Power series and Taylor series.

In the tutorials, do Questions 3(c), 6, 11(d), 11(e), 13, 15 and 20(c), or as many as you can manage in the available time. The other problems are practice exercises for you to do at your own convenience. Full solutions will be placed in Kopystop at the beginning of Week 5.

These questions cover substantially more territory on infinite series than is needed for this course. The most worthwhile questions to study at home are Questions 3 (power series), 8–10 (uniform convergence), 11–16 and 20 (Taylor series) and 22(a) (Laurent series).

1. Are the following series convergent or divergent?

   (a) $\sum_{n=0}^{\infty} \frac{(1 + 2i)^n}{n!}$ [convergent].
   (b) $\sum_{n=0}^{\infty} \frac{n(i/2)^n}{n!}$ [convergent].
   (c) $\sum_{n=1}^{\infty} \frac{(n^2 + i^n)}{n!}$ [divergent].
   (d) $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$ [convergent].

2. Ratio test: Let $\sum z_n$ be a series with nonzero terms. Prove the following:

   (a) If there exist $N$ and $q < 1$ such that $|z_{n+1}/z_n| \leq q$ for all $n > N$, then the series converges absolutely.
   (b) If there exists an $N$ such that $|z_{n+1}/z_n| \geq 1$ for all $n > N$, then the series diverges.

   If $\sum z_n$ converges according to this test, show that the remainder $R_n = z_{n+1} + z_{n+2} + \ldots$ satisfies $|R_n| \leq |z_{n+1}|/(1 - q)$.

Power Series

3. Find the centre and radius of convergence of:

   (a) $\sum_{n=0}^{\infty} (z + 2i)^n$ $[-2i; 1]$.
   (b) $\sum_{n=0}^{\infty} n(z/3)^n$ $[0; 3]$.
   (c) $\sum_{n=0}^{\infty} ((3n + 4)/2^n)(z - 2 - i)^n$ $[2 + i; 2]$.
4. If \( f(z) = \sum_{n=0}^{\infty} z^{2n}/n! \), show that \( f'(z) = 2zf(z) \).

5. Show that if \( R \) (assumed finite) is the radius of convergence of \( \sum a_n z^n \) then the radius of convergence of
   (a) \( \sum a_n z^{2n} \) is \( \sqrt{R} \);
   (b) \( \sum a_n^2 z^n \) is \( R^2 \).

6. By differentiating a suitable series, show that
   \[
   \frac{1}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1)z^n, \quad |z| < 1.
   \]

### Uniform Convergence

7. Prove that the following series converge uniformly in the given region:
   (a) \( \sum_{n=1}^{\infty} (\sin(n|z|))/2^n \), all \( z \);
   (b) \( \sum_{n=1}^{\infty} (\tanh^n x)/(n(n+1)) \), all real \( x \);
   (c) \( \sum_{n=1}^{\infty} (\cos^n |z|)/n^2 \), all \( z \);
   (d) \( \sum_{n=1}^{\infty} 1/(|z| + n^2) \), all \( z \);
   (e) \( \sum_{n=1}^{\infty} z^n/(n\sqrt{n+1}) \), \( |z| \leq 1 \).

8. Prove that \( 1 + \sum_{n=1}^{\infty} (x^n - x^{n-1}) \) is not uniformly convergent in \([0, 1]\). [Hint: the quickest way is to show that the pointwise sum is discontinuous somewhere in \([0, 1]\) and quote a standard theorem. Can you see the nonuniformity directly?]

9. Let \( U \) be the open set \( \{ s \in \mathbb{C} : \text{Re } s > 1 \} \) in the complex \( s \)-plane. Let \( f_n(s) = n^{-s} \) be a sequence of analytic functions on \( U \). Show that the infinite series,
   \[
   \zeta(s) := \sum_{n=1}^{\infty} f_n(s),
   \]
   converges absolutely and uniformly on all compact subsets of \( U \). State the theorem by which you conclude that this infinite series defines an analytic function \( \zeta(s) \) in \( U \).

10. Let \( f_n(z) = \frac{2nz}{1+n^2z^2} \). Show that
    (a) the sequence \( \{f_n(z)\} \) converges for all \( z \in \mathbb{C} \) as \( n \to \infty \);
    (b) the sequence does not converge uniformly for \( z \) in the unit disc \( \{z : |z| \leq 1\} \);
    (c) the sequence converges uniformly in the domain \( \{z : |z| \geq 2\} \).
Taylor and Maclaurin series

A Maclaurin series is just a Taylor series about the origin.

11. Show that

(a) \( \cos z = \sum_{n=0}^{\infty} (-1)^n z^{2n}/(2n)!; \quad \sin z = \sum_{n=0}^{\infty} (-1)^n z^{2n+1}/(2n+1)!; \)

(b) \( \cosh z = \sum_{n=0}^{\infty} z^{2n}/(2n)!; \quad \sinh z = \sum_{n=0}^{\infty} z^{2n+1}/(2n+1)!; \)

(c) \( 1/(1 - z) = \sum_{n=0}^{\infty} z^n, \quad |z| < 1; \)

(d) \( \log(1 + z) = \sum_{n=1}^{\infty} (-1)^{n-1} z^n/n, \quad |z| \leq 1, \quad z \neq -1; \)

(e) \( \log((1 + z)/(1 - z)) = 2 \sum_{n=0}^{\infty} z^{2n+1}/(2n + 1), \quad |z| \leq 1, \quad z \neq \pm 1. \)

In part (a), a suitable starting point would be the identities, \((d/dz) \sin z = \cos z \) and \((d/dz) \cos z = -\sin z\), together with the initial values \(\sin 0 = 0\) and \(\cos 0 = 1\). In parts (d) and (e), \(\log z\) denotes the principal value of the logarithm function. In these cases, convergence extends onto the circle of convergence \(|z| = 1\) (except at the indicated points). The convergence being conditional, the proof requires considerable care. Results for (e) can be deduced from the results for (d).

12. For \(b \neq 0\) and \(c - ab \neq 0\), show that

\[
\frac{1}{c - bz} = \sum_{n=0}^{\infty} b^n (z - a)^n / (c - ab)^{n+1}, \quad |z - a| < |c/b - a|.
\]

13. Show that

\[
\frac{1}{1 + z^2} = \sum_{n=0}^{\infty} (-1)^n z^{2n}, \quad |z| < 1.
\]

By integrating the above series term-by-term, show that

\[
\tan^{-1} z = \sum_{n=0}^{\infty} (-1)^n z^{2n+1} / (2n + 1), \quad |z| \leq 1, \quad z \neq \pm i.
\]

By relating the inverse tangent to the logarithm, give a suitable definition of the principal value of \(\tan^{-1} z\) for all \(z \in \mathbb{C} - \{i, -i\}\). [The convergence of the series on the circle \(|z| = 1\) can be deduced from the corresponding results for the logarithm in Question 11(d).]

14. The binomial series: Prove that, for any real number \(\alpha\),

\[
(1 + z)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n = 1 + \alpha z + \frac{\alpha(\alpha-1)}{2!} z^2 + \ldots, \quad |z| < 1.
\]

Investigate the convergence on the circle \(|z| = 1\).
15. Using the Maclaurin expansions of \( \sin z \) and \( \cos z \), prove that
\[
\tan z = z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \frac{17}{315}z^7 + \ldots , \quad |z| < \frac{1}{2}\pi.
\]
[Hint: \( \tan z \) is odd, so we can assume \( \tan z = c_1z + c_3z^3 + c_5z^5 + \ldots \) and then use the identity \( \sin z = \tan z \cos z \).]

16. Fibonacci numbers: Prove that
\[
\frac{1}{1 - z - z^2} = \sum_{n=0}^{\infty} F_{n+1}z^n, \quad |z| < \frac{1}{2}\sqrt{5} - \frac{1}{2},
\]
where the \( F_n \), \( n = 1, 2, 3, \ldots \), are the Fibonacci numbers, i.e., \( F_1 = F_2 = 1 \) and \( F_n = F_{n-1} + F_{n-2} \) for \( n > 2 \). By resolving the left-hand side into partial fractions and writing each fraction as a convergent geometric series, deduce that
\[
F_n = \frac{1}{\sqrt{5}} \left\{ \left( \frac{1}{2} + \frac{1}{2}\sqrt{5} \right)^n - \left( \frac{1}{2} - \frac{1}{2}\sqrt{5} \right)^n \right\}.
\]

17. Euler numbers: The Euler numbers \( E_{2n} \) are defined by the Maclaurin series,
\[
\sec z = \sum_{n=0}^{\infty} (-1)^nE_{2n} \frac{z^{2n}}{(2n)!}, \quad |z| < \frac{1}{2}\pi.
\]
Show that \( E_0 = 1, E_2 = -1, E_4 = 5, E_6 = -61. \) [According to this convention, all the \( E_n \) with \( n \) odd are zero.]

18. Bernoulli numbers: One of several equivalent definitions of the Bernoulli numbers \( B_n \) is by means of the Maclaurin series,
\[
\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}, \quad |z| < 2\pi.
\]
Show that \( B_0 = 1; B_1 = -\frac{1}{2}; B_2 = \frac{1}{6}; B_3 = 0; B_4 = -\frac{1}{30}. \) Show also that all odd-index \( B_n \), except \( B_1 \), vanish. Derive the recurrence relation, \( B_n = (1 + B)^n \) \( (n \geq 2) \), where the right-hand side is to be expanded by the binomial theorem and then \( B^n \) replaced by \( B_n \). Use the recurrence relation (with assistance from a pocket calculator) to get the next few nonzero Bernoulli numbers: \( B_8 = \frac{1}{6}; B_9 = -\frac{1}{2}; B_{10} = -\frac{5}{66}; B_{12} = \frac{691}{2730}; B_{14} = \frac{7}{6}; B_{16} = \frac{3617}{510}; B_{18} = \frac{43867}{798}; B_{20} = -\frac{174611}{330}. \) [Remark: The first few Bernoulli numbers are fairly small, but after that they grow exponentially with alternating signs—see Question 19(b).]

19. (a) By writing trigonometric functions in exponential form, derive the full Maclaurin series expansions of \( z \cot z \) and \( \tan z \) in terms of Bernoulli numbers.
(b) Given that \( \cot z \) has the infinite partial fraction expansion,

\[
\cot z = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{2z}{z^2 - k^2 \pi^2},
\]

deduce the following famous formula of Euler:

\[
1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \ldots = (-1)^{n-1} \frac{2^{2n-1} B_{2n}}{(2n)!} \pi^{2n}, \quad n \geq 1.
\]

20. Find the Maclaurin expansions of the following special functions:

(a) **Error function**: \( \text{erf} z = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} \, dt; \)

(b) **Sine integral**: \( \text{Si}(z) = \int_0^z \frac{\sin t}{t} \, dt; \)

(c) **Fresnel integrals**: \( S(z) = \int_0^z \sin(t^2) \, dt, \quad C(z) = \int_0^z \cos(t^2) \, dt. \)

In part (a), for example, begin with the standard Maclaurin series for \( e^t \), change \( t \) to \( -t^2 \), and then integrate term by term.

21. Suppose \( f(z) = \sum_{n=0}^{\infty} c_n z^n \) for \( z \in \mathbb{C} \). Show that for all \( R > 0 \),

\[
\sum_{n=0}^{\infty} |c_n| R^n \leq 2M(2R), \quad M(r) = \sup\{|f(z)| : |z| = r\}.
\]

22. **Laurent series**:

(a) For the function \( f(z) = (z+1)^{-2}(z-3)^{-1} \), use partial fractions and the binomial theorem to get the Laurent expansion for \( f(z) \) in the annulus \( 1 < |z| < 3 \). Obtain also the Maclaurin series for \( |z| < 1 \) and the descending Taylor series about \( \infty \) for \( |z| > 3 \).

(b) Obtain the Laurent series for \( \cot z \) in the annulus \( 2\pi < |z| < 3\pi \).

23. **Cauchy-Riemann equations**. Let \( f(z) = u(x, y) + iv(x, y) \) be defined in a domain \( D \) of the complex \( z \)-plane, and let the corresponding domain of the real \( xy \)-plane be also called \( D \), where \( z = x + iy \). Suppose that, at one particular point \( P \in D \) with coordinates \( z_0 = x_0 + iy_0 \) or \( (x_0, y_0) \), the four partial derivatives \( u_x, u_y, v_x \) and \( v_y \) exist and satisfy the Cauchy-Riemann equations \( u_x = v_y \) and \( u_y = -v_x \). Prove that the complex function \( f(z) \) has a derivative \( f'(z_0) \) at \( z = z_0 \) under one of the following additional hypotheses:

(a) \( u \) and \( v \) are differentiable at \( P \);

(b) at least one derivative of \( u \) and one derivative of \( v \) exists in a neighbourhood of \( P \) and is continuous at \( P \) itself. (In fact (b) implies (a).)