G E O M E T R Y
A N D
A S Y M P T O T I C S

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Chapter 1

Algebraic Curves

1.1 Motivation

Given a constant parameter $g_2$, consider the ordinary differential equation (ODE)

$$w'' = 6w^2 - \frac{g_2}{2},$$

(1.1)

where $w$ is a function of $t \in \mathbb{C}$ and primes denote derivatives with respect to $t$.

Multiplying Equation (1.1) by $w'$ and integrating once, we obtain

$$w'^2 = 4w^3 - g_2 w - g_3$$

(1.2)

where $g_3$ is another constant parameter. Integrating once more, by separation of variables, we obtain the well known solutions:

$$w(t) = \wp(t - t_0; g_2, g_3)$$

(1.3)

which are functions of two arbitrary parameters $t_0$ and $g_3$.

Here, $\wp$ is the Weierstrass elliptic function, a doubly periodic, meromorphic function of order 2, which has a double pole at the origin. The equivalent notation $\wp(t) = \wp(t; g_2, g_3)$ is often used for conciseness, when the dependence on $g_2$ and $g_3$ is assumed. Below, we use the fact that it is an even function, i.e., $\wp(-t) = \wp(t)$. (For further information, see a reference on the theory of analytic functions of one complex variable, such as Ahlfors [1].)

Equation (1.2) defines a curve

$$y^2 = 4x^3 - g_2 x - g_3$$

(1.4)

called an elliptic curve (or Weierstrass’ cubic curve), which is parameterised by

$$x = w(t), \quad y = w'(t),$$

where $w(t)$ is given by Equation (1.3).
Let the roots of the cubic on the right of (1.4) be $e_1, e_2, e_3$. If they are real, assume without loss of generality that $e_1 \leq e_2 \leq e_3$. In the real case, the graph of $y$ as a function of $x$, given by (1.4) for generic values of $e_i, i \in \{1, 2, 3\}$, is shown in Figure 1.1.

But solutions of the ODE (1.1) vary as its accompanying initial data vary. Such initial data determine the values of $g_3$ and $t_0$, i.e., the values of $e_1, e_2, e_3$ and a starting point on the corresponding curve, such as the one in Figure 1.1. The values of $g_3$ give a family of level curves of the polynomial

$$f(x, y) = y^2 - 4x^3 + g_2 x \tag{1.5}$$

The collection of corresponding curves, a subset of which is depicted in Figure 1.2, is called a pencil of curves.

As $g_3$ varies, two of the roots $e_1, e_2, e_3$ may coincide. An example is given below.

**Example 1.1.1.** Take $g_2 = 2$, $g_3 = -(2/3)^{3/2}$ and transform variables in Equation (1.4) to

$$x = \frac{\xi}{\sqrt{6}}, \quad y = \left(\frac{2}{3^3}\right)^{1/4} \eta$$

Then the curve becomes

$$\eta^2 = (\xi - 1)^2(\xi + 2)$$

whose graph is depicted in Figure 1.3.
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Figure 1.2: A pencil of Weierstrass cubic curves

Figure 1.3: Singular Weierstrass cubic curve
The Weierstrass elliptic function \( \wp(t-t_0) \) parametrizes the curve (1.4) as a function of a continuous variable \( t \). But, there is also a discrete mapping that parametrizes this curve as a function of a discrete variable \( n \). Geometrically, this mapping is given by taking two distinct points \( P_1 \) and \( P_2 \) on the curve and finding a third point \( P_3 \) also on the curve constructed as follows.

Take the straight line passing through \( P_1 \) and \( P_2 \). (We assume below that the \( x \) coordinates of these points are distinct\(^1\).) As we show below, this line must intersect with the curve again. Take the resulting point of intersection and reflect this point across the \( x \)-axis to obtain \( P_3 \). This construction is depicted graphically in Figure 1.4.

We provide an analytic proof here that the image of this mapping can be expressed rationally in terms of the coordinates of \( P_1 \) and \( P_2 \). Let \( 2\omega_1 \) and \( 2\omega_2 \) be the (smallest) periods of \( \wp(t) \). (By the definition of \( \wp(t) \), \( \omega_1 \) and \( i\omega_2 \) are real.) Denote the fundamental period parallelogram with vertices at the origin, \( 2\omega_1 \), \( 2\omega_2 \) and \( 2(\omega_1 + \omega_2) \) by \( \Pi \). The integer linear combinations of \( 2\omega_1 \) and \( 2\omega_2 \) generate a lattice \( L \) in the complex plane. A typical such \( L \) and \( \Pi \) is drawn in Figure 1.5.

Choose \( t_1, t_2 \in \mathbb{C} \) but not in \( L \) and assume \( t_1 \neq t_2 \mod L \). Let \( a, b \in \mathbb{C} \)

\(^1\)\( P_3 \) can also be constructed when \( P_1 \) and \( P_2 \) have the same \( x \)-coordinate. But, in this case, the line containing these points is vertical and \( P_3 \) will lie at infinity.
Figure 1.5: A period lattice

such that

\[ \wp'(t_1) = a \wp(t_1) + b \]
\[ \wp'(t_2) = a \wp(t_2) + b \]

That is, \( y = ax + b \) is the line through \( P_i = (\wp(t_{i}), \wp'(t_{i})) \), \( i = 1, 2 \).

For any elliptic function \( F(t) \) with period lattice \( L \) and a fundamental period parallelogram \( \Pi \), we have

\[ \frac{1}{2\pi i} \oint_{\Pi} t \frac{F'(t)}{F(t)} dt = \sum_i (z_i - p_i) = 0 \]

by Cauchy’s residue theorem, where \( z_i \) and \( p_i \) are respectively zeroes and poles of \( F \) in \( \Pi \). We take

\[ F(t) = \wp'(t) - a \wp(t) - b \]

which is an elliptic function of order 3, with a triple pole at the origin. So if \( t_1, t_2 \) are zeroes of \( F(t) \), then (because the pole is located at the origin), a third zero must exist at \( t_3 = -(t_1 + t_2) \) modulo \( L \). So we have

\[ \wp'(t_3) = a \wp(t_3) + b . \]

Note that this shows that the straight line \( y = ax + b \) must intersect the Weierstrass cubic curve (1.2) a third time.

At such an intersection between the curve given by (1.2) and the straight line \( y = ax + b \), we also have

\[ 4x^3 - g_2x - g_3 - (ax + b)^2 = 0 \quad (1.6) \]
which has three roots given by \( \wp(t_1), \wp(t_2), \wp(t_3) \). So we get
\[
4 \left( x - \wp(t_1) \right) \left( x - \wp(t_2) \right) \left( x - \wp(t_3) \right) = 0.
\] (1.7)

Comparing the coefficient of \( x^2 \) between Equations (1.6-1.7), we get
\[
\wp(t_1) + \wp(t_2) + \wp(t_3) = \frac{a^2}{4}
\] (1.8)

But also, because \( a \) is the slope of the line through the two points \( P_i = (\wp(t_i), \wp'(t_i)), i = 1, 2 \), we have
\[
a = \frac{\wp'(t_1) - \wp'(t_2)}{\wp(t_1) - \wp(t_2)}.
\] (1.9)

Moreover, \( \wp(t_3) = \wp(-(t_1 + t_2)) = \wp(t_1 + t_2) \) by the evenness of \( \wp(t) \) and \( b = \wp'(t_1) - a \wp(t_1) \). We find therefore from Equation (1.8) that
\[
\wp(t_1 + t_2) = -\wp(t_1) - \wp(t_2) + \frac{1}{4} \left( \frac{\wp'(t_1) - \wp'(t_2)}{\wp(t_1) - \wp(t_2)} \right)^2.
\] (1.10)

If we write \( y = \wp'(t_1 + t_2), y_0 = \wp'(t_1), x = \wp(t_1), x_0 = \wp(t_2) \), then these equations become
\[
\begin{cases}
\overline{x} &= \frac{1}{4} \left( \frac{y - y_0}{x - x_0} \right)^2 - x - x_0 \\
y &= -y - \left( \frac{y - y_0}{x - x_0} \right) (\overline{x} - x)
\end{cases}
\] (1.12)

which provides a discrete mapping on the Weierstrass cubic curve.
Bibliography