Lesson 20
Riemann–Hilbert problems
• In the last lecture, we saw that we could solve Laplace’s equation by phrasing it as a complex boundary value problem – This worked exceptionally well on the unit circle, or domains that can be conformally mapped to the unit circle
• We now look at Riemann–Hilbert problems
• We first discuss the direct scattering transform to motivate the use of Riemann–Hilbert problems
• This will lead to a linear technique for solving the nonlinear KdV equation

\[ u_t + 6uu_x + u_{xxx} = 0 \]
\[ u(0, x) = u_0(x) \]
KdV (Pure solitons)
A more generic solution to KdV
Reflection Coefficients
• The **direct scattering transform** is a map from an initial condition to KdV $q_0(x)$ to the scattering data

$$q_0(x) \rightarrow (\rho(k), \{C_1, \ldots, C_N\}, \{\kappa_1, \ldots, \kappa_N\})$$

where $C_1, \ldots, C_N$ and $\kappa_1, \ldots, \kappa_N$ are constants and $\rho(k)$ is a function defined on the real line
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- This is similar to the Fourier transform

$$q_0(x) \rightarrow \hat{q}_0(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} q_0(x)e^{-ikx} \, dx$$
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• We have shown how to calculate $\{\kappa_1, \ldots, \kappa_N\}$ numerical via discrete spectrum of the Schrödinger equation

• Now, we will see how to calculate $\rho(k)$, which is associated with the continuous spectrum of the Schrödinger equation
• We saw that

\[ u'' + q_0(x)u \]

had continuous spectrum along \((-\infty, 0]\), i.e., in a vague sense, for all \( k \in \mathbb{R} \) we have "eigenfunctions" for

\[ u'' + q_0(x)u = -k^2 u \]
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• Assuming \( q_0(x) \) decays rapidly at \( \pm \infty \), we know that these eigenfunction must satisfy

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  – In other words,

\[ u(x) \sim e^{\pm ikx} \text{ as } x \to \pm \infty \]
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\[ u(x) \sim e^{\pm ikx} \text{ as } x \to \pm \infty \]

• We define the solutions \(\phi_{\pm}(x)\) by

\[ \phi_{\pm}(x) \sim e^{\pm ikx} \text{ as } x \to -\infty \]
• We saw that

\[ u'' + q_0(x)u \]

had continuous spectrum along (−∞, 0], i.e., in a vague sense, for all \( k \in \mathbb{R} \) we have "eigenfunctions" for

\[ u'' + q_0(x)u = -k^2 u \]

• Assuming \( q_0(x) \) decays rapidly at ±∞, we know that these eigenfunction must satisfy

\[ u'' = -k^2 u \]

as \( x \to \pm \infty \)

— In other words,

\[ u(x) \sim e^{\pm ikx} \text{ as } x \to \pm \infty \]

• We define the solutions \( \phi_\pm(x) \) by

\[ \phi_\pm(x) \sim e^{\pm ikx} \text{ as } x \to -\infty \]

and \( \psi_\pm(x) \) by

\[ \psi_\pm(x) \sim e^{\pm ikx} \text{ as } x \to +\infty \]
Now \( \phi_-, \psi_+ \) and \( \psi_- \) all solve

\[
\ddot{u} + q_0(x)u = -k^2 u
\]
• Now $\phi_-, \psi_+ \text{ and } \psi_- \text{ all solve}

\[ u'' + q_0(x)u = -k^2 u \]

• This is a second order equation, and we know that $\psi_\pm$ are linearly independent
• Now $\phi_-, \psi_+$ and $\psi_-$ all solve

$$u'' + q_0(x)u = -k^2 u$$

• This is a second order equation, and we know that $\psi_\pm$ are linearly independent

• Thus there exists $a$ and $b$ such that

$$\phi_-(x) = a\psi_-(x) + b\psi_+(x)$$
Now $\phi_-, \psi_+$ and $\psi_-$ all solve

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This is a second order equation, and we know that $\psi_\pm$ are linearly independent.

Thus there exists $a$ and $b$ such that

$$\phi_-(x) = a\psi_-(x) + b\psi_+(x)$$

We define the reflection coefficient by

$$\rho(k) = \frac{b}{a}$$
Now \( \phi_- \), \( \psi_+ \) and \( \psi_- \) all solve

\[
u'' + q_0(x)u = -k^2 u
\]

This is a second order equation, and we know that \( \psi_\pm \) are linearly independent.

Thus there exists \( a \) and \( b \) such that

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We define the reflection coefficient by

\[
\rho(k) = \frac{b}{a}
\]

Similarly, we have

\[
\phi_+(x) = \tilde{b}\psi_-(x) + \tilde{a}\psi_+(x)
\]

giving

\[
\tilde{\rho}(k) = \frac{\tilde{b}}{\tilde{a}}
\]
• Now \( \phi_-, \psi_+ \) and \( \psi_- \) all solve

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u'' + q_0(x)u = -k^2 u
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• This is a second order equation, and we know that \( \psi_\pm \) are linearly independent

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\]

giving

\[
\tilde{\rho}(k) = \frac{\tilde{b}}{\tilde{a}}
\]

- (Due to symmetries we have \( \tilde{\rho}(k) = \rho(-k) \))
Numerical direct scattering
• We want to calculate $\rho(k)$
• We want to calculate $\rho(k)$

• We do so by calculating:

$$ - \phi_-(x) \text{ on } (-\infty, 0]$$
• We want to calculate $\rho(k)$

• We do so by calculating:

  $- \phi_-(x)$ on $(-\infty, 0]$

  $- \psi_\pm(x)$ on $[0, \infty)$
• We want to calculate $\rho(k)$

• We do so by calculating:

  – $\phi_-(x)$ on $(-\infty, 0]$
  – $\psi_\pm(x)$ on $[0, \infty)$
  – Choosing $a$ and $b$ so that

\[
\begin{pmatrix}
\psi_-(0) & \psi_+(0) \\
\psi'_-(0) & \psi'_+(0)
\end{pmatrix}
\begin{pmatrix}
a \\
b
\end{pmatrix}
= 
\begin{pmatrix}
\phi_-(0) \\
\phi'_-(0)
\end{pmatrix}
We first calculate $\phi_-$ on $(-\infty, 0]$, which satisfies

$$\phi''_- + (q_0 + k^2) \phi_- = 0 \quad \text{and} \quad \phi_-(x) \sim e^{-ikx}$$
• We first calculate \( \phi_- \) on \((-\infty, 0]\), which satisfies

\[
\phi'''_- + (q_0 + k^2)\phi_- = 0 \quad \text{and} \quad \phi_-(x) \sim e^{-ikx}
\]

• We remove the oscillations and let \( \phi_- e^{ikx} = m_+(x) \), giving us \( m_+(-\infty) = 1 \) and \( m'_+(-\infty) = 0 \)
• We first calculate $\phi_-$ on $(-\infty, 0]$, which satisfies

$$\phi''_(- (q_0 + k^2)\phi_- = 0 \quad \text{and} \quad \phi_-(x) \sim e^{-ikx}$$

• We remove the oscillations and let $\phi_- e^{ikx} = m_+(x)$, giving us $m_+(\infty) = 1$ and $m'_+(\infty) = 0$

• Then we obtain

$$m'_+ = (\phi'_- + ik\phi_-) e^{ikx}$$
We first calculate $\phi_-$ on $(-\infty, 0]$, which satisfies

$$\phi'''' + (q_0 + k^2)\phi_0 = 0$$

and

$$\phi_-(x) \sim e^{-ikx}$$

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Then we obtain

$$m'_+ = (\phi'_- + ik\phi_-)e^{ikx}$$

$$m''_+ = (\phi''_- + 2ik\phi'_- - k^2\phi_-)e^{ikx}$$
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$$= \phi''_- e^{ikx} + 2ikm'_+ + k^2m_+$$
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• In other words,

$$0 = \Phi''_- e^{ikx} + (q_0 + k^2)\Phi_+ e^{ikx}$$
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\]

• This equation is amenable to numerics via mapping or truncation
Initial condition  

Reflection coefficient
Initial condition

Reflection coefficient
Initial condition

Reflection coefficient
Initial condition  

Reflection coefficient

---

Graphs showing the change from initial condition to reflection coefficient.
Inverse Scattering
• Now we fix $x$ and look at $\phi_{\pm}(k; x)$ and $\psi_{\pm}(k; x)$ as functions of $k$. 
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• We first remove the oscillations:

$$m_-(k) = \phi_+(k; x)e^{-ikx}, \quad m_+(k) = \phi_-(k; x)e^{ikx}$$

$$n_+(k) = \psi_+(k; x)e^{-ikx}, \quad n_-(k) = \psi_-(k; x)e^{ikx}$$
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• When there are no solitons (i.e., no discrete spectra), we assert that

$$\Phi^+(k) = \left[ \frac{m_+(k)}{a(k)}, n_+(k) \right]$$

is analytic in the upper half plane
• Now we fix $x$ and look at $\phi_{\pm}(k; x)$ and $\psi_{\pm}(k; x)$ as functions of $k$

• We first remove the oscillations:

$$m_{-}(k) = \phi_{+}(k; x)e^{-ikx}, \quad m_{+}(k) = \phi_{-}(k; x)e^{ikx}$$

$$n_{+}(k) = \psi_{+}(k; x)e^{-ikx}, \quad n_{-}(k) = \psi_{-}(k; x)e^{ikx}$$

• When there are no solitons (i.e., no discrete spectra), we assert that

$$\Phi^{+}(k) = \left[ \frac{m_{+}(k)}{a(k)}, n_{+}(k) \right]$$

is analytic in the upper half plane

• Similarly,

$$\Phi^{-}(k) = \left[ n_{-}(k), \frac{m_{-}(k)}{\tilde{a}(k)} \right]$$

is analytic in the lower half plane
• We want to derive a relationship between $\Phi_+$ and $\Phi_-$ for real $k$. 
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We obtain by direct calculation:

$$\Phi_+(k) = \left[ \frac{m_+}{a}, n_+ \right] = \left[ \frac{\phi_- e^{ikx}}{a}, \psi_+ e^{-ikx} \right]$$
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\[
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= \left[ \frac{(a\psi_- + b\psi_+) e^{ikx}}{a}, \psi_+ e^{-ikx} \right] \\
= [n_-, 0] + [\rho e^{ikx}, e^{-ikx}] \psi_+ \\
= [n_-, 0] + [\rho e^{ikx}, e^{-ikx}] \left( \frac{\phi_+}{\tilde{a}} - \tilde{\rho}_- \psi_- \right)
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$$

$$
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$$

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$$= \left[ \frac{(a\psi_- + b\psi_+) e^{ikx}}{a}, \psi_+ e^{-ikx} \right]$$

$$= [n_-, 0] + \left[ \rho e^{ikx}, e^{-ikx} \right] \psi_+$$

$$= [n_-, 0] + \left[ \rho e^{ikx}, e^{-ikx} \right] \left( \frac{\phi_+}{\tilde{a}} - \tilde{\rho} \psi_- \right)$$

$$= [n_-, 0] + \left[ \rho e^{ikx}, e^{-ikx} \right] \left( \frac{m_- e^{ikx}}{\tilde{a}} - \tilde{\rho} n_- e^{-ikx} \right)$$

$$= \left( n_-, \frac{m_-}{\tilde{a}} \right) \left( \begin{pmatrix} 1 & \rho e^{2ikx} & -\rho \tilde{\rho} \\ \rho e^{2ikx} & 1 & -\tilde{\rho} e^{-2ikx} \end{pmatrix} \right)$$
• We want to derive a relationship between $\Phi_+$ and $\Phi_-$ for real $k$

• We obtain by direct calculation:

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\Phi_+(k) = \left[ \frac{m_+}{a}, n_+ \right] = \left[ \frac{\phi_- e^{ikx}}{a}, \psi_+ e^{-ikx} \right] = \left[ \frac{(a\psi_- + b\psi_+) e^{ikx}}{a}, \psi_+ e^{-ikx} \right] = [n_-, 0] + [\rho e^{ikx}, e^{-ikx}] \psi_+ = [n_-, 0] + [\rho e^{ikx}, e^{-ikx}] \left( \frac{\phi_+}{\tilde{a}} - \tilde{\rho} \psi_- \right) = [n_-, 0] + [\rho e^{ikx}, e^{-ikx}] \left( \frac{m_- e^{ikx}}{\tilde{a}} - \tilde{\rho} n_- e^{-ikx} \right) = \left( n_-, \frac{m_-}{\tilde{a}} \right) \left( \begin{pmatrix} 1 & \rho e^{2ikx} & -\rho \tilde{\rho} \\ \rho e^{-2ikx} & 1 & -\tilde{\rho} e^{-2ikx} \end{pmatrix} \right) = \Phi_-(k) \left( \begin{pmatrix} 1 - \rho \tilde{\rho} & -\tilde{\rho} e^{2ikx} \\ \rho e^{-2ikx} & 1 \end{pmatrix} \right)
$$
• In short, we have a Riemann–Hilbert problem:

• Given $\rho(k)$ and $x$, find $\Phi_{\pm}(k)$ such that
• In short, we have a Riemann–Hilbert problem:

• Given $\rho(k)$ and $x$, find $\Phi_{\pm}(k)$ such that

  – $\Phi_{+}(k)$ is analytic in the upper half plane and $\Phi_{-}(k)$ is analytic in the lower half plane
• In short, we have a Riemann–Hilbert problem:

• Given $\rho(k)$ and $x$, find $\Phi_{\pm}(k)$ such that
  
  – $\Phi_{+}(k)$ is analytic in the upper half plane and $\Phi_{-}(k)$ is analytic in the lower half plane
  
  – For all real $k$, they satisfy the jump condition

  \[
  \Phi_{+}(k) = \Phi_{-}(k) \begin{pmatrix}
  1 - \rho(k)\rho(-k) & -\rho(-k)e^{-2ikx} \\
  \rho(k)e^{2ikx} & 1
  \end{pmatrix}
  \]
• In short, we have a Riemann–Hilbert problem:

• Given $\rho(k)$ and $x$, find $\Phi_{\pm}(k)$ such that

  – $\Phi_{+}(k)$ is analytic in the upper half plane and $\Phi_{-}(k)$ is analytic in the lower half plane
  – For all real $k$, they satisfy the jump condition

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\Phi_{+}(k) = \Phi_{-}(k) \begin{pmatrix}
1 - \rho(k)\rho(-k) & -\rho(-k)e^{-2ikx} \\
\rho(k)e^{2ikx} & 1
\end{pmatrix}
$$

  – They satisfy the asymptotic condition

$$
\Phi_{\pm}(k) \sim (1, 1) \quad \text{as} \quad k \to \infty
$$
• How do we recover $q_0$ from $\Phi_\pm$?
• How do we recover $q_0$ from $\Phi_{\pm}$?

• The key: we use the (numerically useful) expression

\[
0 = \partial_x^2 m_+ - 2ik \partial_x m_+ + q_0(x)m_+
\]
• How do we recover $q_0$ from $\Phi_{\pm}$?

• The key: we use the (numerically useful) expression

$$0 = \partial_x^2 m_+ - 2ik \partial_x m_+ + q_0(x)m_+$$

• By matched asymptotics we derive

$$m_+(k; x) = 1 + \frac{q'_0(x)}{ik} + \mathcal{O}\left(\frac{1}{k^2}\right)$$
• How do we recover $q_0$ from $\Phi_\pm$?

• The key: we use the (numerically useful) expression

\[ 0 = \partial_x^2 m_+ - 2ik \partial_x m_+ + q_0(x)m_+ \]

• By matched asymptotics we derive

\[ m_+(k; x) = 1 + \frac{q_0'(x)}{ik} + \mathcal{O}\left(\frac{1}{k^2}\right) \]

- Thus as $k \to \infty$, $m_+ \to 1$ and $m_+'' \to 0$
• How do we recover \( q_0 \) from \( \Phi \)?

• The key: we use the (numerically useful) expression

\[
0 = \partial_x^2 m_+ - 2ik \partial_x m_+ + q_0(x) m_+
\]

• By matched asymptotics we derive

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m_+(k; x) = 1 + \frac{q_0'(x)}{ik} + \mathcal{O}\left(\frac{1}{k^2}\right)
\]

  – Thus as \( k \to \infty \), \( m_+ \to 1 \) and \( m_+'' \to 0 \)

• Thus we get

\[
q_0(x) = 2i \lim_{k \to \infty} km_+'(k; x) = 2i \frac{\partial}{\partial x} \lim_{k \to \infty} k\Phi_+(k)
\]
Time evolution
• Why bother moving from initial conditions to reflection coefficients?
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• Because time-evolution is trivial!
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• Because time-evolution is trivial!

• Suppose \( q \) satisfies

\[
qt + 6qq_x + q_{xxx} = 0 \quad \text{and} \quad q(0, x) = q_0(x)
\]
• Why bother moving from initial conditions to reflection coefficients?
• Because time-evolution is trivial!
• Suppose \( q \) satisfies

\[
q_t + 6qq_x + q_{xxx} = 0 \quad \text{and} \quad q(0, x) = q_0(x)
\]

• If \( q(0, x) = q_0(x) \) has reflection coefficient \( \rho(k) \)
• Then \( q(t, x) \) has reflection coefficient \( \rho(k) e^{8ik^3 t} \)
Inverse scattering for KdV

$q_0(x)$
Inverse scattering for KdV

$q_0(x)$ \hspace{3cm} \text{Direct transform} \hspace{3cm} \rho(k)$
Inverse scattering for KdV

\[ q_0(x) \quad \text{Direct transform} \quad \rho(k) \]

\[ \rho(k) e^{8i k^3 t} \]

Time evolution
Inverse scattering for KdV

$q_0(x)$ \quad \rightarrow \quad \rho(k)$

$q(t, x)$ \quad \leftarrow \quad \rho(k)e^{8ik^3t}$

Direct transform

Inverse transform

Time evolution
Inverse scattering for KdV

$q_0(x)$ \rightarrow \rho(k)$

$q(t, x) \leftrightarrow \rho(k)e^{8ik^3t}$

Direct transform

Inverse transform

Time evolution
In short, we have a Riemann–Hilbert problem that we want to solve numerically:

- **Given** \( \rho(k) \) and \( x \) and \( t \), find \( \Phi_\pm(k) \) such that
  - \( \Phi_+(k) \) is analytic in the upper half plane and \( \Phi_-(k) \) is analytic in the lower half plane
  - For all real \( k \), the satisfy the jump condition
    \[
    \Phi_+(k) = \Phi_-(k) \begin{pmatrix}
    1 - \rho(k) \rho(-k) & -\rho(-k) e^{-2ikx - 8ik^3t} \\
    \rho(k) e^{2ikx + 8ik^3t} & 1
    \end{pmatrix}
    \]
  - The satisfy the asymptotic condition
    \[
    \Phi_\pm(k) \sim (1, 1) \text{ as } k \to \infty
    \]
- Calculate
  \[
  q(t, x) = 2i \lim_{k \to \infty} k \Phi_+(k)
  \]
Scalar Riemann–Hilbert problems
• We have reduced calculating KdV to a Riemann–Hilbert problem on the real axis
• Just like Laplace's equation, we will map the problem to the unit circle
• Thus, we investigate numerical solution of the following scalar Riemann–Hilbert problem on the unit circle:
• We have reduced calculating KdV to a Riemann–Hilbert problem on the real axis
• Just like Laplace's equation, we will map the problem to the unit circle
• Thus, we investigate numerical solution of the following scalar Riemann–Hilbert problem on the unit circle:
  – Given a "nice" function $G(w)$ on the unit circle
  – Find a function $\Phi_+$ analytic inside the disk and $\Phi_-$ analytic outside a disk
  – That satisfy the jump condition on the unit circle

\[
\Phi_+(w) = \Phi_-(w) G(w)
\]

  – and the asymptotic condition

\[
\Phi(\infty) = 1
\]
Input: a “nice” function $G$ defined on the unit circle
Input: a “nice” function $G$ defined on the unit circle

Output: two analytic functions, inside and outside the disk

$G(w)$

$\Phi^+(z)$

$\Phi^-(z)$
Input: a “nice” function $G$ defined on the unit circle

Output: two analytic functions, inside and outside the disk

Equation: the relationship we want on the unit circle is

$$\Phi^+(w) = \Phi^-(w) G(w), \quad \Phi^-(\infty) = 1$$
Input: a “nice” function $G$ defined on the unit circle

Output: two analytic functions, inside and outside the disk

Equation: the relationship we want on the unit circle is

$\Phi^+(w) = \Phi^-(w) G(w)$, $\Phi^-(\infty) = 1$

(For practical applications, $G$ and $\Phi$ are matrix-valued)
Idea: express the unknown in coefficient space

\[ \Phi^+(z) \]

\[ \Phi^-(z) \]
Idea: express the unknown in coefficient space

\[ \Phi^+(z) \quad \leftrightarrow \quad \Phi^-(z) \]

\[ 1 + \sum_{k=0}^{\infty} u_k z^k \]

\[ 1 - \sum_{k=-\infty}^{-1} u_k z^k \]
(In other words: $\Phi$ is $1 +$ the **Cauchy transform** of an unknown function $u$)
(In other words: $\Phi$ is $1 +$ the Cauchy transform of an unknown function $u$)

\[
\Phi(z) = 1 + \frac{1}{2\pi i} \oint \frac{u(w)}{w - z} \, dw \quad \text{for}
\]

\[
u(w) = \sum_{k=-\infty}^{\infty} u_k w^k
\]
• We now reformulate

\[ \Phi^+(w) = \Phi^-(w)G(w) \]

in coefficient space:
• We now reformulate

\[ \Phi^+(w) = \Phi^-(w) G(w) \]

in coefficient space:

\[ 1 + \sum_{k=0}^{\infty} u_k w^k = \left( 1 - \sum_{k=-\infty}^{-1} u_k w^k \right) G(w) \]
We now reformulate

\[ \Phi^+(w) = \Phi^-(w)G(w) \]

in coefficient space:

\[
1 + \sum_{k=0}^{\infty} u_k w^k = \left( 1 - \sum_{k=-\infty}^{-1} u_k w^k \right) G(w) \Rightarrow
\]

\[
\sum_{k=0}^{\infty} u_k w^k + G(w) \sum_{k=-\infty}^{-1} u_k w^k = G(w) - 1
\]
We define the operator that selects the negative coefficients:

\[
I_\neg = \begin{pmatrix} I \\ 0 \end{pmatrix} = \begin{pmatrix} \cdots & 1 & 1 \\ 0 & 0 & \ddots \end{pmatrix}
\]
So that

- We define the operator that selects the negative coefficients:

\[
I_- = \begin{pmatrix} I & 0 \\ 0 & \end{pmatrix} = \begin{pmatrix} \vdots & 1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots & \end{pmatrix}
\]

\[
I_- \begin{pmatrix} \vdots \\ u_{-2} \\ u_{-1} \\ u_0 \\ u_1 \\ u_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ u_{-2} \\ u_{-1} \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}
\]
• We define the operator that selects the negative coefficients:

\[ I_- = \begin{pmatrix} I & 0 \\ \end{pmatrix} = \begin{pmatrix} \vdots & 1 & 1 & 0 & 0 & \cdots \\ \end{pmatrix} \]

\[ I_- = \begin{pmatrix} u_{-2} \\ u_{-1} \\ u_0 \\ u_1 \\ u_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ u_{-2} \\ u_{-1} \\ u_0 \\ u_1 \\ u_2 \\ \vdots \end{pmatrix} \]

• Similarly,

\[ I_+ = \begin{pmatrix} 0 & I \\ \end{pmatrix} = \begin{pmatrix} \vdots & 0 & 0 & 1 & 1 & \cdots \\ \end{pmatrix} \]

\[ I_+ = \begin{pmatrix} \vdots \\ u_{-2} \\ u_{-1} \\ u_0 \\ u_1 \\ u_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ 0 \\ 0 \\ u_0 \\ u_1 \\ u_2 \\ \vdots \end{pmatrix} \]
• Recall that multiplication by $G$ in coefficient space is the Laurent operator

$$
\mathcal{L}[G] = \begin{pmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \hat{G}_0 & \hat{G}_{-1} & \hat{G}_{-2} & \cdots \\
\cdots & \hat{G}_1 & \hat{G}_0 & \hat{G}_{-1} & \cdots \\
\cdots & \hat{G}_2 & \hat{G}_1 & \hat{G}_0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}
$$
• Thus our equation

\[
\sum_{k=0}^{\infty} u_k w^k + G(w) \sum_{k=-\infty}^{-1} u_k w^k = G(w) - 1
\]
• Thus our equation

$$\sum_{k=0}^{\infty} u_k w^k + G(w) \sum_{k=-\infty}^{-1} u_k w^k = G(w) - 1$$

Becomes

$$(I_+ + \mathcal{L}[G]I_-) \mathbf{u} = \mathbf{F}$$
• Thus our equation

\[
\sum_{k=0}^{\infty} u_k w^k + G(w) \sum_{k=-\infty}^{-1} u_k w^k = G(w) - 1
\]

Becomes

\[
(I_+ + \mathcal{L}[G]I_-) \mathbf{u} = \mathbf{F}
\]

where

\[
\mathbf{u} = \begin{pmatrix}
\vdots \\
\hat{G}_2 \\
\hat{G}_1 \\
\hat{G}_0 - 1 \\
u_0 \\
u_1 \\
u_2 \\
\vdots
\end{pmatrix}
\quad \text{and} \quad
\mathbf{F} = \begin{pmatrix}
\vdots \\
\hat{G}_2 \\
\hat{G}_1 \\
\hat{G}_0 - 1 \\
\vdots
\end{pmatrix}
\]
• Or in matrix form, the operator

\[ I_+ + \mathcal{L}[G]I_- \]

becomes:
• Or in matrix form, the operator

\[ I_+ + \mathcal{L}[G]I_- \]

becomes:

\[
\begin{pmatrix}
\ddots \\
0 & 0 \\
0 & 1 \\
1 & 1 \\
\ddots \\
\end{pmatrix} +
\]
• Or in matrix form, the operator

\[ I_+ + \mathcal{L}[G]I_- \]

becomes:

\[
\begin{pmatrix}
\vdots \\
0 \\
0 \\
1 \\
1 \\
\vdots
\end{pmatrix}
\begin{pmatrix}
\vdots \\
\hat{G}_0 \\
\hat{G}_{-1} \\
\hat{G}_{-2} \\
\vdots
\end{pmatrix}
\begin{pmatrix}
\vdots \\
\hat{G}_1 \\
\hat{G}_0 \\
\hat{G}_{-1} \\
\vdots
\end{pmatrix}
\begin{pmatrix}
\vdots \\
\hat{G}_2 \\
\hat{G}_1 \\
\hat{G}_0 \\
\vdots
\end{pmatrix}
\]
• Or in matrix form, the operator

\[ I_+ + \mathcal{L}[G]I_- \]

becomes:

\[
\begin{pmatrix}
\cdots & 0 & 0 \\
0 & 1 & 1 \\
1 & 1 & 1 \\
\end{pmatrix}
\begin{pmatrix}
\hat{G}_0 & \hat{G}_1 & \hat{G}_2 \\
\hat{G}_1 & \hat{G}_0 & \hat{G}_1 \\
\hat{G}_2 & \hat{G}_1 & \hat{G}_0 \\
\end{pmatrix}
\begin{pmatrix}
\cdots & 1 & 1 \\
1 & 1 & 0 \\
0 & 0 & 0 \\
\end{pmatrix}
\]
• Or in matrix form, the operator

\[ I_+ + \mathcal{L}[G]I_- \]

becomes:

\[
\begin{pmatrix}
\vdots & 0 & 0 & \hat{G}_0 & \hat{G}_{-1} & \hat{G}_{-2} \\
0 & \hat{G}_1 & \hat{G}_0 & \hat{G}_{-1} \\
1 & \hat{G}_2 & \hat{G}_1 & \hat{G}_0 & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{pmatrix}
\begin{pmatrix}
\vdots \\
1 \\
1 \\
0 \\
0 \\
\vdots
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\hat{G}_1 & \hat{G}_0 & \hat{G}_{-1} & \ddots & \ddots & \ddots \\
\hat{G}_2 & \hat{G}_1 & \hat{G}_0 & \ddots & \ddots & \ddots \\
\hat{G}_3 & \hat{G}_2 & \hat{G}_1 & 1 & \ddots & \ddots \\
\hat{G}_4 & \hat{G}_3 & \hat{G}_2 & 1 & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{pmatrix}
\]
• In practice, we approximate $G$ by a finite Laurent series:

$$G(w) \approx \sum_{k=\alpha}^{\beta} \hat{G}_k w^k$$
In practice, we approximate $G$ by a finite Laurent series:

$$G(w) \approx \sum_{k=\alpha}^{\beta} \hat{G}_k w^k$$

Thus

$$I_+ + \mathcal{L}[G]I_-$$

becomes a banded operator:

$$
\begin{pmatrix}
\vdots & \vdots & \vdots & \vdots \\
\vdots & \hat{G}_1 & \hat{G}_0 & \hat{G}_{-1} \\
\hat{G}_2 & \hat{G}_1 & \hat{G}_0 & \vdots \\
\hat{G}_2 & \hat{G}_1 & 1 & \hat{G}_2 \\
\hat{G}_2 & \hat{G}_1 & 1 & \ddots \\
\end{pmatrix}
$$
• The operator is bounded (by the assignment) and stable:

\[ \| [P_n (I_+ + L[G] I_-) P_n^T]^{-1} \| \]

is bounded as \( n \to \infty \)

- **Excercise:** Show that the operator is bounded, and use stability to prove convergence when the operator is invertible

- Proving stability is beyond the scope of the course, but follows from the Toeplitz structure of the operator