Geometry and Asymptotics

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@monsoon0

Supported by the Australian Research Council

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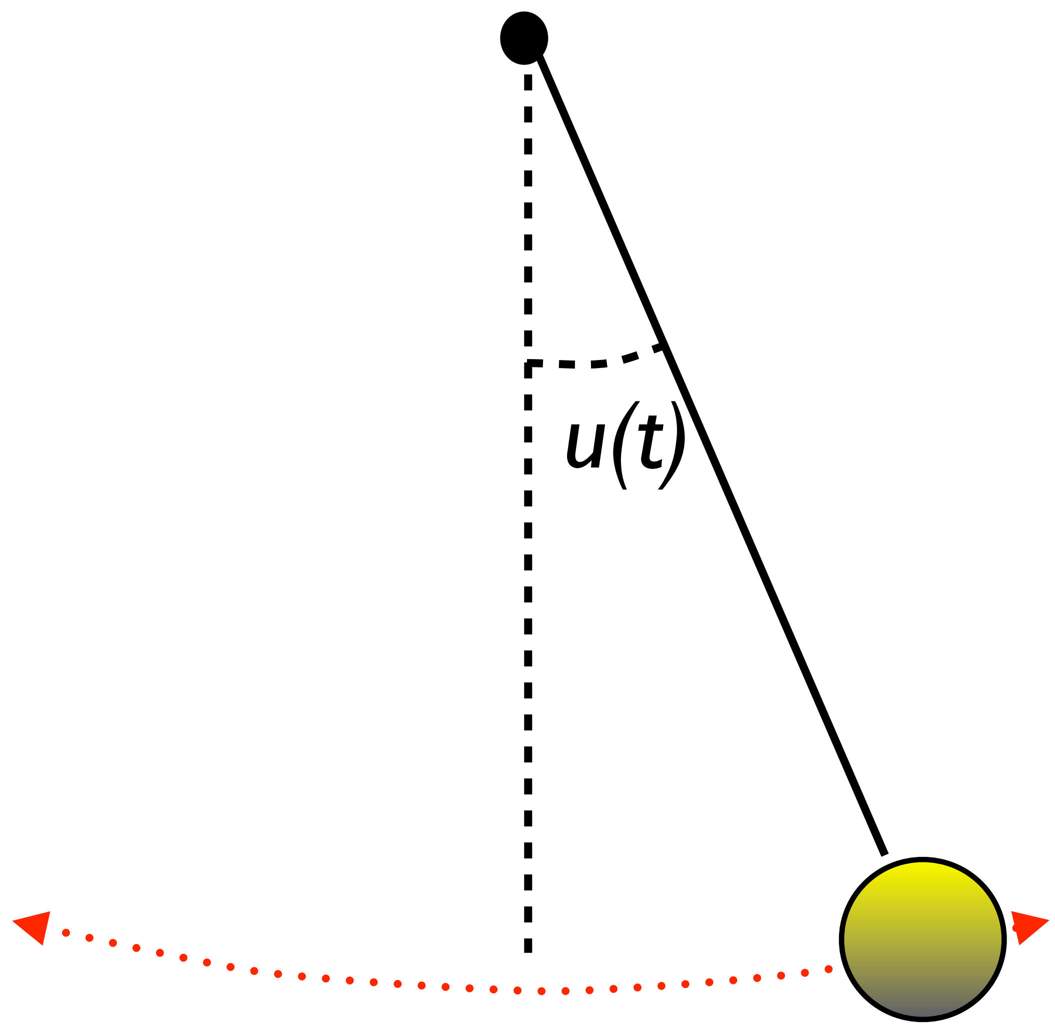
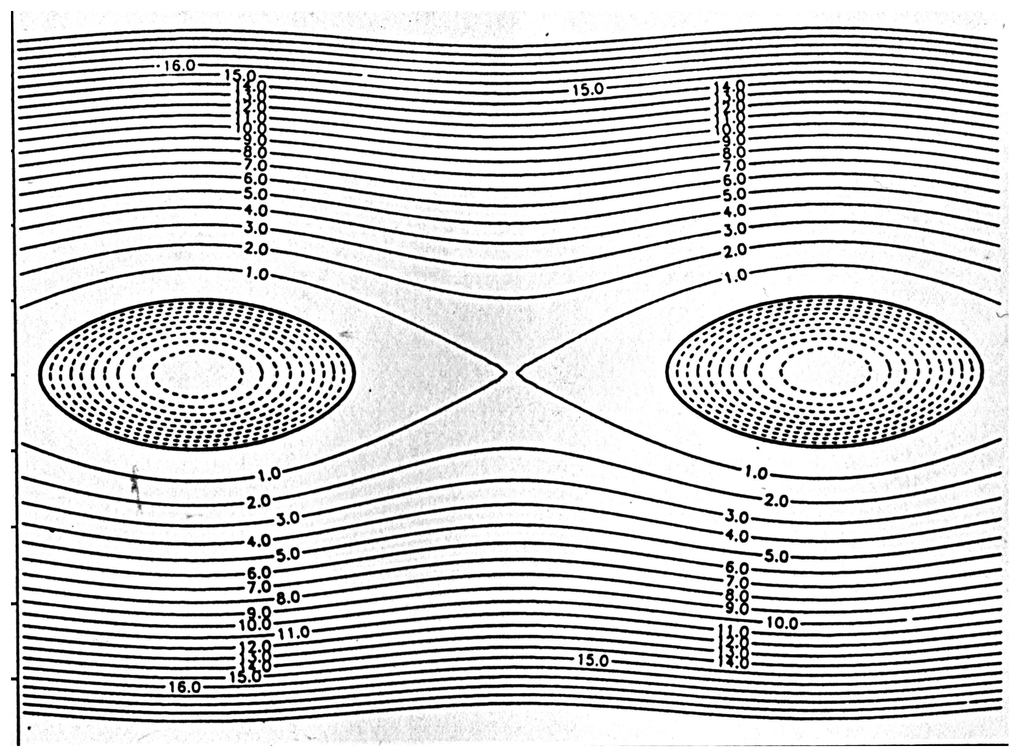
Dynamical Systems



Phase Space

$$\begin{cases} \dot{u}(t) = v(t) \\ \dot{v}(t) = -\sin\big(u(t)\big) \end{cases}$$



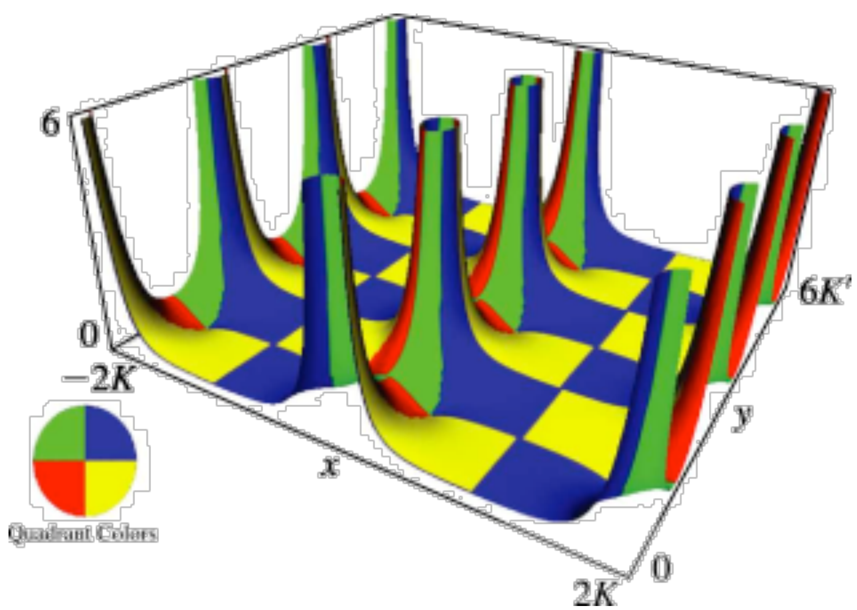
$$H(u,v) = \frac{v^2}{2} - \cos\big(u(t)\big)$$

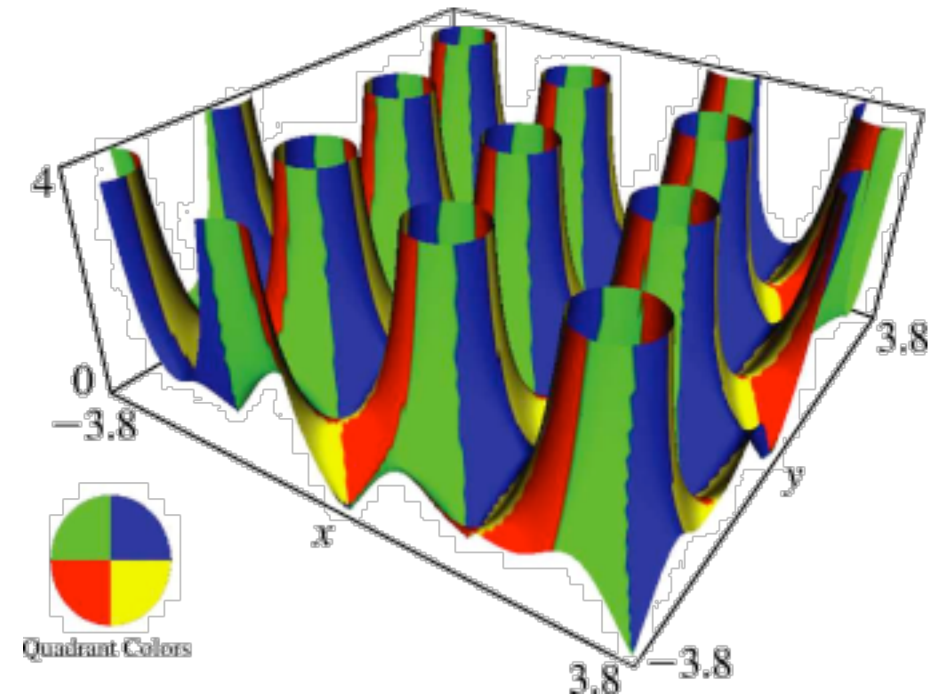
Noncompact Phase Space

- The pendulum is completely integrable. Liouville's theorem guarantees that the solution can be obtained by quadratures.
- However, this theorem requires that the phase space be compact.
- What can we do if it isn't?
- Elliptic functions provide instructive examples.

Elliptic Functions

- Weierstrass elliptic functions $\wp(t)$





$$\wp''(t) = 6\,\wp^2(t) - \frac{g_2}{2}$$

$$\wp'^2(t) = 4\,\wp^3(t) - g_2\,\wp(t) - g_3$$

Elliptic Functions

$$\ddot{w} = 6\,w^2 - \frac{g_2}{2}$$

$$\Rightarrow \qquad \frac{\dot{w}^2}{2} = 2\,w^3 - \frac{g_2}{2}\,w - \frac{g_3}{2}$$

$$\Rightarrow \qquad w(t) = \wp\big(t - t_0; g_2, g_3\big)$$

We have a conserved Hamiltonian, but the phase space is not compact, due to the poles of the solutions.

Level Curves

In phase space, $\dot{w} = y, w = x,$ the conserved quantity becomes

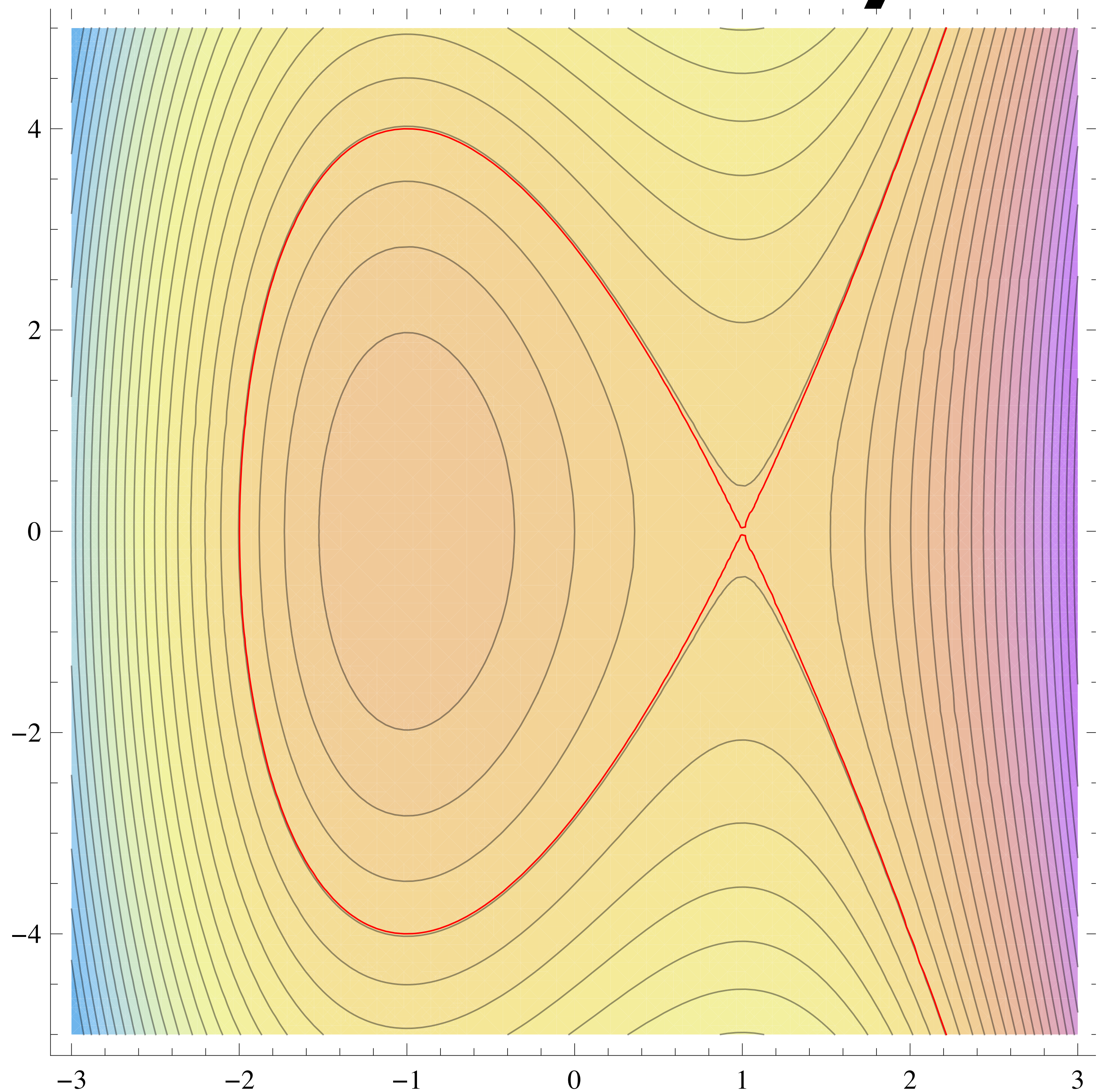
$$y^2 = 4\,x^3 - g_2\,x - g_3$$

defining an elliptic curve, or, Weierstrass' cubic.

- Initial values determine g_3 and provide the phase.
- Each value of g_3 defines a level curve of

$$f(x, y) = y^2 - 4\,x^3 + g_2\,x$$

Geometry



Level curves of $y^2 - 4x^3 + 12x$

Algebraic Curves

- Curves provide a geometric way of studying the functions $(x(t), y(t))$ that parametrize them.
- But the unique continuation of $(x(t), y(t))$ can fail at two kinds of points:
  - (i) singularities
  - (ii) base points

Singularities

- On a curve given by $f(x, y)=0$, a point (x_0, y_0) is called a singularity if

$$\nabla f \Big|_{(x_0, y_0)} = 0$$

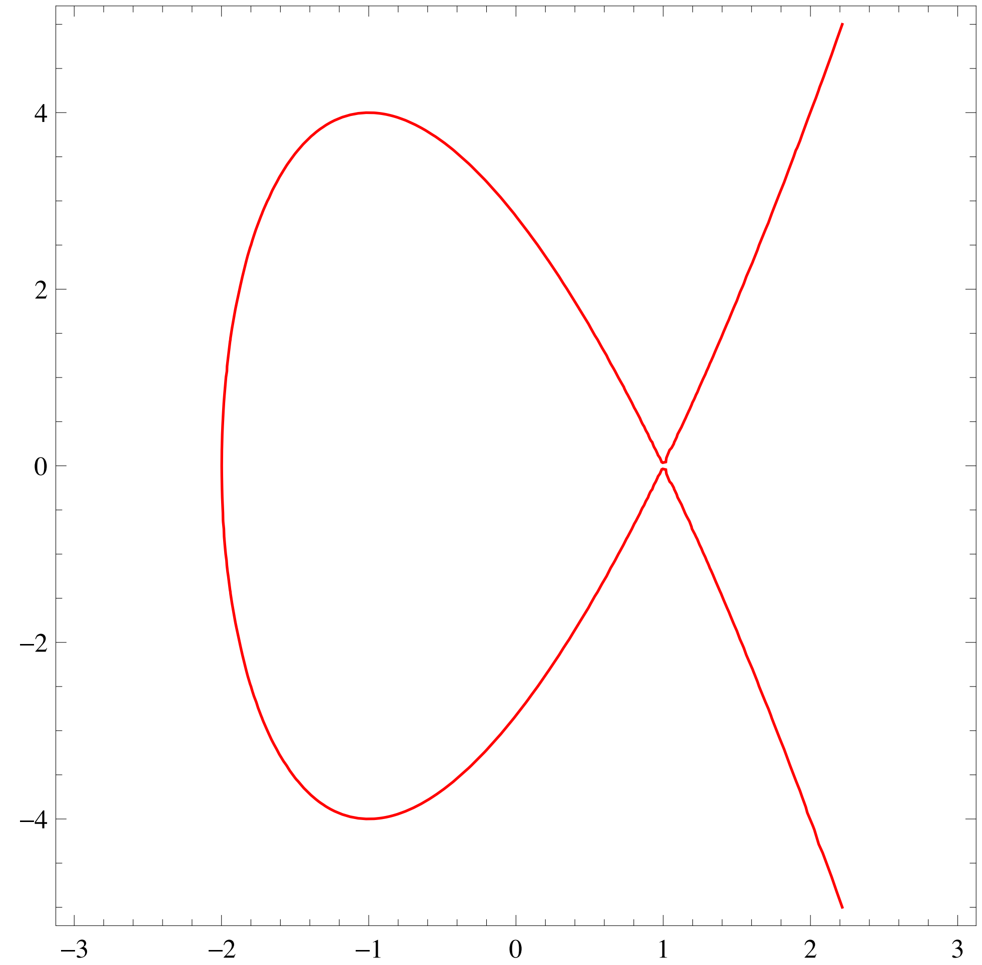
- For example

$$f(x, y) = y^2 - 4\,x^3$$
$$\nabla f = (-12\,x^2, 2\,y)$$
$$\Rightarrow \;\; (x_0, y_0) = (0, 0)$$

Singular Elliptic Curve

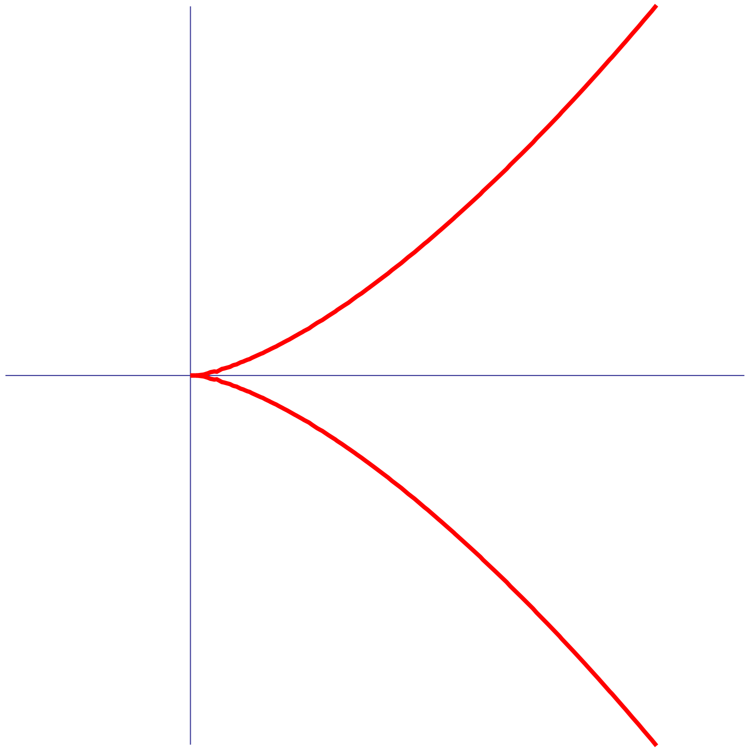
$$f(x,y) = y^2 - 4\,x^3 + 12\,x \;\Rightarrow \nabla f = (-12\,x^2 + 12, 2\,y)$$

$$\nabla f = 0 \text{ on } f(x,y) = g_3 \;\Rightarrow\; (x_0, y_0) = (\pm 1, 0),\; g_3 = \mp 8$$



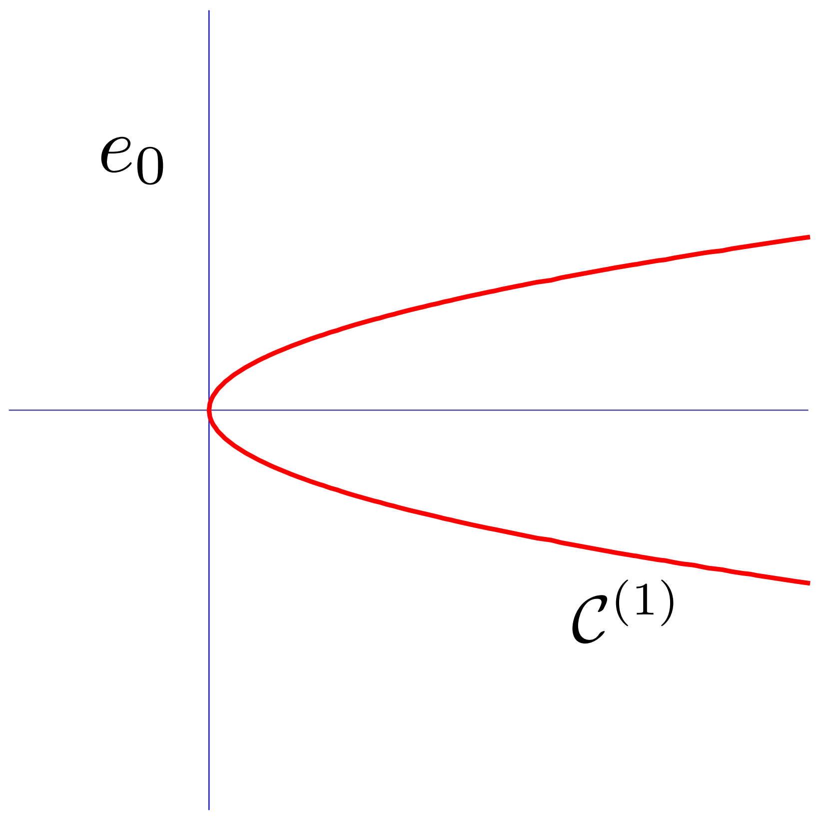
Resolution of Singularities

$$y^2 = x^3$$



$$(x, y) = (x_1, x_1\, y_1) \;\Rightarrow\; f(x, y) = y^2 - x^3$$
$$= x_1^2\, y_1^2 - x_1^3$$
$$= x_1^2\, (y_1^2 - x_1)$$

$$e_0 :\; x_1^2 = 0\;,\, f^{(1)} = y_1^2 - x_1$$

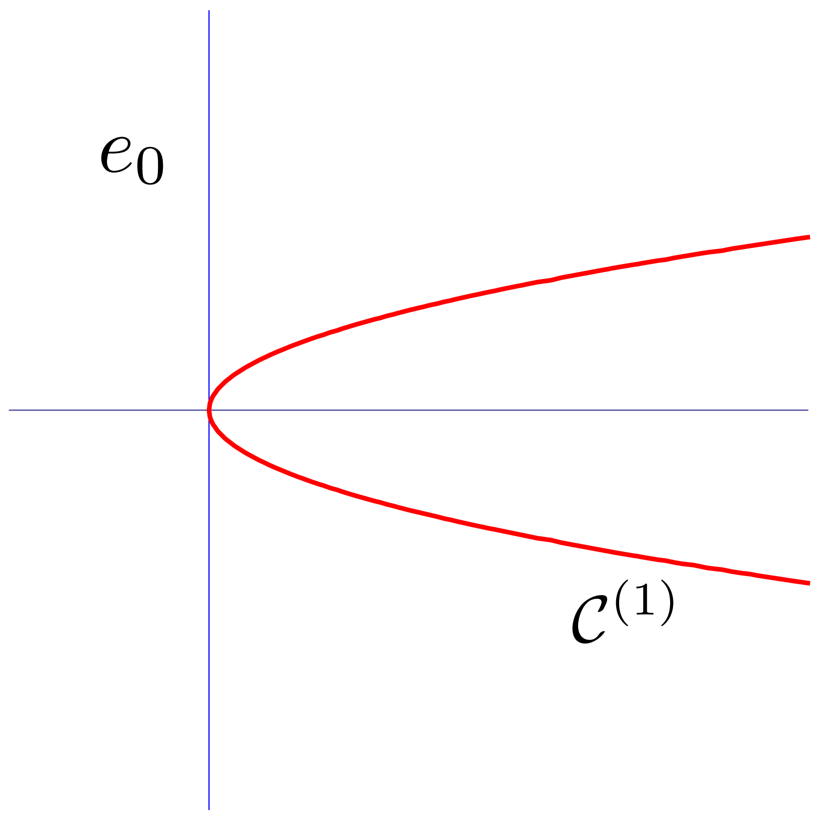


$$(x, y) = (x_1, x_1 y_1) \;\Rightarrow\; f(x, y) = y^2 - x^3$$

$$x_1 = x, y_1 = y/x \qquad = x_1^2 y_1^2 - x_1^3$$

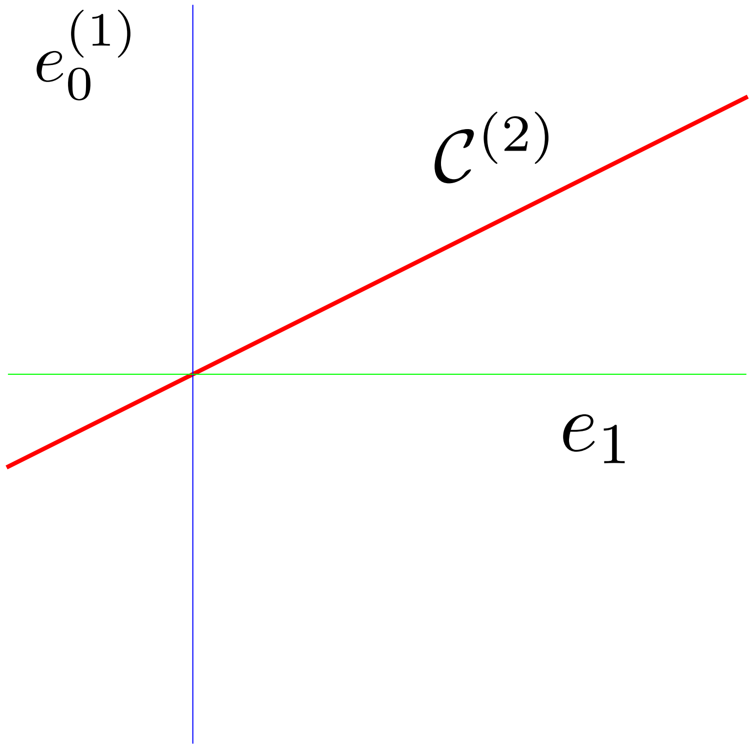
$$= x_1^2 (y_1^2 - x_1)$$

$$e_0 : \; x_1^2 = 0 \;, f^{(1)} = y_1^2 - x_1$$



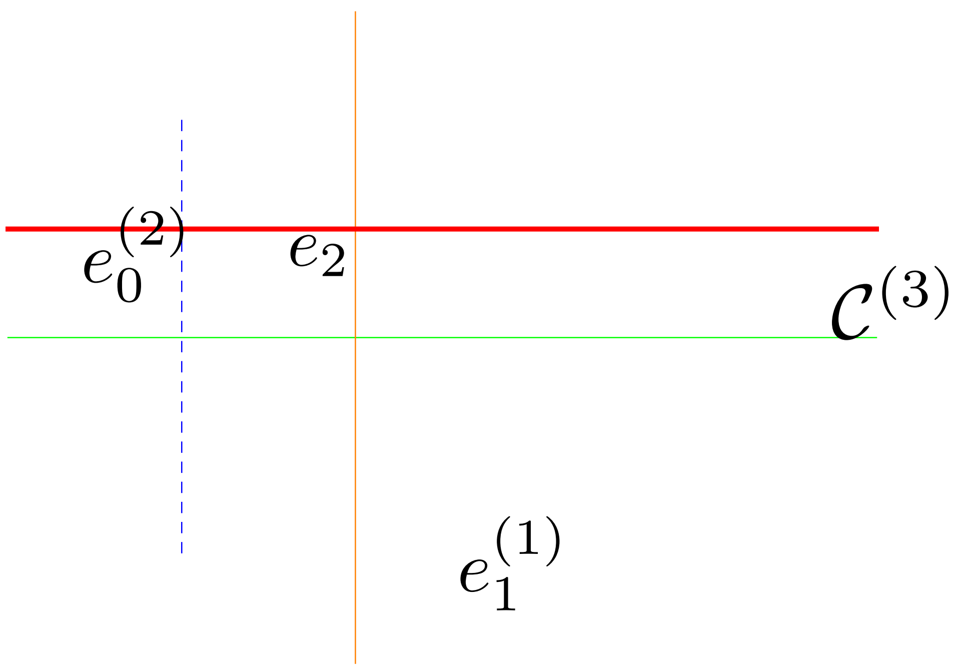
$$(x_1, y_1) = (x_2\, y_2, y_2) \;\Rightarrow\; f^{(1)}(x_1, y_1) = y_2^2 - x_2\, y_2$$
$$= y_2\,(y_2 - x_2)$$

$$e_1 : \; y_2 = 0, \; f^{(2)} = y_2 - x_2$$



$$(x_2, y_2) = (x_3, x_3\, y_3) \;\Rightarrow\; f^{(2)}(x_2, y_2) = x_3\, y_3 - x_3$$
$$= x_3\,(y_3 - 1)$$

$$e_2 : \; x_3 = 0, \; f^{(3)} = y_3 - 1$$

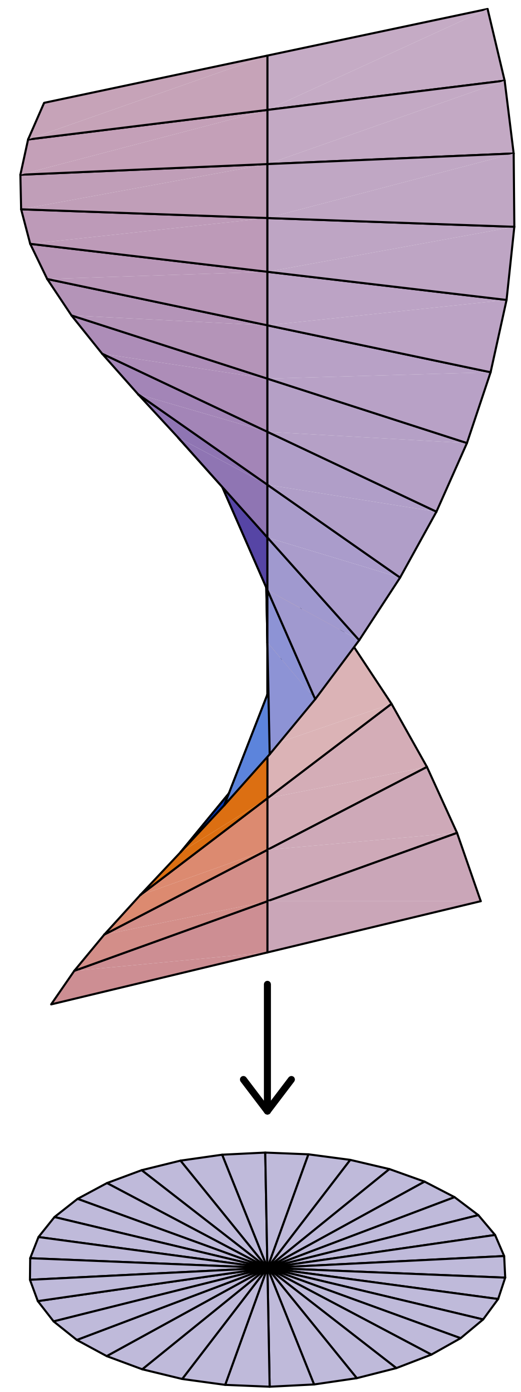


Good Resolution

- When all curves intersect each other transversally at distinct points, the result is called a "good resolution".
- Hironaka's theorem guarantees this in complex projective space.
- Note: each transformation had the form

$$x_1 = x, y_1 = y/x$$

Blowing up at a base pt



From JJ Duistermaat, QRT Maps and Elliptic Surfaces, Springer Verlag, 2010

Resolved Dynamics

- The phase space of the system after a “good resolution” is compact.
- The motion is globally regularised on the curves in the blown-up space.
- Is this useful for all non-compact phase spaces ?

Motivation

- Dubrovin, Grava and Klein *J. Nonlin. Sci* (2009) analysed critical behaviour of a fluid system in an elliptic region

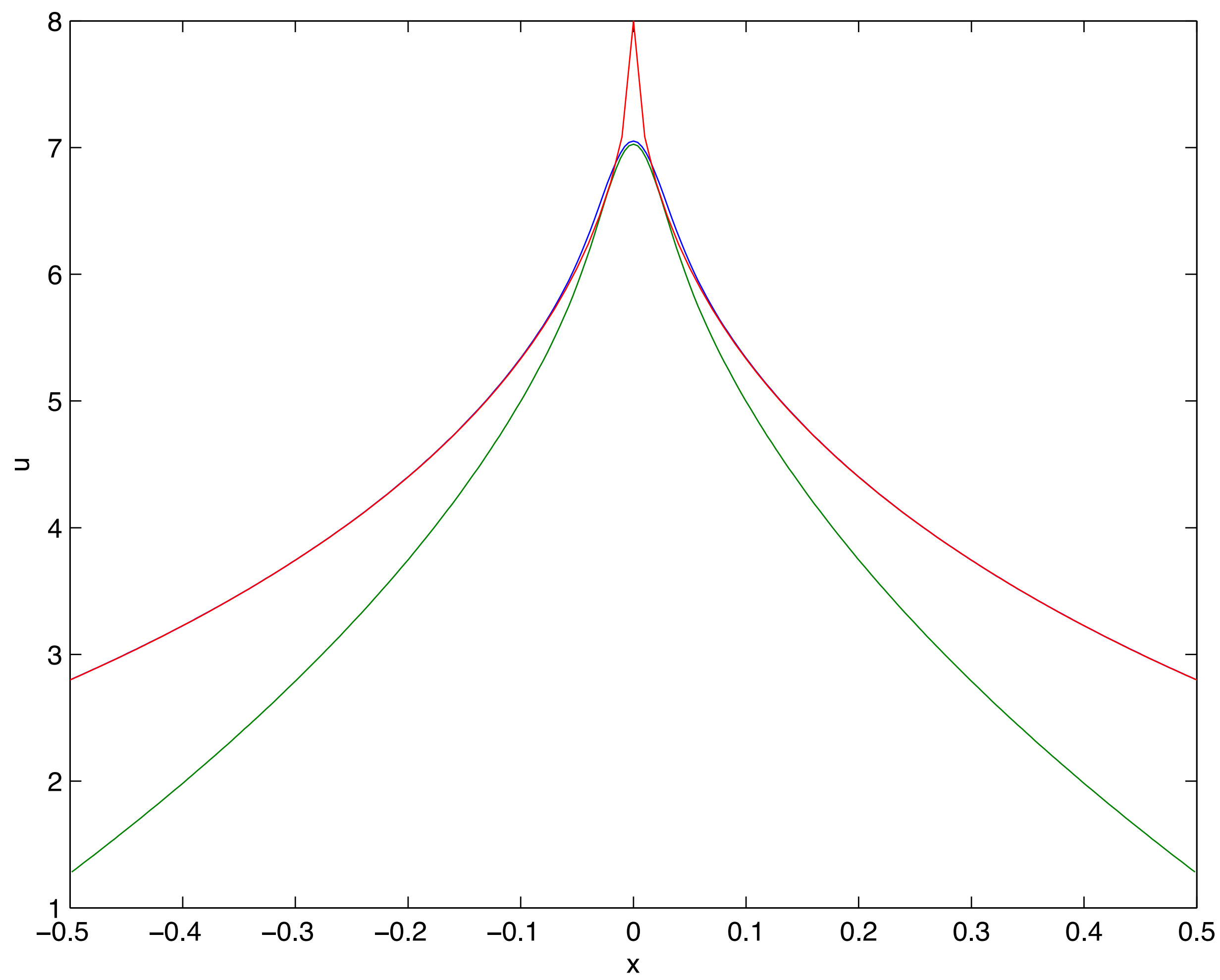


Figure 8: The blue line is the function u of the solution to the focusing NLS equation for the initial data $u(x,0) = 2\,\mathrm{sech}\,x$ and $\epsilon = 0.04$ at the critical time, and the red line is the corresponding semiclassical solution given by formulas (2.4). The green line gives the multiscales solution via the tritronquée solution of the Painlevé I equation.

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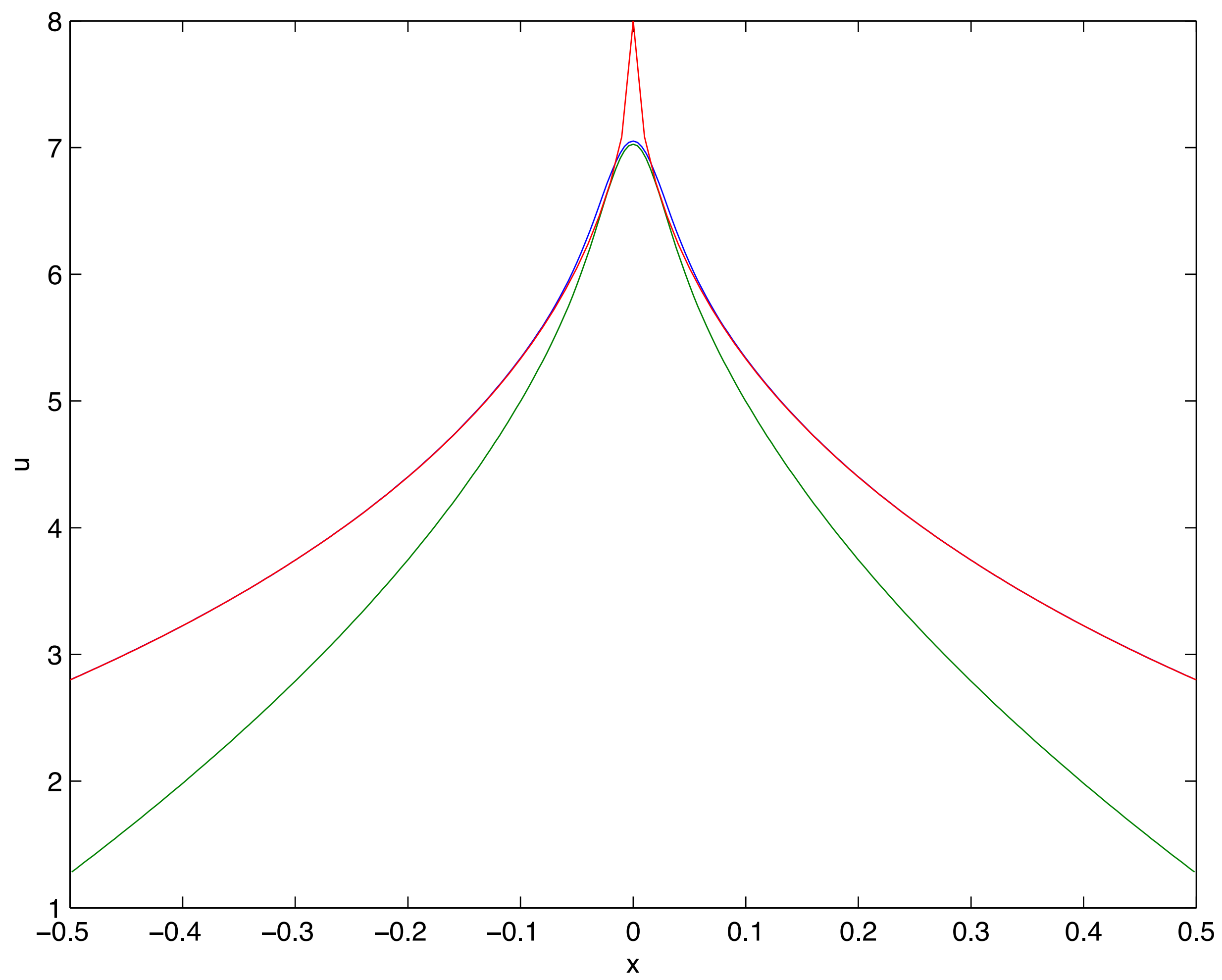

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The First Painlevé Equation

- $\mathrm{P_I}$: $\ddot{w} = 6\,w^2 - t$
- In system form $\mathrm{P_I}$ becomes

$$\frac{d}{dt}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} w_2 \\ 6\,w_1^2 - t \end{pmatrix}$$

- $\mathrm{P_I}$ has *t*-dependent Hamiltonian

$$H = \frac{w_2^2}{2} - 2\,w_1^3 + t\,w_1$$

Perturbed Form

- Or, in Boutroux's coordinates:

$$w_1 = t^{1/2}\, u_1(z),\ \ w_2 = t^{3/4}\, u_2(z) \quad z = \frac{4}{5} t^{5/4}$$

$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix} = \begin{pmatrix} u_2 \\ 6\, u_1^2 - 1 \end{pmatrix} - \frac{1}{(5z)} \begin{pmatrix} 2\, u_1 \\ 3\, u_2 \end{pmatrix}$$

- A perturbation of a Hamiltonian system:

$$E = \frac{u_2^2}{2} - 2\, u_1^3 + u_1 \ \Rightarrow\ \frac{dE}{dt} = \frac{1}{5t}\left(6E + 4u_1\right)$$

Perturbed Form

- Or, in Boutroux's coordinates:

$$w_1 = t^{1/2}\, u_1(z),\;\; w_2 = t^{3/4}\, u_2(z) \quad z = \frac{4}{5} t^{5/4}$$

$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \end{pmatrix} = \begin{pmatrix} u_2 \\ 6\, u_1^2 - 1 \end{pmatrix} - \frac{1}{(5z)} \begin{pmatrix} 2\, u_1 \\ 3\, u_2 \end{pmatrix}$$

- A perturbation of a Hamiltonian system:

$$E = \frac{u_2^2}{2} - 2\, u_1^3 + u_1 \;\Rightarrow\; \frac{dE}{dt} = \frac{1}{5t}(6E + 4u_1)$$

Initial Value Space

- To describe all solutions in the limit, we study them in the (unsteady) phase space or the *space of initial values* first constructed by Okamoto (1979).
- Underlying this geometric construction is the theory of complex algebraic curves.

Projective Space

$$\mathbb{CP}^2 \;:\; \overbrace{[\frac{u}{w} : \frac{v}{w} : 1]}^{Affine\ coordinates} \;\Leftrightarrow\; \overbrace{[u : v : w]}^{Homogeneous\ coordinates}$$

$$\mathbb{CP}^1 \times \mathbb{CP}^1 \;:$$

$$\overbrace{\left(\left[\frac{u}{w} : 1\right], \left[\frac{v}{z} : 1\right]\right)}^{Affine\ coordinates} \;\Leftrightarrow\; \overbrace{([u : w], [v, z])}^{Homogeneous\ coordinates}$$

Homogeneous Curve

- The level curves $\phi(x,y) = y^2 - 4\,x^3 + g_2\,x + g_3$

- Become in $\mathbb{P}^2$:

$$F = w\,v^2 - 4\,u^3 + g_2\,u\,w^2 + g_3\,w^3$$

all intersecting at [0, 1, 0], which lies at infinity.

- [0,1,0] is an example of a *base point*.

Base Points

- When all the curves in a pencil (one-parameter family) intersect at one point, it is called a *base point*.
- The flow through it can also be disentangled by using the process of *resolution*.

First Step

- First embed into the projective plane:

$$[1 : u_1 : u_2] = [1 : u_{011} : u_{012}]$$

$$[u_1^{-1} : 1 : u_1^{-1} u_2] = [u_{021} : 1 : u_{022}]$$

$$[u_2^{-1} : u_1 u_2^{-1} : 1] = [u_{031} : u_{032} : 1]$$

- Line at infinity:

$$u_{021} = 0$$

$$u_{031} = 0 \qquad L_0$$

The Projective Plane $\mathbb{CP}^2$

The Projective Plane $\mathbb{CP}^2$

First chart: $[u_1^{-1} : 1 : u_1^{-1}\, u_2] = [u_{021} : 1 : u_{022}]$

$$\dot{u}_{021} = -u_{021}u_{022} + 2(5z)^{-1}u_{021}$$

$$\dot{u}_{022} = u_{021} + 6u_{021}^{-1} - u_{022}^2 - (5z)^{-1}u_{022}$$

The Projective Plane $\mathbb{CP}^2$

First chart: $[u_1^{-1} : 1 : u_1^{-1}\, u_2] = [u_{021} : 1 : u_{022}]$

$$\dot{u}_{021} = -u_{021}u_{022} + 2(5z)^{-1}u_{021}$$

$$\dot{u}_{022} = u_{021} + 6u_{021}^{-1} - u_{022}^2 - (5z)^{-1}u_{022}$$

Second chart: $[u_2^{-1} : u_1\, u_2^{-1} : 1] = [u_{031} : u_{032} : 1]$

$$\dot{u}_{031} = -u_{031}^2 - 6u_{032}^2 + 3(5z)^{-1}u_{031}$$

$$\dot{u}_{032} = -u_{031}u_{032} - 6u_{031}^{-1}u_{032}^3 + 1 + (5z)^{-1}u_{032}$$

The Projective Plane $\mathbb{CP}^2$

First chart: $[u_1^{-1} : 1 : u_1^{-1}\, u_2] = [u_{021} : 1 : u_{022}]$

$$\dot{u}_{021} = -u_{021}u_{022} + 2(5z)^{-1}u_{021}$$

$$\dot{u}_{022} = u_{021} + 6u_{021}^{-1} - u_{022}^2 - (5z)^{-1}u_{022}$$

Second chart: $[u_2^{-1} : u_1\, u_2^{-1} : 1] = [u_{031} : u_{032} : 1]$

$$\dot{u}_{031} = -u_{031}^2 - 6u_{032}^2 + 3(5z)^{-1}u_{031}$$

$$\dot{u}_{032} = -u_{031}u_{032} - 6u_{031}^{-1}u_{032}^3 + 1 + (5z)^{-1}u_{032}$$

The Projective Plane $\mathbb{CP}^2$

First chart: $[u_1^{-1} : 1 : u_1^{-1}\, u_2] = [u_{021} : 1 : u_{022}]$

$$\dot{u}_{021} = -u_{021} u_{022} + 2(5z)^{-1} u_{021}$$

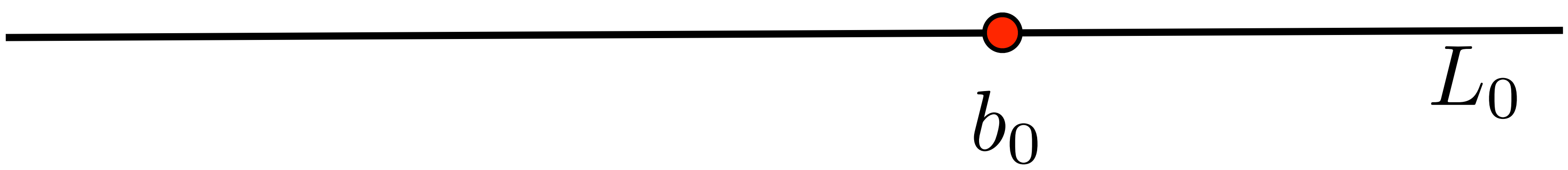
$$\dot{u}_{022} = u_{021} + 6u_{021}^{-1} - u_{022}^2 - (5z)^{-1} u_{022}$$

Second chart: $[u_2^{-1} : u_1\, u_2^{-1} : 1] = [u_{031} : u_{032} : 1]$

$$\dot{u}_{031} = -u_{031}^2 - 6u_{032}^2 + 3(5z)^{-1} u_{031}$$

$$\dot{u}_{032} = -u_{031} u_{032} - 6u_{031}^{-1} u_{032}^3 + 1 + (5z)^{-1} u_{032}$$

→ base pt $b_0 : \; u_{031} = 0, u_{032} = 0$



- So we need to blow-up at this point. There are two charts:

$$[1 : u_{111} : u_{112}] = [1 : u_{031}/u_{032} : u_{032}]$$

$$[1 : u_{121} : u_{122}] = [1 : u_{031} : u_{032}/u_{031}]$$

First Blow-up

- Chart (1,1):

$$\dot{u}_{111} = -u_{111}u_{112}^{-1} + 2(5z)^{-1}u_{111}$$
$$\dot{u}_{112} = 1 - u_{111}u_{112}^{2} - 6u_{111}^{-1}u_{112}^{2} + (5z)^{-1}u_{112}$$

- Chart (1,2):

$$\dot{u}_{121} = u_{121}^{2}\left(-6u_{122}^{2} - 1\right) + 3\left(5z\right)^{-1}u_{121}$$
$$\dot{u}_{122} = u_{121}^{-1} - 2\left(5z\right)^{-1}u_{122}$$

First Blow-up

- Chart (1,1):

$$\dot{u}_{111} = -u_{111}u_{112}^{-1} + 2(5z)^{-1}u_{111}$$
$$\dot{u}_{112} = 1 - u_{111}u_{112}^{2} - 6u_{111}^{-1}u_{112}^{2} + (5z)^{-1}u_{112}$$

- Chart (1,2):

$$\dot{u}_{121} = u_{121}^{2}\left(-6u_{122}^{2} - 1\right) + 3\left(5z\right)^{-1}u_{121}$$
$$\dot{u}_{122} = u_{121}^{-1} - 2\left(5z\right)^{-1}u_{122}$$

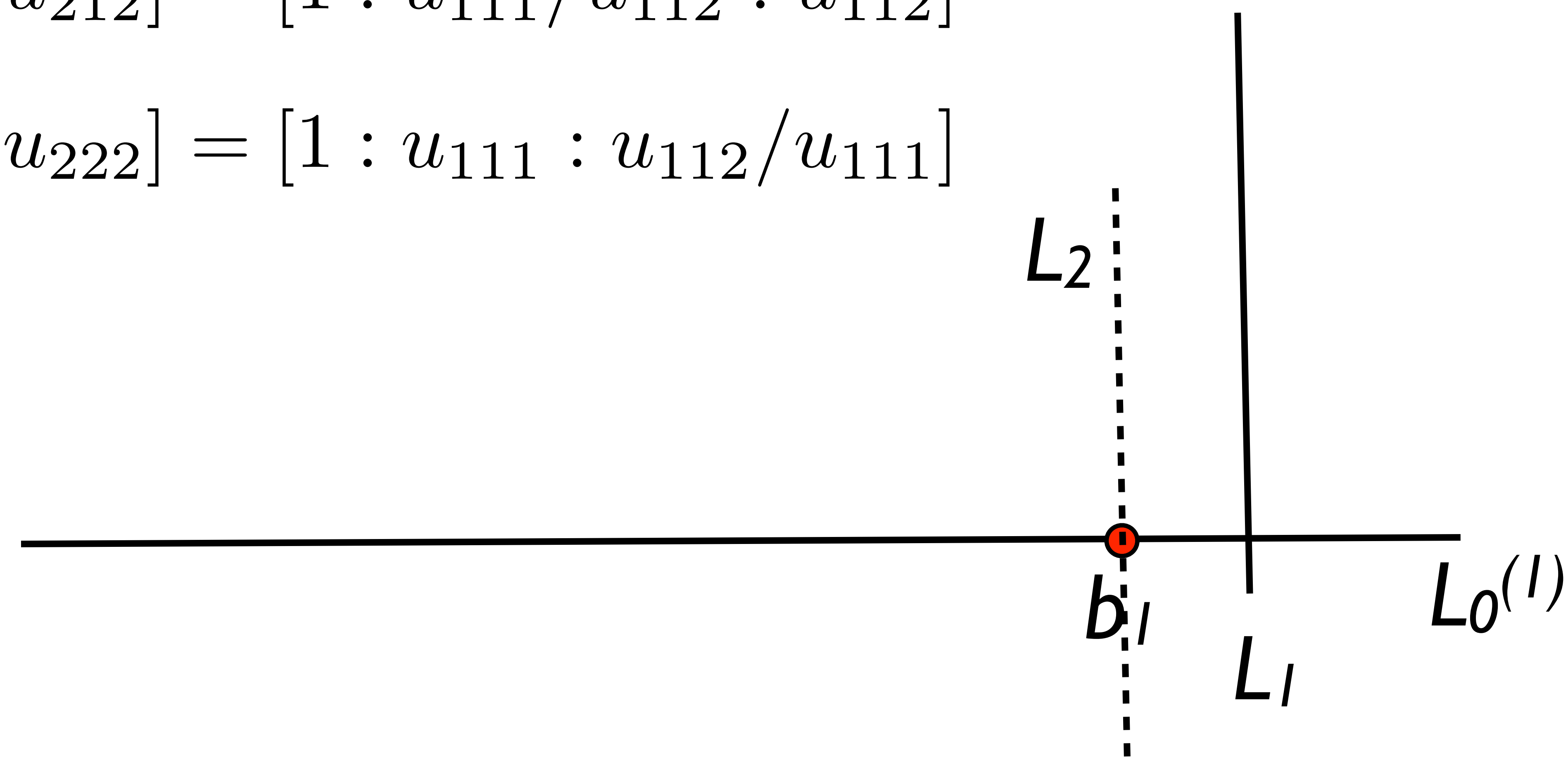
→ **base pt** $b_1 : \ u_{111} = 0, u_{112} = 0$

Exceptional Lines

- $u_{111}=0$ defines the proper transform $L_0^{(1)}$, while $u_{112}=0$ is L_1.
- Blowing up at b_1 we have two charts:

$$[1 : u_{211} : u_{212}] = [1 : u_{111}/u_{112} : u_{112}]$$

$$[1 : u_{221} : u_{222}] = [1 : u_{111} : u_{112}/u_{111}]$$



Second Blow-Up

- Chart (2,1):

- Chart (2,2):

Second Blow-Up

- Chart (2,1):

$$\dot{u}_{211} = u_{211}^2 u_{212}^2 - 2u_{211}u_{212}^{-1} + 6 + (5z)^{-1}u_{211}$$
$$\dot{u}_{212} = -u_{211}u_{212}^3 - 6u_{211}^{-1}u_{212} + 1 + (5z)^{-1}u_{212}$$

- Chart (2,2):

Second Blow-Up

- Chart (2,1):

$$\dot{u}_{211} = u_{211}^2 u_{212}^2 - 2u_{211}u_{212}^{-1} + 6 + (5z)^{-1}u_{211}$$

$$\dot{u}_{212} = -u_{211}u_{212}^3 - 6u_{211}^{-1}u_{212} + 1 + (5z)^{-1}u_{212}$$

- Chart (2,2):

$$\dot{u}_{221} = -u_{222}^{-1} + 2(5z)^{-1}u_{221}$$

$$\dot{u}_{222} = -u_{221}^2 u_{222}^2 + 2u_{221}^{-1} - 6u_{222}^2 - (5z)^{-1}u_{222}$$

Second Blow-Up

- Chart (2,1):

$$\dot{u}_{211} = u_{211}^2 u_{212}^2 - 2u_{211}u_{212}^{-1} + 6 + (5z)^{-1}u_{211}$$

$$\dot{u}_{212} = -u_{211}u_{212}^3 - 6u_{211}^{-1}u_{212} + 1 + (5z)^{-1}u_{212}$$

- Chart (2,2):

$$\dot{u}_{221} = -u_{222}^{-1} + 2(5z)^{-1}u_{221}$$

$$\dot{u}_{222} = -u_{221}^2 u_{222}^2 + 2u_{221}^{-1} - 6u_{222}^2 - (5z)^{-1}u_{222}$$

Second Blow-Up

- Chart (2,1):

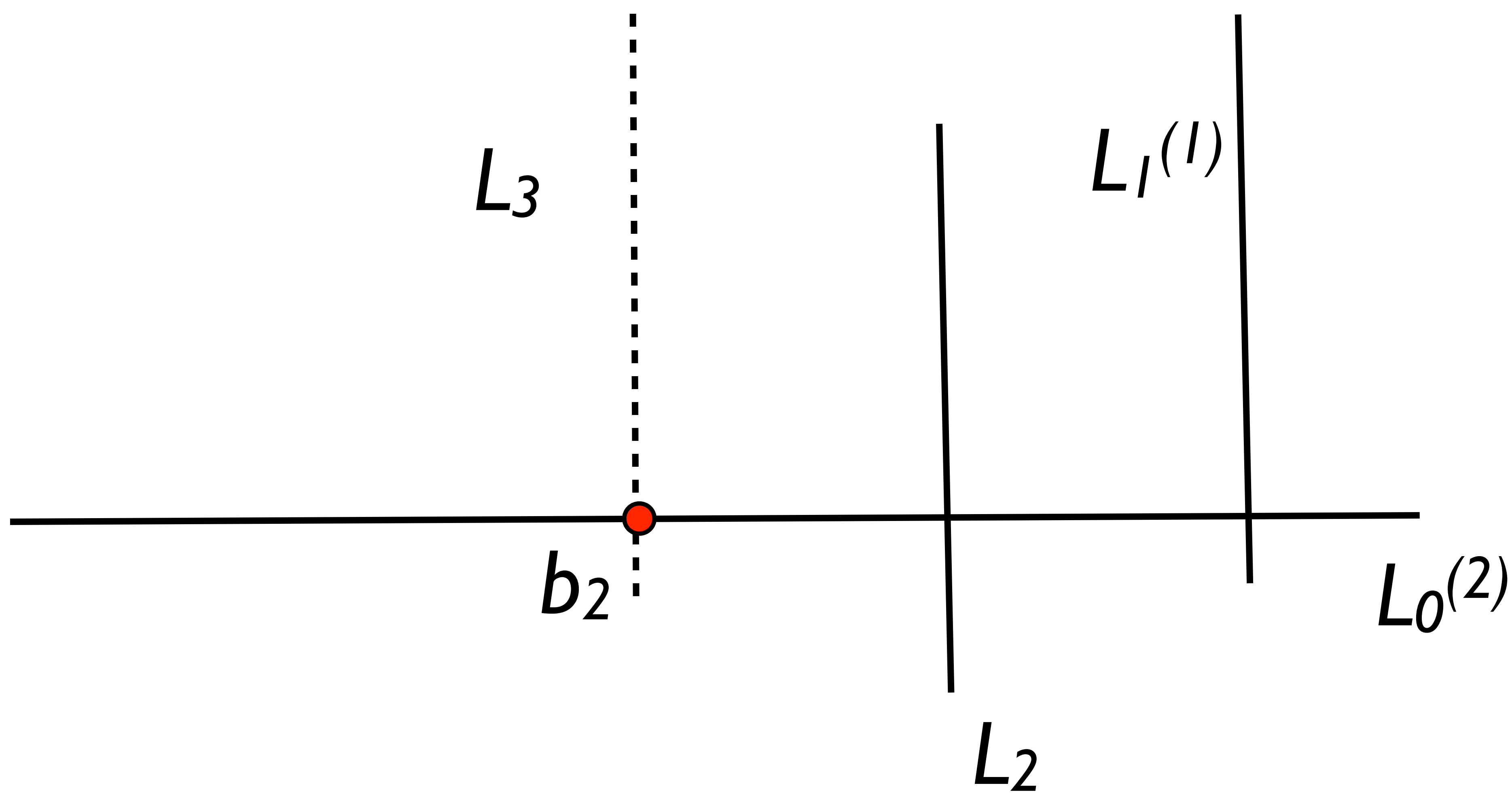
$$\dot{u}_{211} = u_{211}^2 u_{212}^2 - 2u_{211}u_{212}^{-1} + 6 + (5z)^{-1}u_{211}$$
$$\dot{u}_{212} = -u_{211}u_{212}^3 - 6u_{211}^{-1}u_{212} + 1 + (5z)^{-1}u_{212}$$

- Chart (2,2):

$$\dot{u}_{221} = -u_{222}^{-1} + 2(5z)^{-1}u_{221}$$
$$\dot{u}_{222} = -u_{221}^2 u_{222}^2 + 2u_{221}^{-1} - 6u_{222}^2 - (5z)^{-1}u_{222}$$

→ $b_2 : u_{211} = 0, u_{212} = 0$

Exceptional Lines



From First to Ninth

- Altogether there are nine blow-ups:

$$b_0 : u_{031} = 0, u_{032} = 0$$
$$b_1 : u_{111} = 0, u_{112} = 0$$
$$b_2 : u_{211} = 0, u_{212} = 0$$
$$b_3 : u_{311} = 4, u_{312} = 0$$
$$b_4 : u_{411} = 4, u_{412} = 0$$
$$b_5 : u_{511} = 0, u_{512} = 0$$
$$b_6 : u_{611} = 0, u_{612} = 0$$
$$b_7 : u_{711} = 32, u_{712} = 0$$
$$b_8 : u_{811} = -\frac{2^8}{(5\,z)}, u_{812} = 0$$

- Only the last one differs from the elliptic case.

Ninth Blow-Up

Ninth Blow-Up

- Chart (9,1):

$$\dot{u}_{911} = \left(4 + 32u_{912}^4 + u_{911}u_{912}^6 - \frac{2^8}{(5z)}u_{912}^5\right)^{-1}$$
$$\times \Big[u_{912}\Big(-2^{11} - 2^6 \cdot 5\, u_{911}u_{912}^2 + 2^{13} \cdot 7u_{912}^4$$
$$- 3^2u_{911}^2u_{912}^4 + 2^{12}u_{911}u_{912}^6 + 2^{16} \cdot 3u_{912}^8 + 2^3 \cdot 3^2u_{911}^2u_{912}^8$$
$$+ 2^{12} \cdot 5u_{911}u_{912}^{10} + 2^6 \cdot 11u_{911}^2u_{912}^{12} + 2^3u_{911}^3u_{912}^{14}\Big)$$
$$- \frac{2}{(5z)}\Big(2^2 \cdot 3u_{911} - 2^{12} \cdot 3^2u_{912}^2 - 2^5 \cdot 3^2 \cdot 7u_{911}u_{912}^4$$
$$+ 2^{15} \cdot 3 \cdot 5u_{912}^6 + 3u_{911}^2u_{912}^6 + 2^{10} \cdot 17u_{911}u_{912}^8$$
$$+ 2^{17} \cdot 19u_{912}^{10} + 2^{13} \cdot 3 \cdot 7u_{911}u_{912}^{12} + 2^7 \cdot 23u_{911}^2u_{912}^{14}\Big)$$
$$+ 2^9\,(5z)^{-2}u_{912}^3\Big(-2^6 \cdot 3 \cdot 5 + 3u_{911}u_{912}^2 + 2^{13}u_{912}^4$$
$$+ 2^{14} \cdot 5u_{912}^8 + 2^8 \cdot 11u_{911}u_{912}^{10}\Big) - 2^{24} \cdot 7(5z)^{-3}u_{912}^{12}\Big]$$

Ninth Blow-Up Ct'd

$$
\begin{aligned}
\dot{u}_{912} = -&\left(4 + 32u_{912}^4 + u_{911}u_{912}^6 - \frac{2^8}{(5z)}u_{912}^5\right)^{-1} \\
&\times \Big[2 - 2^4u_{912}^4 - u_{911}u_{912}^6 + 2^8u_{912}^8 \\
&\quad + 2^3u_{911}u_{912}^{10} + 2^{10}u_{912}^{12} + 2^6u_{911}u_{912}^{14} + u_{911}^2u_{912}^{16} \\
&- (5z)^{-1}u_{912}\big(2^2 - 2^5 \cdot 7u_{912}^4 - u_{911}u_{912}^6 + 2^{11}u_{912}^8 \\
&\quad + 2^{14}u_{912}^{12} + 2^9u_{911}u_{912}^{14}\big) + 2^8(5z)^{-2}u_{912}^6\left(1 + 2^8u_{912}^8\right)\Big]
\end{aligned}
$$

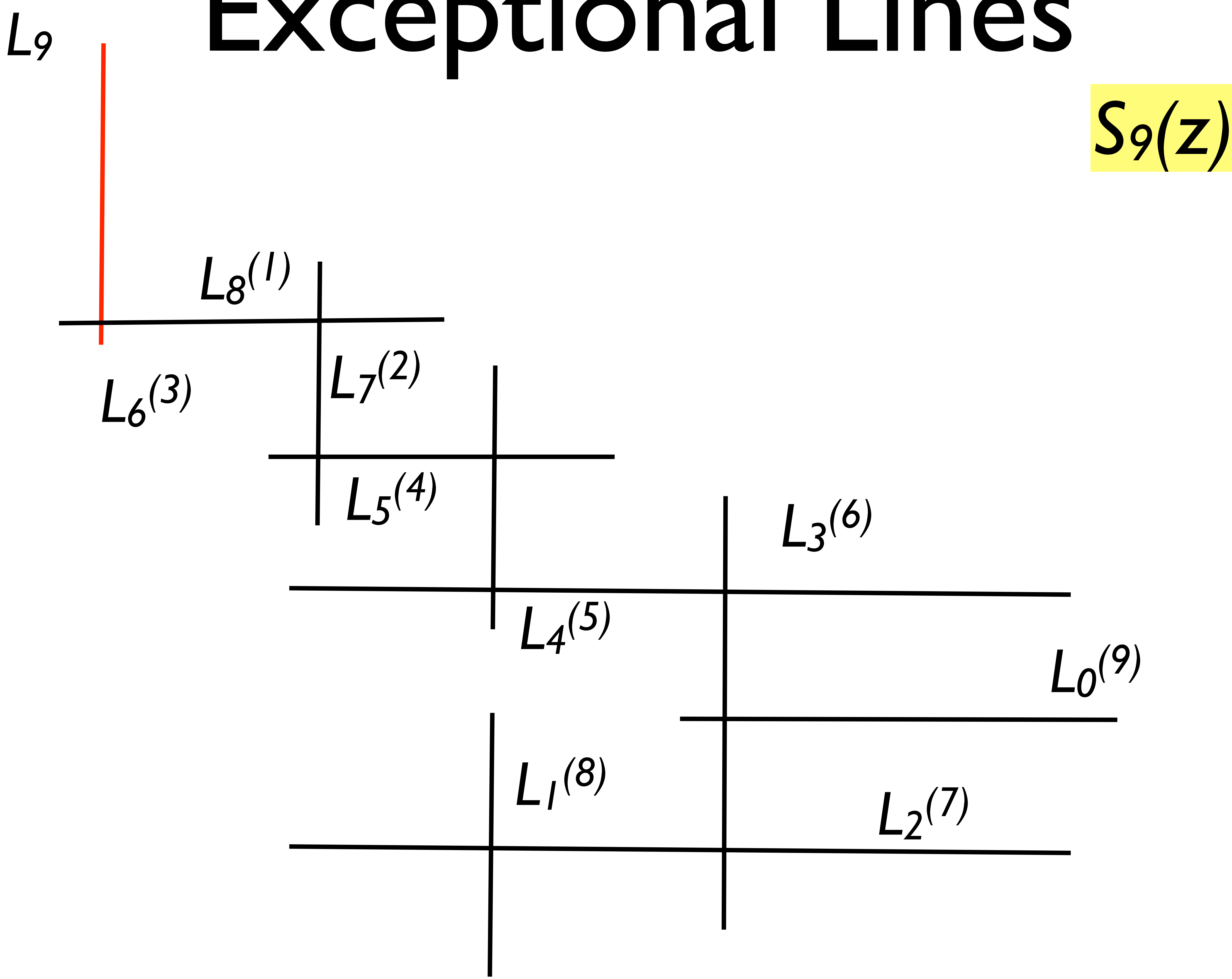
Ninth Blow-Up Ct'd

$$\dot{u}_{912} = -\left(4 + 32u_{912}^4 + u_{911}u_{912}^6 - \frac{2^8}{(5z)}u_{912}^5\right)^{-1}$$
$$\times \Big[2 - 2^4 u_{912}^4 - u_{911}u_{912}^6 + 2^8 u_{912}^8$$
$$+ 2^3 u_{911}u_{912}^{10} + 2^{10}u_{912}^{12} + 2^6 u_{911}u_{912}^{14} + u_{911}^2 u_{912}^{16}$$
$$- (5z)^{-1}u_{912}\big(2^2 - 2^5\cdot 7u_{912}^4 - u_{911}u_{912}^6 + 2^{11}u_{912}^8$$
$$+ 2^{14}u_{912}^{12} + 2^9 u_{911}u_{912}^{14}\big) + 2^8(5z)^{-2}u_{912}^6\left(1 + 2^8 u_{912}^8\right)\Big]$$

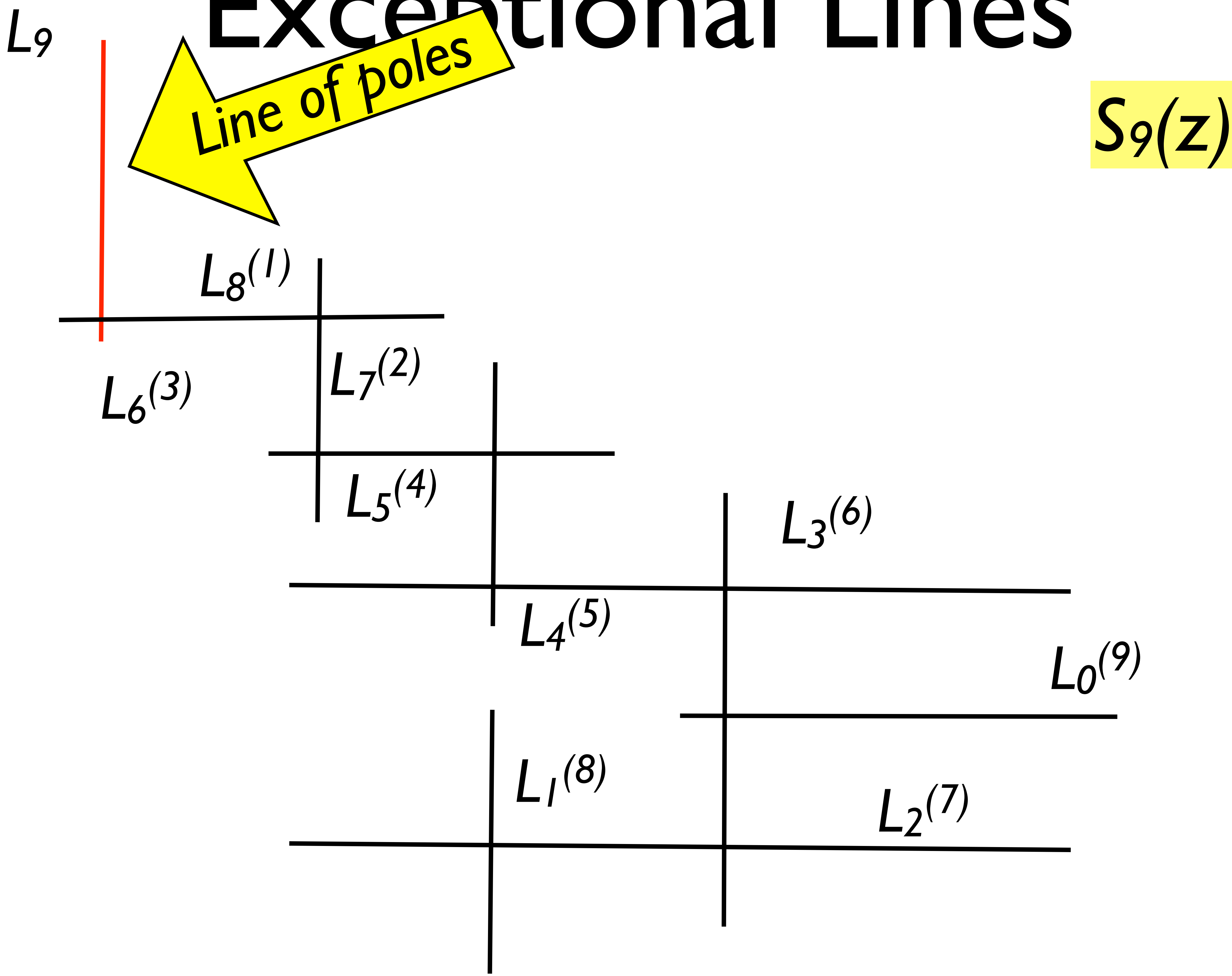
- The equation $4 + 32u_{912}^4 + u_{911}u_{912}^6 - \frac{2^8}{(5z)}u_{912}^5 = 0$

  is the proper transform $L_0^{(9)}$ of the line at infinity.

Exceptional Lines



Exceptional Lines



Regularised Dynamics in Okamoto's Space

- *Definition*: For $z \in \mathbb{C} \backslash \{0\}$, let *S* denote the fibre bundle of the Okamoto surfaces $S_9(z)$ and

$$I(z) := \cup_{i=0}^{8} L_i^{(9-i)}(z)$$

This is the *infinity set.*

- *Proposition*: $I(z)$ is a repellor for the flow.

Elements of Proof

- The "energy" function $E := \frac{u_2^2}{2} - 2\,u_1^3 - u_1$ and the Jacobian of the coordinate change to each chart

$$w_{ij} = \frac{\partial u_{ij1}}{\partial u_1}\,\frac{\partial u_{ij2}}{\partial u_2} - \frac{\partial u_{ij1}}{\partial u_2}\,\frac{\partial u_{ij2}}{\partial u_1}$$

provide a "distance" function to *I* which allow us to bound the flow near *I*.

- Near $I \backslash L_8^{(1)}$ we use $1/E$ while near $L_8^{(1)}$ we use w_{92}. In the overlap, $2\,E\,w_{92} \to 1$

The Limit Set

- *Definition*: For every solution *U(z)* $\in$ *S₉(z)\I(z)*, let

$$\Omega_U = \Big\{ s \in S_9(\infty)\backslash I(\infty) \;\Big|\; \exists\ \{z_j\} \text{ s.t. } z_j \to \infty, \; U(z_j) \to s \text{ as } j \to \infty \Big\}$$

  This is the *limit set*.

- *Lemma*: Ω_U is a non-empty, connected and compact subset of Okamoto's space.

How many poles?

- *Lemma*: Every solution of the first Painlevé equation has infinitely many poles.

  If Ω_U intersects L_9 then we get infinitely many poles. If not, then Ω_U must be a compact subset of $S_9 \backslash \{S_{9,\infty} \cup L_9\}$. Since holomorphic, the limit set must equal one point. But the autonomous system has two points ⇒ *contradiction.*

Near Equilibria

- Perturbing around the equilibria of the limiting system

$$\dot{u}_1 = u_2 - 2\,(5\,z)^{-1}\,u_1,$$
$$\dot{u}_2 = 6\,{u_1}^2 + 1 - 3\,(5\,z)^{-1}\,u_2.$$

- We study a rescaled system

$$\frac{\mathrm{d}p}{\mathrm{d}t} = v(t^{-1},\,p)$$

and find solutions analytic in a half-plane, i.e., *tronquée* and *tritronquée* solutions.

Summary

- Phase spaces of dynamical systems can be non-compact, e.g., when solutions become unbounded. Such spaces can be compactified and regularised.
- This powerful geometric process provides a global description of all solutions, including in the neighbourhoods of singular points.
- Particularly useful for systems that asymptote to elliptic functions; ; *deserves to be used more widely in applied mathematics.*