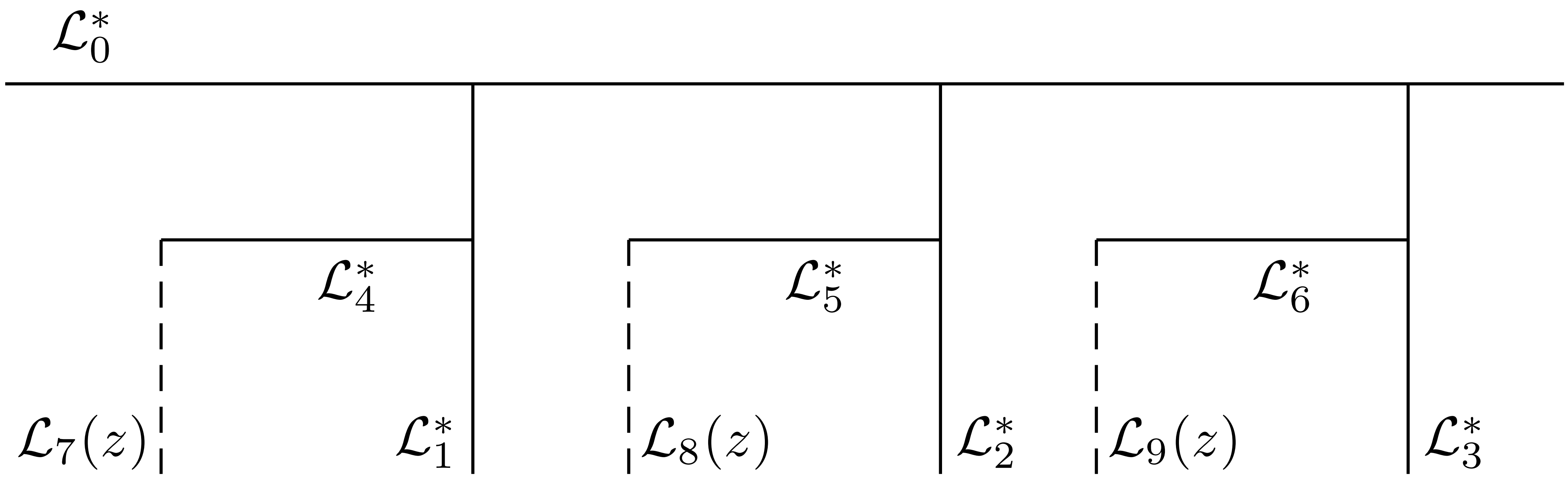
Howes & Joshi, 2014

P_{II}



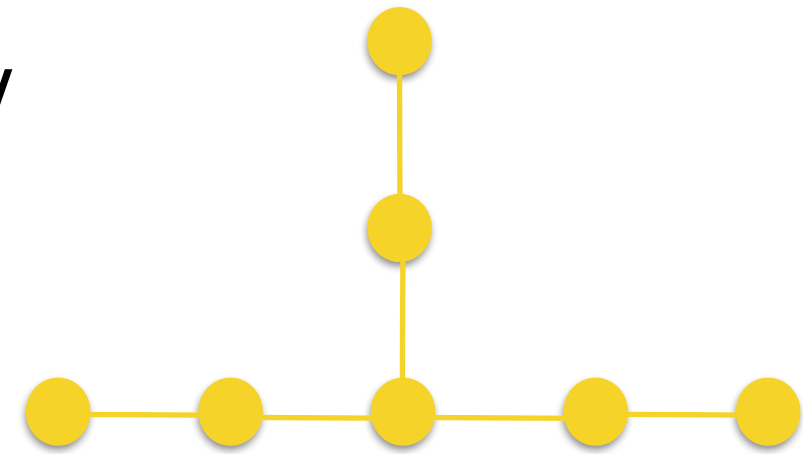
Howes & Joshi, 2014

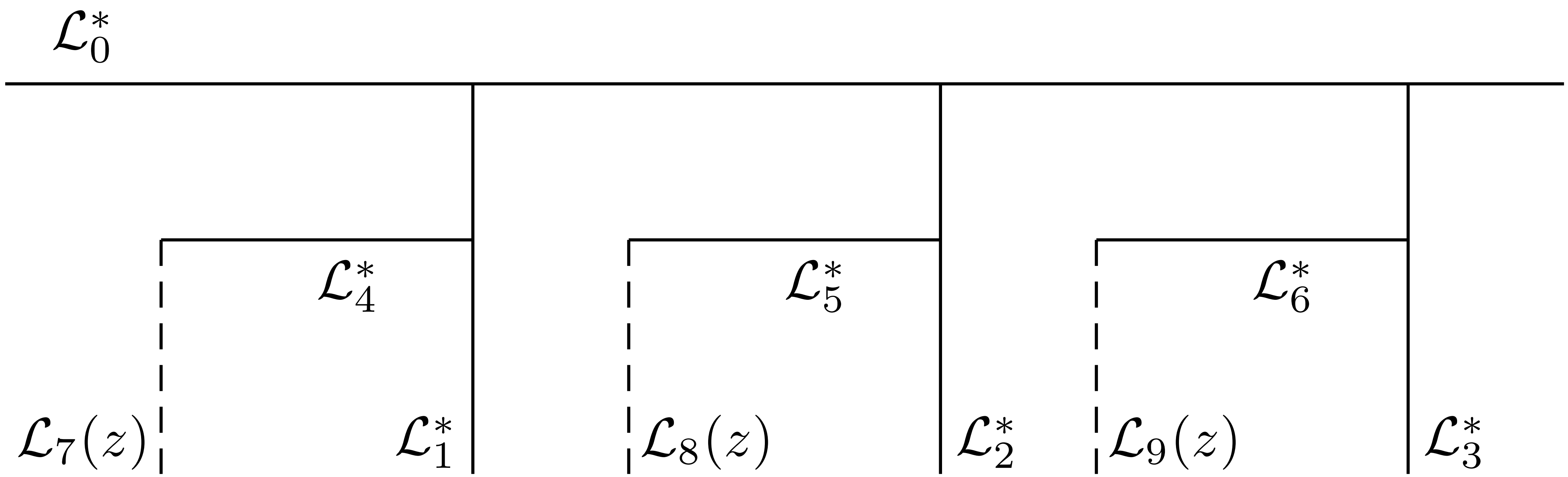
P_{IV}



Joshi & Radnovic, 2015

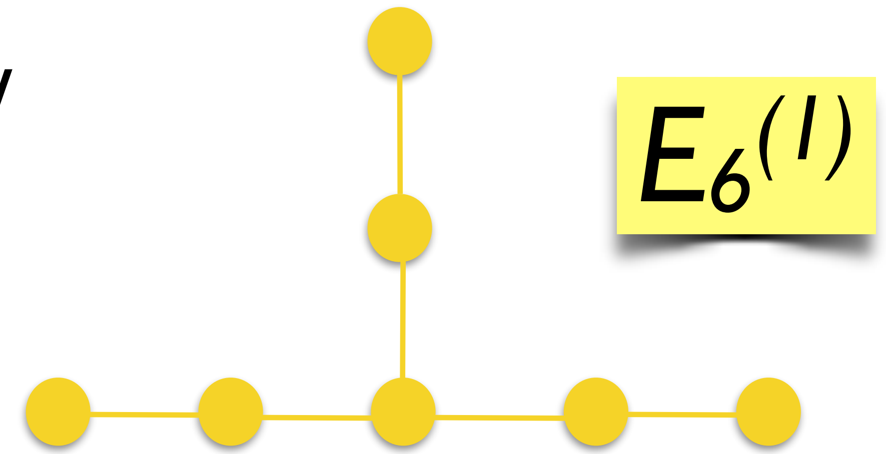
P_{IV}

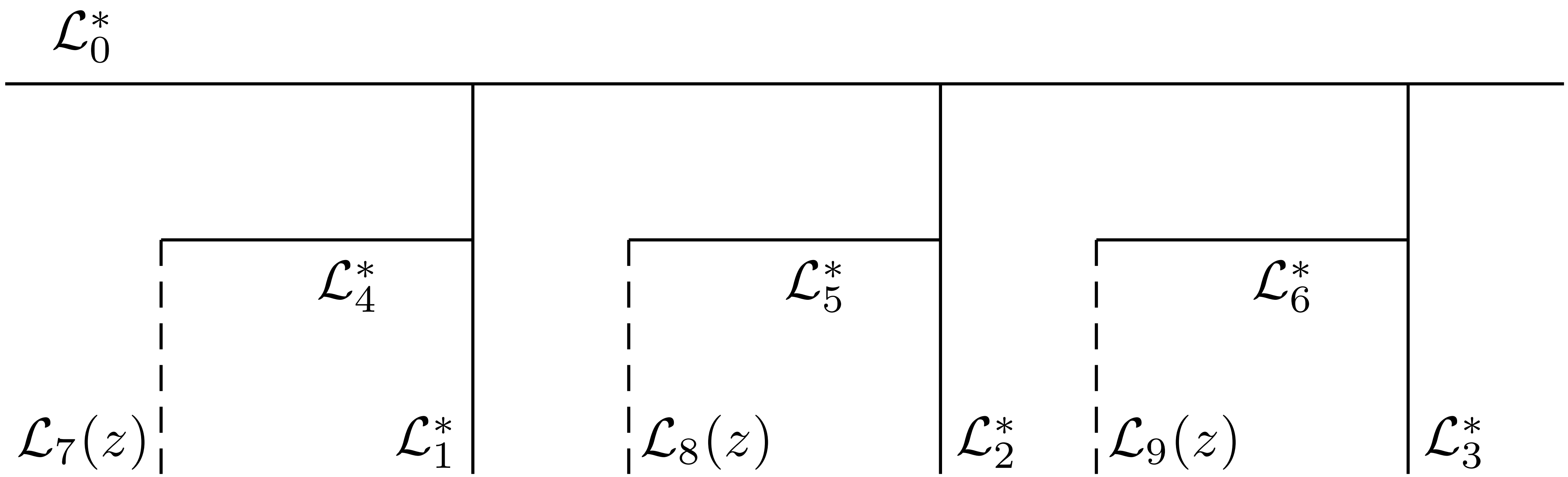




Joshi & Radnovic, 2015

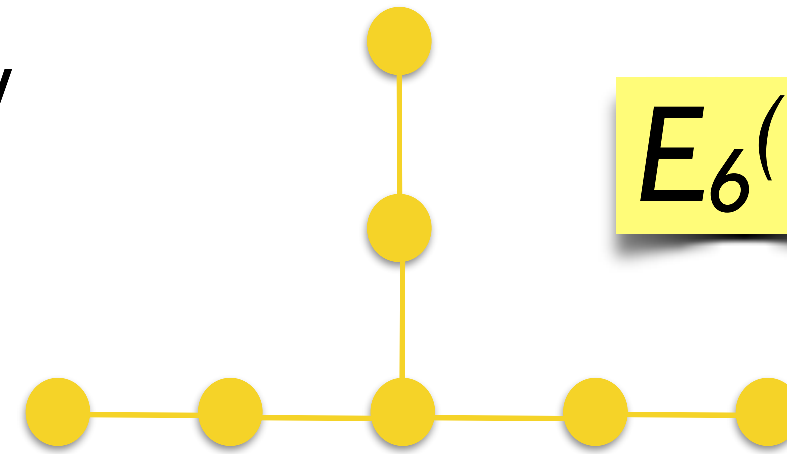
P_{IV}



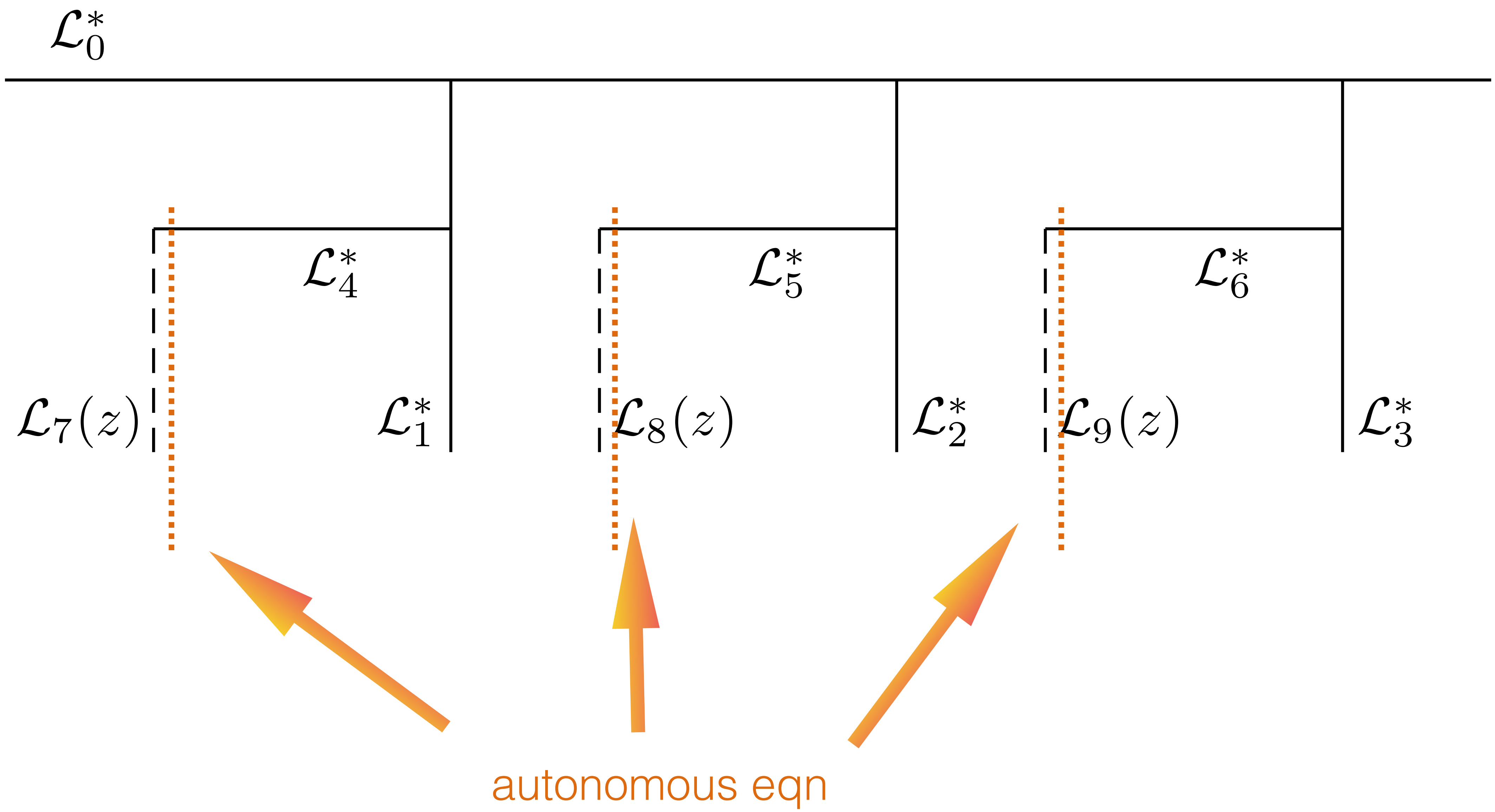


Joshi & Radnovic, 2015

P_{IV}



$E_6^{(1)}$



Joshi & Radnovic, 2015

Sakai's Description I

$$
\begin{tikzcd}
\text{Ell:} & A_0^{(1)} \arrow[d, dashed] & & & & & & & A_7^{(1)} \arrow[dr] & \\
\text{Mul:} & A_0^{(1)} \arrow[r] \arrow[d, dashed] & A_1^{(1)} \arrow[r] \arrow[d, dashed] & A_2^{(1)} \arrow[r] \arrow[d, dashed] & A_3^{(1)} \arrow[r] \arrow[dr] & A_4^{(1)} \arrow[r] \arrow[dr] & A_5^{(1)} \arrow[r] \arrow[dr] & A_6^{(1)} \arrow[r] \arrow[ur] \arrow[dr] & {A_7^{(1)}}' \arrow[r] \arrow[dr] \arrow[dd, bend left] & A_8^{(1)} \arrow[dd, bend left] \\
\text{Add:} & A_0^{(1)} \arrow[r] & A_1^{(1)} \arrow[r] & A_2^{(1)} \arrow[rr] & & D_4^{(1)} \arrow[r] & D_5^{(1)} \arrow[r] \arrow[dr] & D_6^{(1)} \arrow[r] \arrow[dr] & D_7^{(1)} \arrow[r] \arrow[dr] & D_8^{(1)} \\
 & & & & & & & E_6^{(1)} \arrow[r] & E_7^{(1)} \arrow[r] & E_8^{(1)}
\end{tikzcd}
$$

Initial-value spaces of all continuous and discrete Painlevé equations

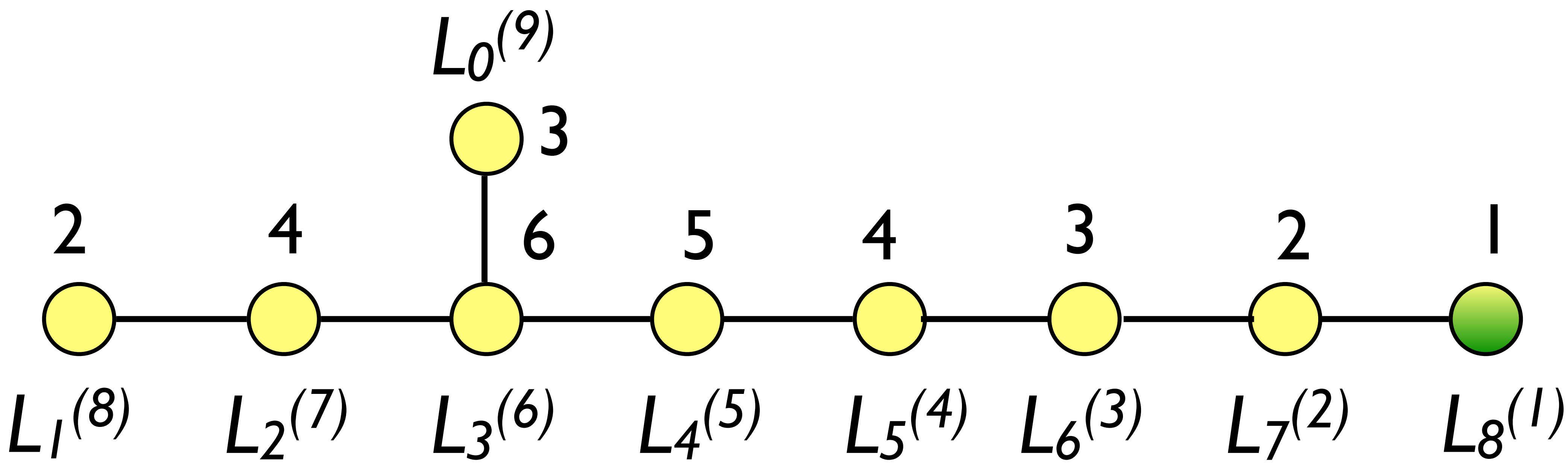
Sakai 2001

Global results for P_{I} , P_{II} , P_{IV}

- The union of exceptional lines is a repeller for the flow.
- There exists a complex limit set, which is non-empty, connected and compact.
- Every solution of P_{I} , every solution of P_{II} whose limit set is not {0}, and every non-rational solution of P_{IV} intersects the last exceptional line(s) infinitely many times => infinite number of movable poles and movable zeroes.

Duistermaat & J (2011); Howes & J (2014); J & Radnovic (2014)

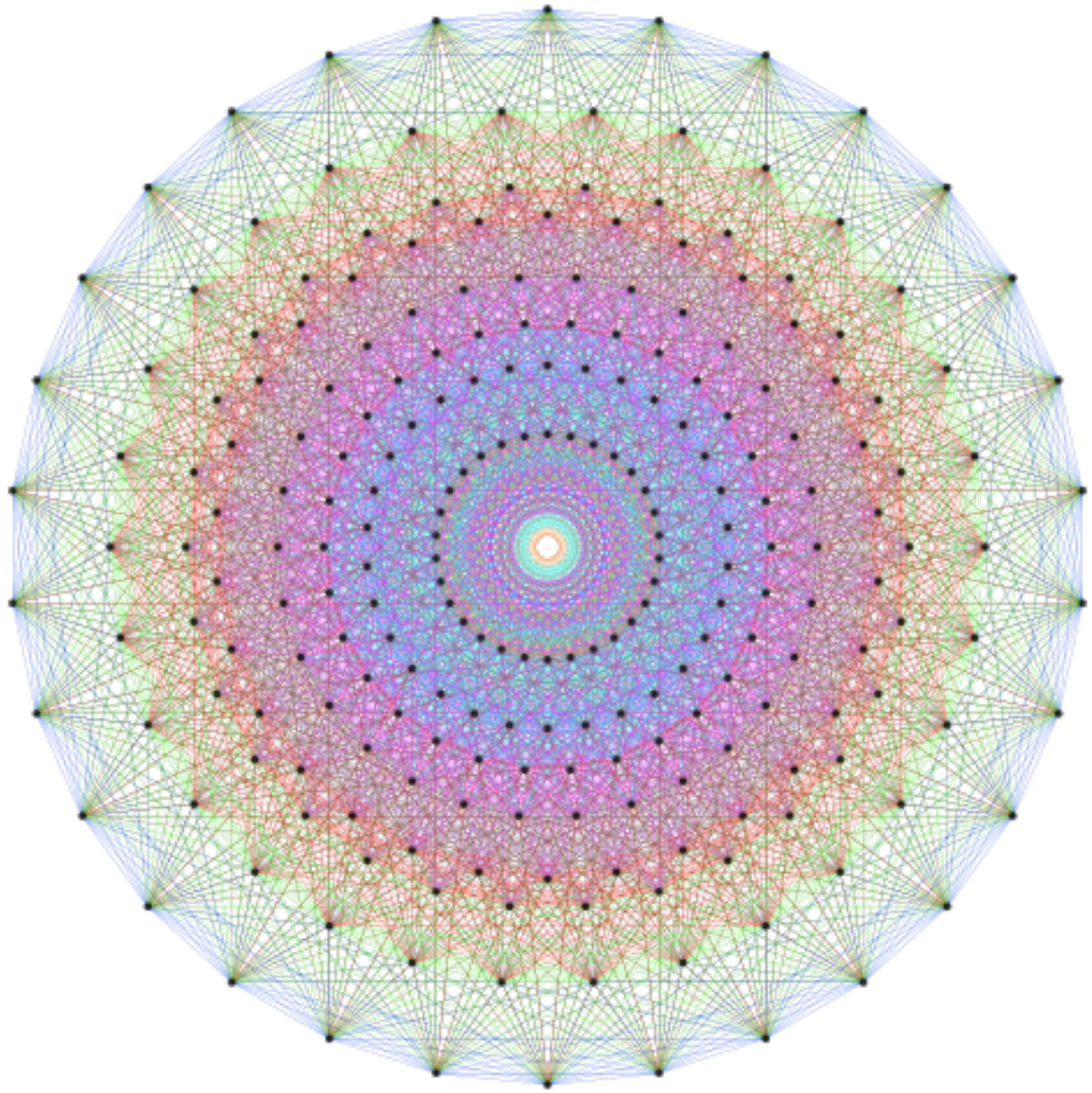
Inside P_I



Affine extended E_8

At the heart

At the heart



E_8

Symmetry groups

- Affine Weyl groups:
  - Natural lattice translations
  - Cremona isometries

$\Leftrightarrow$ Painlevé equations

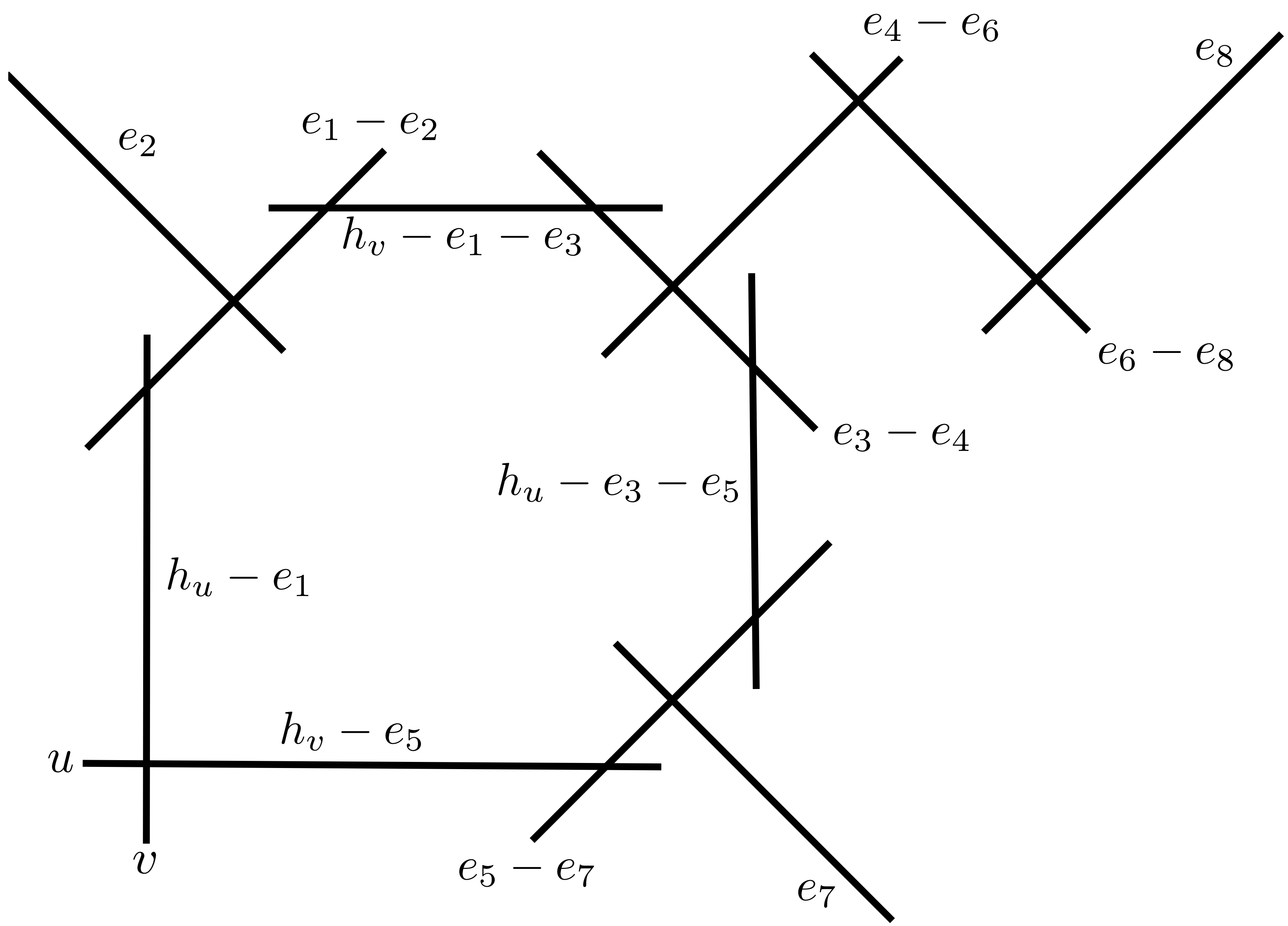
Sakai’s Description II

$$
\begin{tikzcd}
\text{Ell:} & E_8^{(1)} \arrow[d] & & & & & & & {A_1^{(1)}}' \arrow[dr] & \\
\text{Mul:} & E_8^{(1)} \arrow[r]\arrow[d] & E_7^{(1)} \arrow[r]\arrow[d] & E_6^{(1)} \arrow[r]\arrow[d] & D_5^{(1)} \arrow[r]\arrow[dr] & A_4^{(1)} \arrow[r]\arrow[dr] & (A_2+A_1)^{(1)} \arrow[r]\arrow[dr] & (A_1+A_1)^{(1)} \arrow[r]\arrow[ur]\arrow[dr] & A_1^{(1)} \arrow[r]\arrow[dr]\arrow[dd, bend left] & A_0^{(1)} \arrow[dd, bend left] \\
\text{Add:} & E_8^{(1)} \arrow[r] & E_7^{(1)} \arrow[r] & E_6^{(1)} \arrow[rr] & & D_4^{(1)} \arrow[r] & A_3^{(1)} \arrow[r]\arrow[dr] & 2A_1^{(1)} \arrow[r]\arrow[dr] & {A_1^{(1)}}' \arrow[r]\arrow[dr] & A_0^{(1)} \\
 & & & & & & & A_2^{(1)} \arrow[r] & A_1^{(1)} \arrow[r] & A_0^{(1)}
\end{tikzcd}
$$

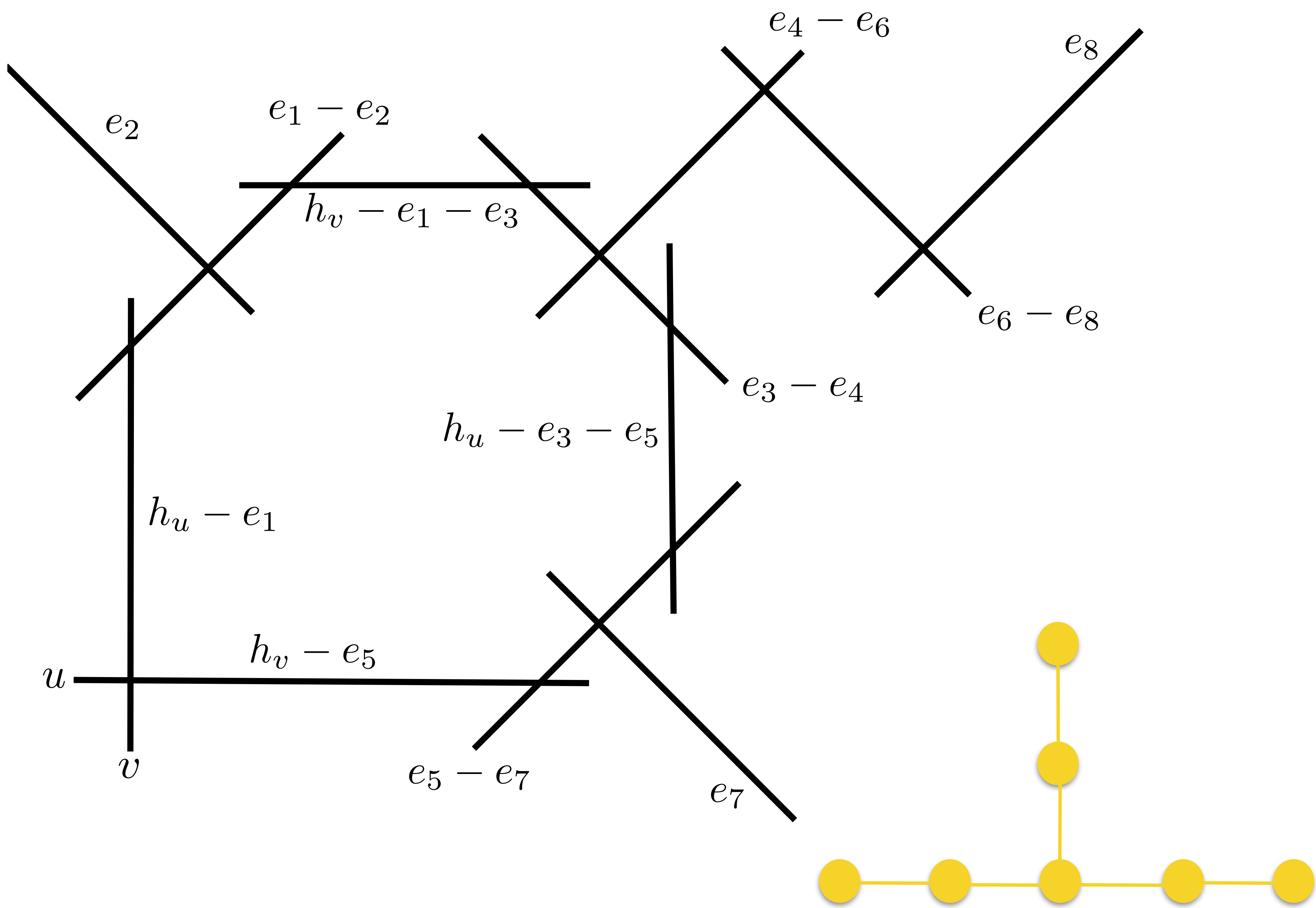
Symmetry groups of Painlevé equations

Sakai 2001

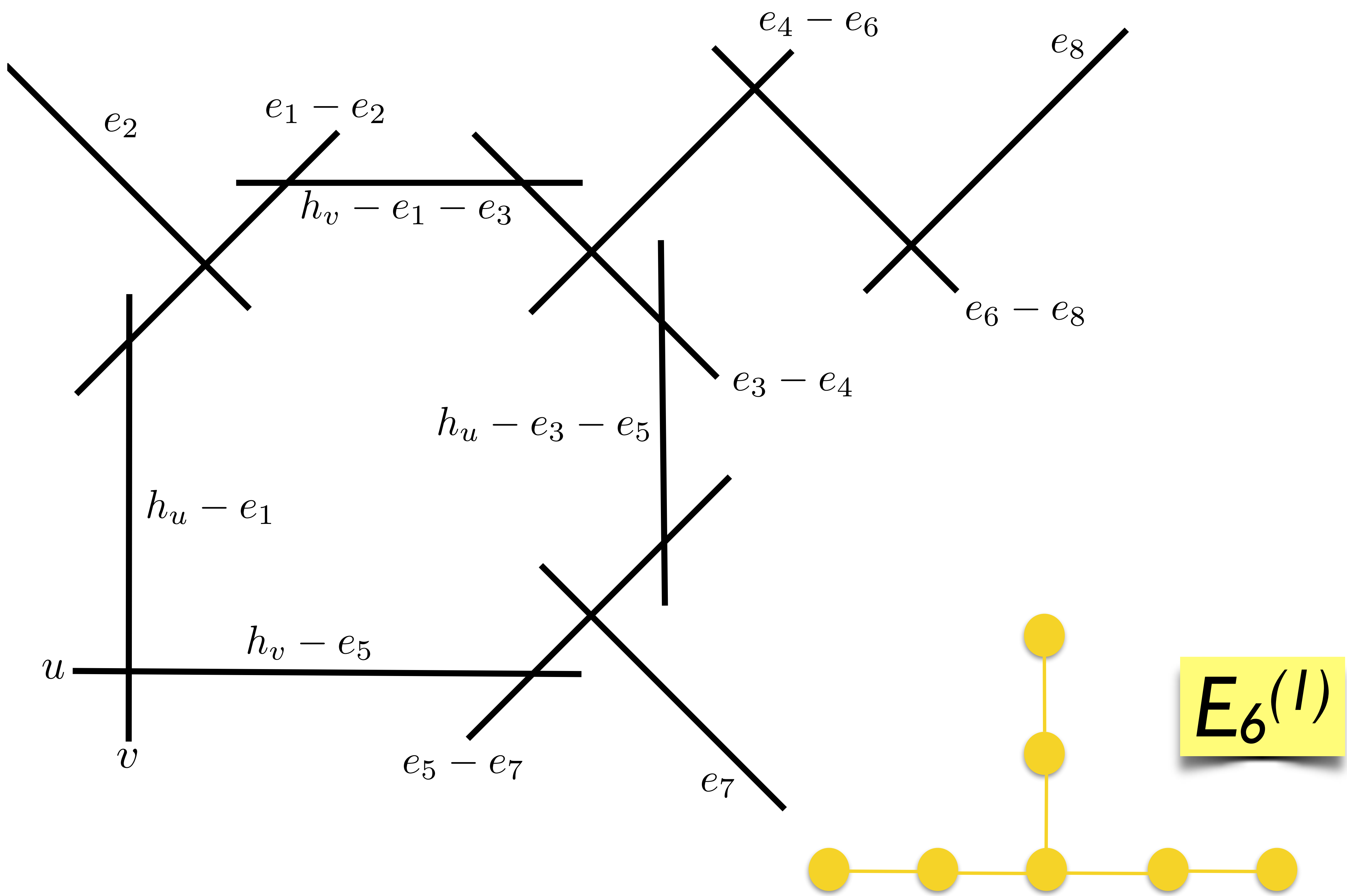
$\mathrm{dP_I}$



$\mathrm{dP_I}$

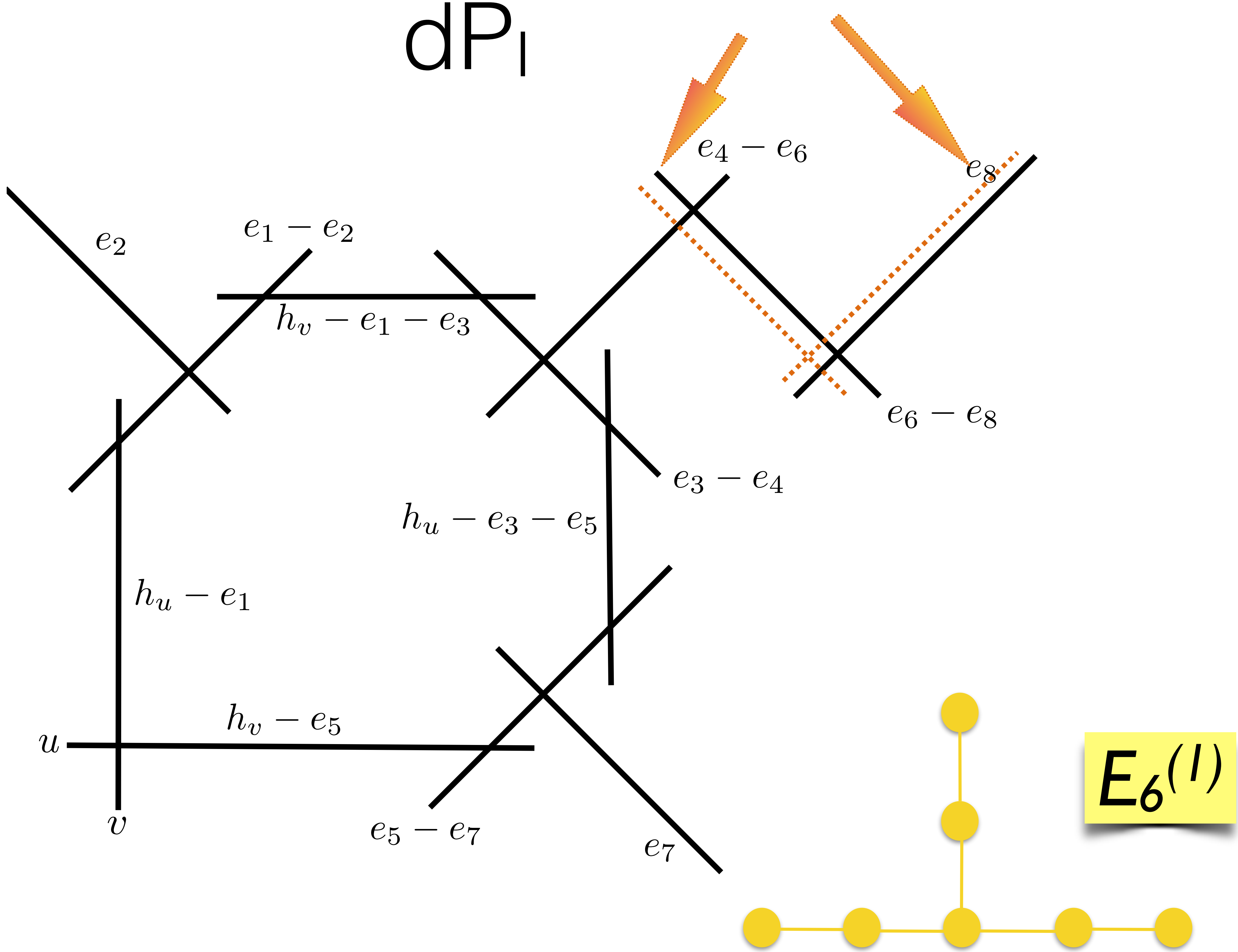


dP$_I$



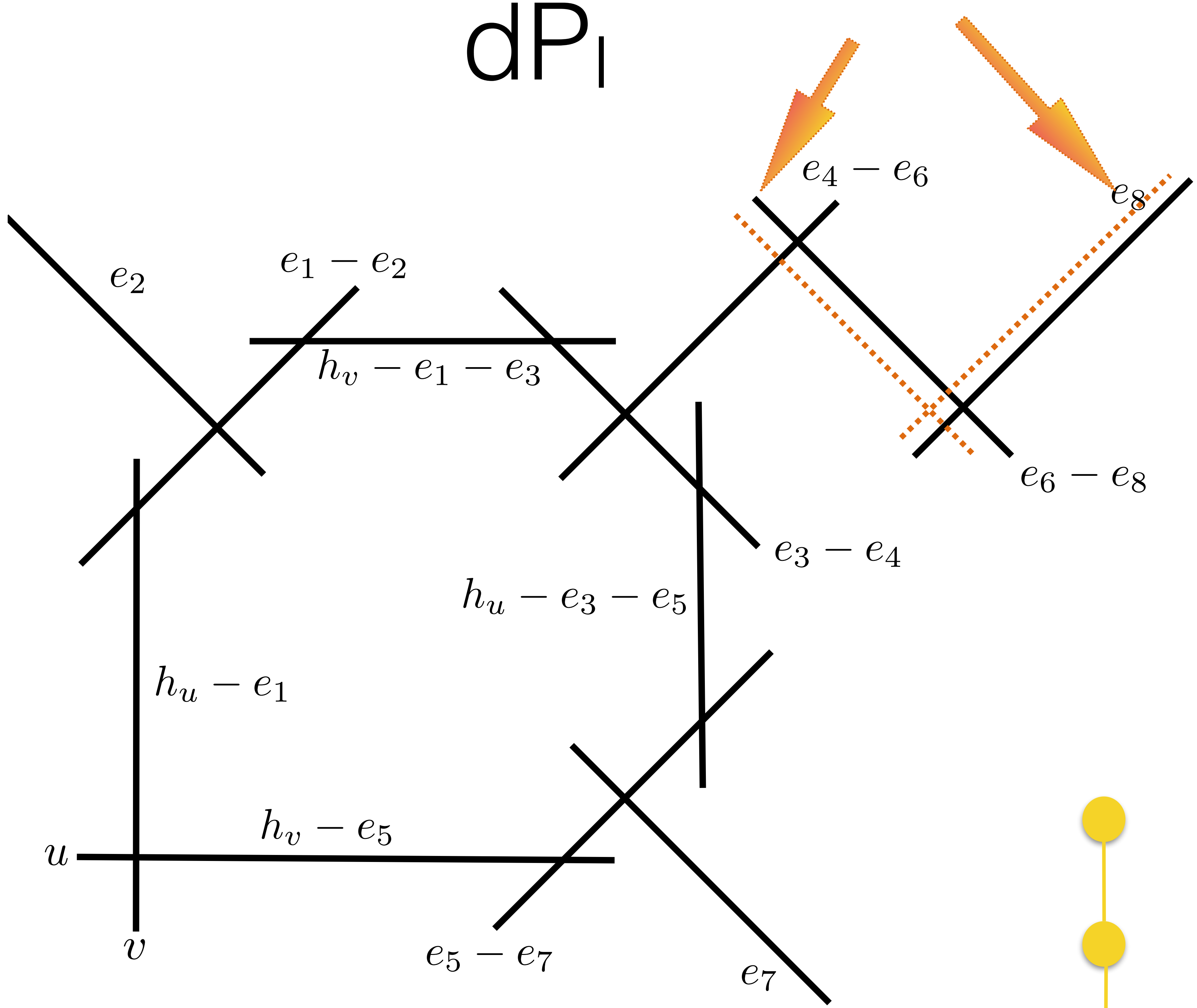
e_2
$e_1 - e_2$
$h_v - e_1 - e_3$
$e_4 - e_6$
e_8
$e_6 - e_8$
$e_3 - e_4$
$h_u - e_3 - e_5$
$h_u - e_1$
$h_v - e_5$
u
v
$e_5 - e_7$
e_7
$E_6^{(1)}$

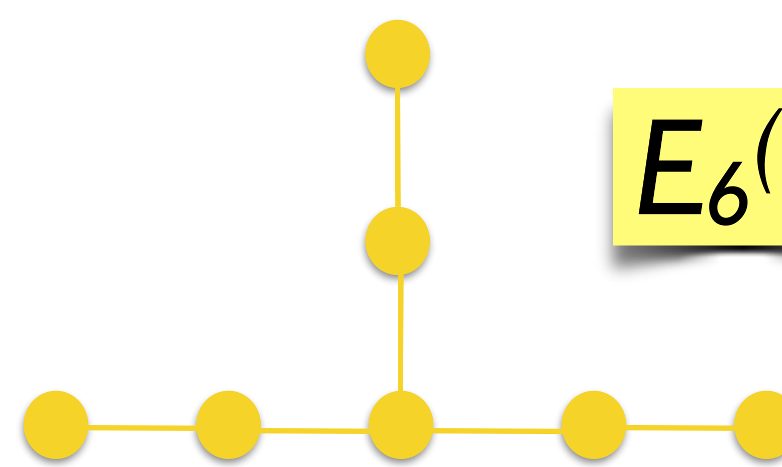
dP$_{\mathrm{I}}$


$e_4 - e_6$
e_8
e_2
$e_1 - e_2$
$h_v - e_1 - e_3$
$e_6 - e_8$
$e_3 - e_4$
$h_u - e_3 - e_5$
$h_u - e_1$
$h_v - e_5$
u
v
$e_5 - e_7$
e_7
$E_6^{(I)}$

dP$_{\mathrm{I}}$

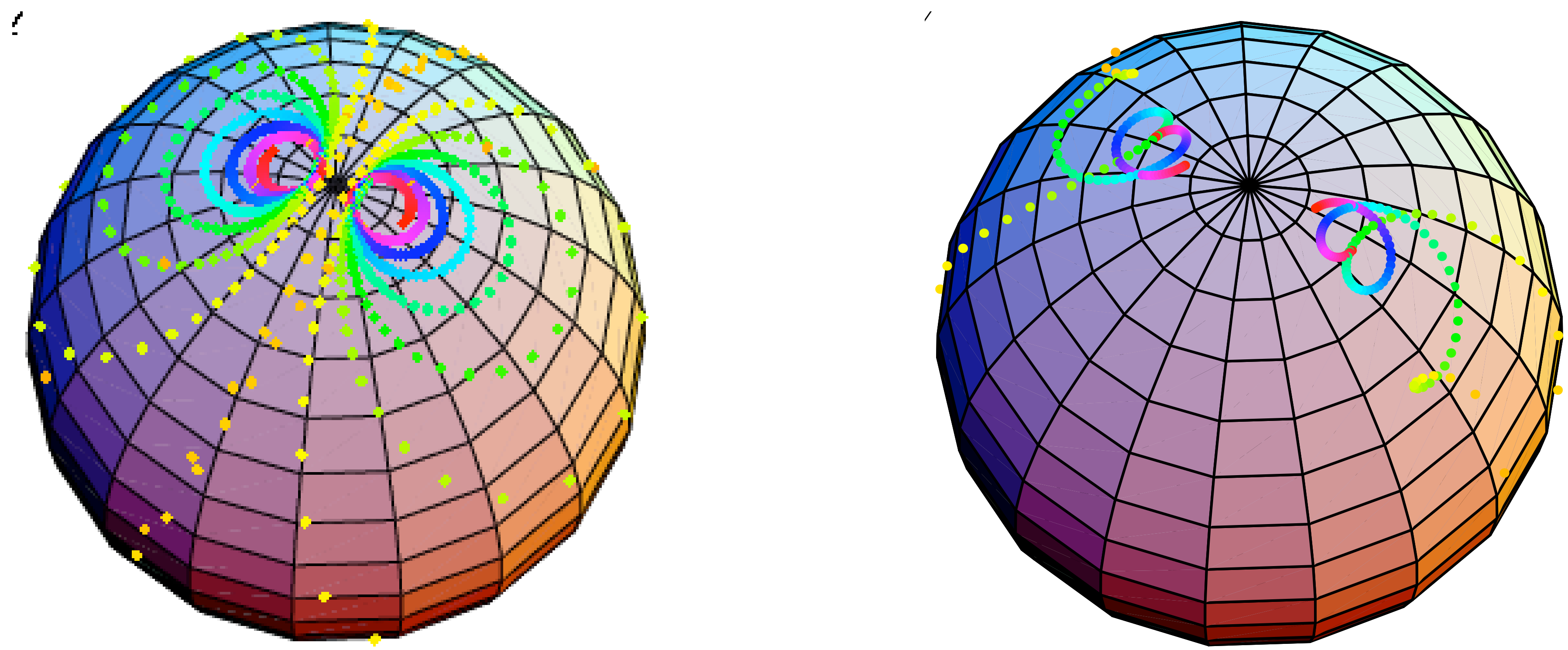
degenerate autonomous limit





$E_6^{(1)}$

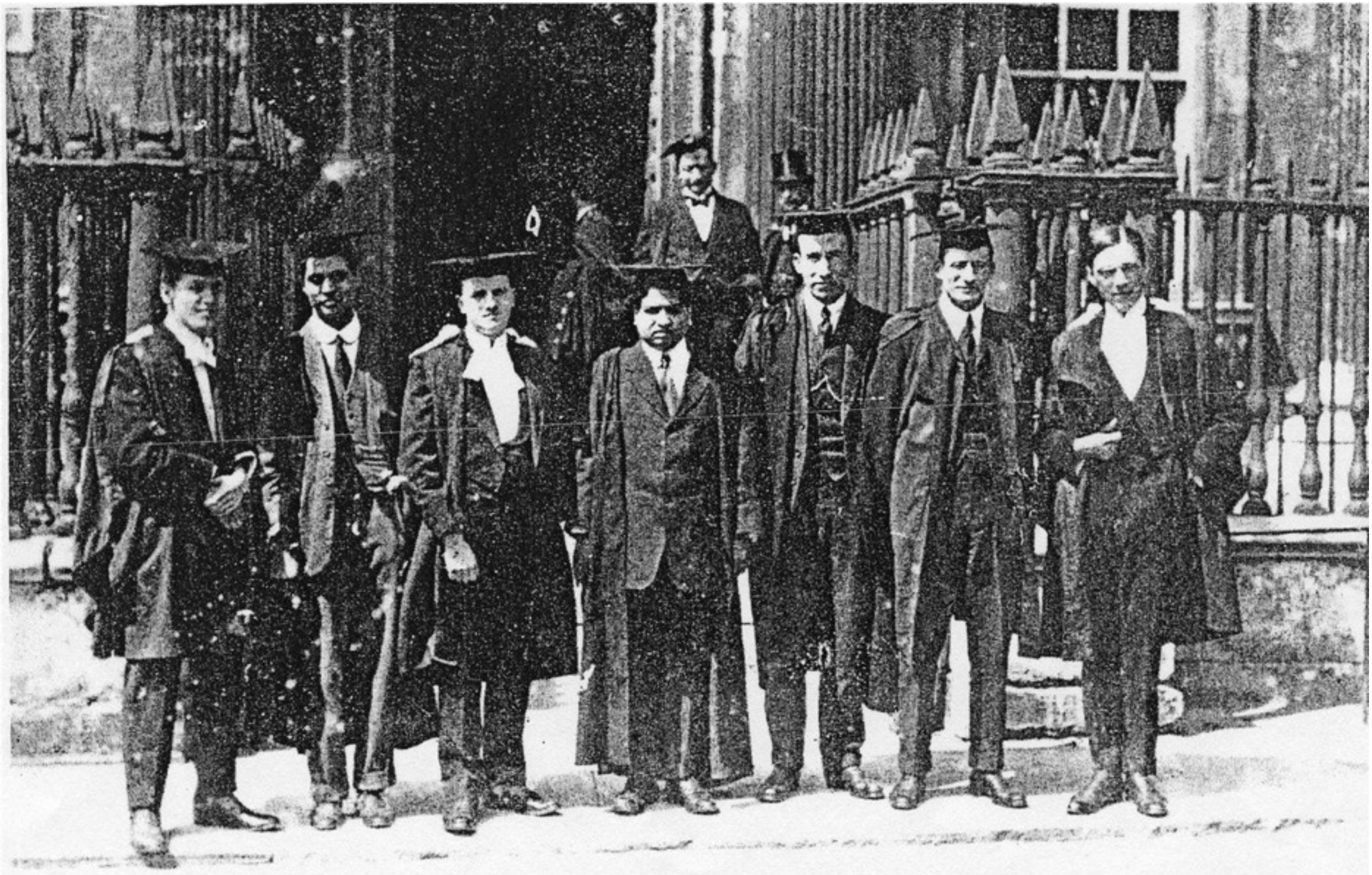
Solutions



Solution orbits of scalar dP1 on the Riemann sphere (where the north pole is infinity).

Summary

- New mathematical models of physics pose new questions for applied mathematics
- Global dynamics of solutions of non-linear equations, whether they are differential or discrete, can be found through geometry.
- Geometry provides the only analytic approach available in $\mathbb{C}$ for discrete equations.
- Tantalising questions about finite properties of solutions remain open.



The mathematician's pattern's, like those of the painter's or the poet's, must be beautiful, the ideas, like the colours or the words, must fit together in a harmonious way. *GH Hardy, A Mathematician's Apology, 1940*