1. (a) The fitted regression model is

\[ Y = 35.8255 - 0.6765x_1 + 1.2811x_2. \]

The error standard deviation estimate is \( \hat{\sigma} = 22.98 \).

(b) A 95% CI for \( \beta_1 \) is

\[ -0.6765 \pm 2.039 \times 1.4359 = (-3.605, 2.252). \]

The interval contains 0 and so we can conclude that temperature does not add any extra information to that provided by humidity for explaining the variability in \( Y \).

(c) The multiple correlation coefficient is

\[ R = \sqrt{0.55} = 0.7416. \]

(d) From the lm summary the \( p \)-value for testing \( H_0 : \beta_2 = 0 \) is 0.005. Thus there is strong evidence against \( H_0 \) and we conclude that humidity cannot be dropped from the model. The test statistic is \( \hat{\beta}_2 \) divided by its estimated standard error, \( 1.2811/0.4243 = 3.019 \). The \( p \) value is \( P(|t_{31}| \geq 3.019) \).

(e) The fitted regression line is

\[ Y = 10.98 + 1.45x_2. \]

The predicted value for \( Y \) when \( x_2 = 60 \) is 97.818. Thus a 95% CI for the expected infestation rate is

\[ 97.818 \pm t \times 6.9675 = (83.63, 112.01). \]

R-code:

```r
lm1<-lm(y~x1+x2, aphid)
summary (lm1)
c<-summary(lm1)$coeff
t<-qt(0.975,31)
c[2,1] -t*c[2,2]
c[2,1] +t*c[2,2]

lm2<-lm(y~x2, aphid)
summary(lm2)
p<-predict(lm2, data.frame(x2=60),se.fit =T)
t1<-qt(0.975,32)
c(p$fit - t1 * p$se.fit, p$fit + t1 * p$se.fit)
```
2. (a) The simple linear regression line is

\[ Y = 20.5705 + 3.1857x. \]

The residual plot shows that the variability increases with \( x \) and so the classical regression assumptions do not hold. The estimated standard error of the slope parameter is 0.404.

(b) The fitted transformed model leads to an estimate for \( \beta_1 \) of 3.3944 with estimated standard error 0.3749, and an estimate for \( \beta_0 \) of 17.5859.

The constant \( k \) is estimated to be \( 0.2792^2 = 0.0780 \).

```
x<-c(16,14,22,10,14,17,10,13,19,12)
y<-c(77,70,85,50,62,70,52,63,88,57)
lm2<-lm(y~x)
summary(lm2)
```

Coefficients:

| Estimate  | Std. Error | t value | Pr(>|t|) |
|-----------|------------|---------|----------|
| (Intercept)| 20.5705    | 6.1269  | 3.357    |
|           | 3.1857     | 0.4044  | 7.877    |

r<-lm2$resid

```
> r
      1      2      3      4      5      6      7      8      9     10
 5.4586130 4.8299776 -5.6554810 -2.4272931 -3.1700224 -4.7270694 -0.4272931
```

```
plot(x,r, main="Residuals vs x")
y1<-y/x
x1<-1/x
lm3<-lm(y1~x1)
summary(lm3)
```

Coefficients:

| Estimate  | Std. Error | t value | Pr(>|t|) |
|-----------|------------|---------|----------|
| (Intercept)| 3.3944     | 0.3749  | 9.053    |
|           | 17.5859    | 5.0434  | 3.487    |

Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
Residual standard error: 0.2792 on 8 degrees of freedom
3. The least squares estimator for the intercept \( \beta_0 \) is

\[
\hat{\beta}_0 = Y - \hat{\beta}_1 \bar{x} = \sum_{i=1}^{n} \left( \frac{1}{n} - \frac{(x_i - \bar{x})\bar{x}}{S_{XX}} \right) Y_i,
\]

where \( S_{XX} = \sum_{i=1}^{n} (x_i - \bar{x})^2 \).

Thus

\[
E(\hat{\beta}_0) = \sum_{i=1}^{n} \left( \frac{1}{n} - \frac{(x_i - \bar{x})\bar{x}}{S_{XX}} \right) E(Y_i),
\]

\[
= \sum_{i=1}^{n} \left( \frac{1}{n} - \frac{(x_i - \bar{x})\bar{x}}{S_{XX}} \right) (\beta_0 + \beta_1 x_i),
\]

\[
= \beta_0 + \beta_1 \bar{x} - \bar{x}\beta_1 \sum_{i=1}^{n} \frac{(x_i - \bar{x})x_i}{S_{XX}}
\]

\[
= \beta_0,
\]

as \( \sum_{i=1}^{n} (x_i - \bar{x}) = 0 \) and \( \sum_{i=1}^{n} (x_i - \bar{x})x_i = S_{XX} \).

To obtain the variance note

\[
Var(\hat{\beta}_0) = \sum_{i=1}^{n} \left( \frac{1}{n} - \frac{(x_i - \bar{x})\bar{x}}{S_{XX}} \right)^2 Var(Y_i),
\]

\[
= \sigma^2 \sum_{i=1}^{n} \left( \frac{1}{n} - \frac{(x_i - \bar{x})\bar{x}}{S_{XX}} \right)^2
\]

\[
= \sigma^2 \left( \frac{1}{n} + \frac{(\bar{x})^2 S_{XX}}{S_{XX}^2} \right)
\]

\[
= \sigma^2 \left( \frac{\sum_{i=1}^{n} x_i^2}{nS_{XX}} \right),
\]

as \( S_{XX} = \sum_{i=1}^{n} x_i^2 - n\bar{x}^2 \).
4. (STAT3912 Only)

(a) The matrix \( X \) is \( n \times p \) of rank \( p \) and so \( X^T X \) has an inverse. Thus the least squares estimator for \( \beta \) is \( \hat{\beta} = (X^T X)^{-1} X^T Y \).

The residual vector is

\[
R = Y - X \hat{\beta} = (I - X(X^T X)^{-1} X^T) Y = (I - H) \epsilon,
\]

as \( Y = X \beta + \epsilon \) and \( (I - H) X = X - X = 0 \).

(b) \[
\hat{Y} = X \hat{\beta} = X(X^T X)^{-1} X^T Y = H(X \beta + \epsilon) = X \beta + H \epsilon.
\]

(c) Note \( E(\hat{Y}) = X \beta \). Thus

\[
\text{Cov}(Y, \hat{Y}) = E(Y(\hat{Y} - E(\hat{Y}))^T) = E(X \beta + \epsilon)(H \epsilon)^T = \sigma^2 H
\]

as \( H = H^T \) and \( E \epsilon \epsilon^T = \sigma^2 I \).

Next

\[
\text{Cov}(\hat{Y}, \hat{Y}) = E(H \epsilon)(H \epsilon)^T = H \sigma^2 I H^T = \sigma^2 H,
\]

as \( H^2 = H \). Thus \( \text{Cov}(Y, \hat{Y}) = \text{Cov}(\hat{Y}, \hat{Y}) \).

(d) \[
\sum_{i=1}^{n} \text{Var} \hat{Y}_i = \text{trace} \left( \text{Cov}(\hat{Y}, \hat{Y}) \right) = \sigma^2 \text{trace} (H) = \sigma^2 p,
\]

as \( \text{trace} H = \text{trace} ((X^T X)^{-1} X^T X) \).