Part III
Complex boundary value problems

Lesson 19
Laplace’s equation
• We have used function approximation to solve ODEs and compute their spectra
  – The key idea: phrase the differential equation as an operator acting on unknown coefficients

• We have also solved PDEs on a rectangle

• But what about other domains?

• Today, we will discuss the canonical PDE: Laplace’s equation
  – This will be phrased as a complex boundary value problem
Rectangle

Funny domains?
Laplace’s equation

\[ u_{xx} + u_{yy} = 0 \quad \text{for} \quad (x, y) \in \Omega, \]
\[ u(x, y) = f(x, y) \quad \text{for} \quad (x, y) \in \partial\Omega \]

- Solutions represent (among other things) temperature equilibrium
- A function which satisfies Laplace’s equation is called harmonic
• Recall if \( f(x+iy) = u(x, y)+iv(x, y) \) is analytic, then it satisfies the Cauchy–Riemann conditions

\[
\begin{align*}
  u_x &= v_y \\
  u_y &= -v_x
\end{align*}
\]
• Recall if $f(x + iy) = u(x, y) + iv(x, y)$ is analytic, then it satisfies the Cauchy–Riemann conditions

\[ u_x = v_y \quad \text{and} \quad u_y = -v_x \]

• It follows that

\[ u_{xx} = v_{xy} \quad \text{and} \quad u_{yy} = -v_{xy} \]
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  $$u_{xx} = v_{xy} \quad \text{and} \quad u_{yy} = -v_{xy}$$

• In other words:
  
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    u_x &= v_y & \text{and} & \quad u_y &= -v_x \\
    u_{xx} &= v_{xy} & \text{and} & \quad u_{yy} &= -v_{xy}
\end{align*}
\]

• It follows that

\[
\begin{align*}
    u_{xx} + u_{yy} &= 0 \\
    v_{xx} + v_{yy} &= -u_{xy} + u_{xy} = 0
\end{align*}
\]

• In other words:

  – Similarly,

\[
\begin{align*}
    v_{xx} + v_{yy} &= -u_{xy} + u_{xy} = 0
\end{align*}
\]

• Thus the real and imaginary parts of an analytic function are harmonic
Laplace’s equation on the unit disk
• Suppose we are given a real-function $f(z)$ on the unit circle
• We want to find $u(x, y)$ which solves Laplace's equation in the interior of the disk
• And $u$ equals $f$ on the boundary:
  \[ u(x, y) = f(x + iy) \quad \text{for} \quad \| (x, y) \| = 1 \]
• We can solve this by finding an analytic function $\phi(z)$ whose real part is $u$
• In other words, we want to find analytic $\phi$ such that
  \[ \Re[\phi(z)] = f(z) \]
  for $|z| = 1$
• For differential equations, we solved them by constructing a infinite dimensional matrix operating on coefficients

• Lets try this for Laplace equation

• I.e., our unknown $\phi$ is analytic, hence it has a Taylor series:

$$\phi(z) = \sum_{k=0}^{\infty} \hat{\phi}_k z^k$$

• The question: how does the operator $\hat{R}$ act on each term

$$\hat{\phi}_k z^k$$

individually?
• Let \( \phi_k = p + iq \)

• We separate the operator \( \Re \) by its actions on the real and imaginary parts:

\[
\Re p z^k = p \Re z^k = p \cos k \arg z
\]
• Let $\hat{\phi}_k = p + iq$

• We separate the operator $\mathcal{R}$ by its actions on the real and imaginary parts:

$$\mathcal{R}pz^k = p\mathcal{R}z^k = p \cos k \arg z$$

$$= p \frac{e^{ik \arg z} + e^{-ik \arg z}}{2}$$
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$$= p \frac{z^k + z^{-k}}{2}$$

• Similarly:

$$\mathcal{R}iqz^k = -q\mathcal{S}z^k = -q \sin k \arg z = -q \frac{z^k - z^{-k}}{2i}$$
• We thus separate the real and imaginary parts of the unknowns \( \hat{u}_k = p_k + iq_k \):

\[
\begin{pmatrix}
\hat{u}_0 \\
\hat{u}_1 \\
\vdots
\end{pmatrix} \to \begin{pmatrix} p_0 \\
p_1 \\
\vdots
\end{pmatrix} + i \begin{pmatrix} q_0 \\
q_1 \\
\vdots
\end{pmatrix} = p + iq
• We thus separate the real and imaginary parts of the unknowns \( \hat{u}_k = p_k + iq_k \):

\[
\begin{pmatrix}
\hat{u}_0 \\
\hat{u}_1 \\
\vdots
\end{pmatrix}
\rightarrow
\begin{pmatrix}
p_0 \\
p_1 \\
\vdots
\end{pmatrix} + i \begin{pmatrix}
q_0 \\
q_1 \\
\vdots
\end{pmatrix} = p + iq
\]

• The operator \( \mathcal{R} \) then maps the vector of real and imaginary parts to a Laurent series:

\[
\mathcal{R}\hat{u} = \mathcal{R}_R r + \mathcal{R}_I q
\]
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\[
\begin{pmatrix}
\hat{u}_0 \\
\hat{u}_1 \\
\vdots
\end{pmatrix}
\rightarrow
\begin{pmatrix}
p_0 \\
p_1 \\
\vdots
\end{pmatrix}
+ i
\begin{pmatrix}
q_0 \\
q_1 \\
\vdots
\end{pmatrix}
= \begin{pmatrix} p \end{pmatrix} + i \begin{pmatrix} q \end{pmatrix}
\]

• The operator \( \mathcal{R} \) then maps the vector of real and imaginary parts to a Laurent series:
\[
\mathcal{R}\hat{u} = \mathcal{R}^R r + \mathcal{R}^I q
\]

– \( \mathcal{R}^R \) acts on the real vector \( p \) and \( \mathcal{R}^I \) acts on the imaginary vector \( q \):
\[
\mathcal{R}^R = \begin{pmatrix}
1 & \frac{1}{2} & \cdots \\
\frac{1}{2} & \frac{1}{2} & \cdots \\
\frac{1}{2} & \frac{1}{2} & \cdots \\
\vdots & \vdots & \ddots
\end{pmatrix}
\text{ and } \mathcal{R}^I = i \begin{pmatrix}
0 & -\frac{1}{2} & -\frac{1}{2} & \cdots \\
\frac{1}{2} & 0 & \frac{1}{2} & \cdots \\
\frac{1}{2} & \frac{1}{2} & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]
• Assuming \( f \) is "nice", we can expand it in Laurent series

\[
f(z) = \sum_{k=-\infty}^{\infty} \hat{f}_k z^k
\]
• Assuming \( f \) is "nice", we can expand it in Laurent series

\[
f(z) = \sum_{k=-\infty}^{\infty} \hat{f}_k z^k
\]

• Since \( f \) is real, we have that \( \hat{f}_k \) is the complex conjugate of \( \hat{f}_{-k} \):

\[
\overline{\hat{f}_k} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-ik\theta} \, d\theta
\]
Assuming $f$ is "nice", we can expand it in Laurent series

$$f(z) = \sum_{k=-\infty}^{\infty} \hat{f}_k z^k$$

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$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) e^{ik\theta} \, d\theta$$

$$= \hat{f}_{-k}$$
• The equation for $p$ and $q$ becomes:

$$\mathcal{R}_R^p + \mathcal{R}^i q = \hat{f}$$
• The equation for $p$ and $q$ becomes:

$$\Re^R p + \Re^i q = \hat{f}$$

• Or in operator form:

$$\left( \begin{array}{ccc} 1 & \frac{1}{2} & \vdots \\ \frac{1}{2} & \frac{1}{2} & \vdots \\ \vdots & \vdots & \ddots \end{array} \right) p + i \left( \begin{array}{ccc} 0 & -\frac{1}{2} & \vdots \\ \frac{1}{2} & \frac{1}{2} & \vdots \\ \vdots & \vdots & \ddots \end{array} \right) q = \left( \begin{array}{c} \vdots \\ \frac{\hat{f}_2}{f_2} \\ \frac{\hat{f}_1}{f_1} \\ \frac{\hat{f}_0}{f_0} \\ \vdots \end{array} \right)$$
• The equation for \( p \) and \( q \) becomes:

\[
R^R p + R^i q = \hat{f}
\]

• Or in operator form:

\[
\begin{pmatrix}
1 & \frac{1}{2} & \cdots \\
\frac{1}{2} & 1 & \cdots \\
\vdots & \vdots & \ddots
\end{pmatrix} p + i \begin{pmatrix}
0 & -\frac{1}{2} & \cdots \\
\frac{1}{2} & 0 & \cdots \\
\vdots & \vdots & \ddots
\end{pmatrix} q = \begin{pmatrix}
\vdots \\
\hat{f}_2 \\
\hat{f}_1 \\
\hat{f}_0 \\
\vdots
\end{pmatrix}
\]

• We can write down the exact result, the first coefficient is simple:

\[
p_0 = \hat{f}_0
\]
• The equation for \( p \) and \( q \) becomes:

\[
\mathbb{R}^R p + \mathbb{R}^i q = \hat{f}
\]

• Or in operator form:

\[
\begin{pmatrix}
1 & \frac{1}{2} & \cdots \\
\frac{1}{2} & & \\
& \ddots & \\
\frac{1}{2} & & 1
\end{pmatrix}
\begin{pmatrix}
p \\
p + i q
\end{pmatrix}
\begin{pmatrix}
\frac{1}{2} & \cdots \\
-\frac{1}{2} & & \\
& \ddots & \\
\frac{1}{2} & & \frac{1}{2}
\end{pmatrix}
\begin{pmatrix}
q \\
q
\end{pmatrix}
\begin{pmatrix}
\hat{f}_2 \\
\hat{f}_1 \\
\hat{f}_0 \\
\hat{f}_1 \\
\hat{f}_2
\end{pmatrix}
\]

• We can write down the exact result, the first coefficient is simple:

\( p_0 = \hat{f}_0 \)

• For the second coefficients \( p_1 \) and \( q_1 \), we have two equations with two unknowns:

\[
p_1 - i q_1 = 2\hat{f}_1 \quad \text{and} \quad p_1 + i q_1 = 2\hat{f}_1
\]
\[
\begin{pmatrix}
1 & \frac{1}{2} & \\
\frac{1}{2} & \frac{1}{2} & \\
\frac{1}{2} & \frac{1}{2} & \ddots
\end{pmatrix}
\begin{pmatrix}
p + \text{i}q
\end{pmatrix}
+ \begin{pmatrix}
0 & -\frac{1}{2} & \\
\frac{1}{2} & \frac{1}{2} & \\
\frac{1}{2} & \frac{1}{2} & \ddots
\end{pmatrix}
\begin{pmatrix}
q
\end{pmatrix}
= \begin{pmatrix}
\vdots
\hat{f}_2 \\
\hat{f}_1 \\
\hat{f}_0
\end{pmatrix}
\]

- We can write down the exact result, the first coefficient is simple:
  \[
p_0 = \hat{f}_0
\]

- For the second coefficients \(p_1\) and \(q_1\), we have two equations with two unknowns:
  \[
p_1 - \text{i}q_1 = 2\hat{f}_1 \quad \text{and} \quad p_1 + \text{i}q_1 = 2\hat{f}_1
\]
  - These are satisfied if
  \[
  \hat{u}_1 = 2\hat{f}_1
  \]

- In general we get
  \[
  \hat{u}_k = 2\hat{f}_k
  \]
• In short, if $f(z)$ is real-valued and "nice", then

$$\hat{f}_0 + 2\Re \sum_{k=1}^{\infty} \hat{f}_k z^k$$

solves Laplace's equation in the unit disk

• By the same logic as Taylor series, etc., the decay in $\hat{f}_k$ dictates the convergence of the approximation
$$f(x, y) = \frac{2 \sin(10x^2y^2 - y)}{2x^2 + 1}$$
\[ f(x, y) = \frac{2 \sin(10x^2y^2 - y)}{2x^2 + 1} \]
Conformal maps
• Suppose we have a conformal map $M$ from the domain $\Omega$ to the unit disk
  
  – Recall $M$ is conformal if it is analytic and one-to-one
  – In other words, $M^{-1}$ is analytic in the unit disk
  – An equivalent condition: $M'(w) \neq 0$
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  – Recall $M$ is conformal if it is analytic and one-to-one
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• Given $f$ defined on the boundary of $\Omega$, $f(M^{-1}(z))$ is defined on the unit circle, and we can expand

\[
f(M^{-1}(z)) = \sum_{k=-\infty}^{\infty} \hat{f}_k z^k
\]
• Suppose we have a **conformal map** $M$ from the domain $\Omega$ to the unit disk
  
  – Recall $M$ is conformal if it is **analytic** and **one-to-one**
  
  – In other words, $M^{-1}$ is analytic in the unit disk
  
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f(M^{-1}(z)) = \sum_{k=-\infty}^{\infty} \hat{f}_k z^k
\]

• Let

\[
\phi(z) = \hat{f}_0 + 2 \sum_{k=0}^{\infty} \hat{f}_k z^k
\]
• Suppose we have a conformal map $M$ from the domain $\Omega$ to the unit disk
  
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• Given $f$ defined on the boundary of $\Omega$, $f(M^{-1}(z))$ is defined on the unit circle, and we can expand

$$f(M^{-1}(z)) = \sum_{k=-\infty}^{\infty} \hat{f}_k z^k$$

• Let

$$\phi(z) = \hat{f}_0 + 2 \sum_{k=0}^{\infty} \hat{f}_k z^k$$

• Then $\mathcal{R}\phi(M(w))$ is holomorphic and satisfies, for $w \in \partial \Omega$,

$$\mathcal{R}u(M(w)) = f(M^{-1}(M(w))) = f(w)$$
• Consider the map

\[ M(z) = \frac{z - i}{z + i} \]

• This maps the real line to the unit circle

  – More precisely: it **conformally maps** the upper half plane to the interior of the circle
Real line
\[ M^{-1}(z) = \frac{1}{i} \frac{1 - z}{z + 1} \]
Unit circle
Unit circle

Real line

\[ M(z) = \frac{z - i}{z + i} \]
Laplace’s equation on other domains
• On the unit circle, we used an approach called **boundary matching**:

  – We approximated the solution

\[
u(x, y) = \sum_{k=0}^{n-1} u_k \psi_k(x, y)\]

where the basis \(\psi_k\) itself (in this case either \(Rz^j\) or \(Sz^j\)) satisfies Laplace's equation.

  – The unknown coefficients are then chosen to satisfy the boundary condition:

\[
u(x, y) \approx f(x, y) \quad \text{for} \quad \|(x, y)\| = 1\]

• We will adapt this approach to other domains
• Step 1: Choose a basis
  – We will use monomials again

\[ u(x, y) \approx \Re \left[ \sum_{k=0}^{n-1} \phi_k z^k \right] \]
• Step 1: Choose a basis
  
  – We will use monomials again

\[
  u(x, y) \approx \Re \left[ \sum_{k=0}^{n-1} \phi_k z^k \right]
\]

• Step 2: Choose points \( o_m \) on \( \Omega \)
  
  – In the previous case, we chose evenly spaced points on the circle \( z_m \)
  
  – In general, we assume that a set of \( m \) points \( o_m \) is constructible that lie on \( \partial \Omega \)
• Step 1: Choose a basis
  – We will use monomials again
    \[ u(x, y) \approx \mathbb{R} \left[ \sum_{k=0}^{n-1} \phi_k z^k \right] \]

• Step 2: Choose points \( o_m \) on \( \Omega \)
  – In the previous case, we chose evenly spaced points on the circle \( z_m \)
  – In general, we assume that a set of \( m \) points \( o_m \) is constructible that lie on \( \partial\Omega \)

• Step 3: Choose unknowns to match the boundary conditions
  – Before we obtained a simple form by phrasing in coefficient space
  – We cannot do this in general: we don't have a nice representation like Laurent series
Example 1: mapped circle

\[ M(z) = \frac{1}{6} e^{\frac{z}{2} + 2z} - 1 \]

- Step 1: Choose a basis
  - We will use monomials again
- Step 2: Choose points on \( \Omega \)
  - We choose the points \( o_m = M(z_m) \)
- Step 3: Choose unknowns to match the boundary conditions
  - ...
• We will express the right-hand side in value-space by evaluating

$$ u(z_m) = \Re \left[ \sum_{k=0}^{n-1} \phi_k o_m^k \right] $$

• In other words, we want to choose the unknowns $u = p + iq$ so that

$$ (1 \mid \Re o_m \mid \cdots \mid \Re o_m^{n-1}) p - (0 \mid \Im o_m \mid \cdots \mid \Im o_m^{n-1}) q \approx f(o_m) $$

• We have two options (we'll choose option 2):
  – Try to choose $m \approx n$ and hope the resulting system is nonsingular
  – Solve as a least squares problem
Example: error at \((0.1, 0)\) for \(u(x, y) = \Re e^{x+iy}\)

\[
m = 2n
\]
Example: error at $(0.1, 0)$ for $u(x, y) = \Re e^{x+iy}$

$$m = n^2$$

Number of unknowns $n$
Example: error at $(0.1, 0)$ for $u(x, y) = \Re e^{x+iy}$

$m = n^3$
Example: error at $z$ for $u(x, y) = \Re e^{x+iy} \sqrt{x+iy+1.1}$ with $m = n^2$

$z = .1$
Example: error at $z$ for $u(x, y) = \Re e^{x+iy} \sqrt{x+iy+1.1}$ with $m = n^2$

$z = 1.3i$

Number of unknowns $n$
Example: error at $z$ for $u(x, y) = \Re e^{x+iy} \sqrt{x + iy + 1.1}$ with $m = n^2$

The approximation is converging, even though the Taylor series does not!

$z = 1.3i$
Example 2: square

- Step 1: Choose a basis
  - We will use monomials again

- Step 2: Choose points on $\Omega$
  - We choose mapped Chebyshev points

- Step 3: Choose unknowns to match the boundary conditions
  - Use least squares again
Example: error at $(0.1, 0)$ for $u(x, y) = \Re e^{x+iy}$

$m = 2n$
Example: error at $z$ for $u(x, y) = \Re e^{x+iy} \sqrt{x + iy + 1.1}$ with $m = 2n$

$z = .1$
Example: error at \( z \) for \( u(x, y) = \Re e^{x+iy} \sqrt{x + iy + 1.1} \) with \( m = 2n \)

\[ z = .1 \]