Complex Numerical Analysis: Assignment 2, due Thursday 31 October 2013

In the previous assignment we constructed a numerically stable scheme to represent a function f analytic in the annulus $A_{r,R} = \{z : r \leq |z| \leq R\}$. This took the form of:

1) Compute the coefficients of

$$f(Rz) = \sum_{k=-\infty}^{\infty} \hat{f}_k^+ z^k$$

using the DFT

2) Compute the coefficients of

$$f(rz) = \sum_{k=-\infty}^{\infty} \hat{f}_k^- z^k$$

using the DFT

3) Approximate

$$f(z) = \sum_{k=-\infty}^{-1} \hat{f}_k^- (z/r)^k + \sum_{k=0}^{\infty} \hat{f}_k^+ (z/R)^k$$

Problem 1.1 [7 points] Show that the eigenvalues of

$$\begin{pmatrix} 0 & r & & & & & & & & \\ & & r & & & & & & & \\ & & & \ddots & & & & & & \\ & & & & r & & & & & \\ & & & & & r & & & & \\ & & & & & & R & & & \\ & & & & & & & \ddots & & \\ & & & & & & & & R & \\ & & & & & & & & & R^2 \\ -\frac{\hat{f}_\alpha^-}{\hat{f}_\beta^+} & -\frac{\hat{f}_{\alpha+1}^-}{\hat{f}_\beta^+} & -\frac{\hat{f}_{\alpha+2}^-}{\hat{f}_\beta^+} & \cdots & -\frac{\hat{f}_{-1}^-}{\hat{f}_\beta^+} & -\frac{\hat{f}_0^+}{\hat{f}_\beta^+} & -\frac{\hat{f}_1^+}{\hat{f}_\beta^+} & \cdots & -\frac{\hat{f}_{\beta-2}^+}{\hat{f}_\beta^+} & -\frac{\hat{f}_{\beta-1}^+}{\hat{f}_\beta^+} R \end{pmatrix}$$

are the roots of

$$p(z) = \sum_{k=\alpha}^{-1} \hat{f}_k^- (z/r)^k + \sum_{k=0}^{\beta} \hat{f}_k^+ (z/R)^k$$

Problem 1.2 [7 points] Construct the differentiation operator as an operator on the coefficients

$$\begin{pmatrix} \vdots \\ \hat{f}_{-2}^- \\ \hat{f}_{-1}^- \\ \hat{f}_0^+ \\ \hat{f}_1^+ \\ \vdots \end{pmatrix}$$

Problem 1.3 [6 points] Represent the equation

$$u'' + u = f \qquad \text{and} \qquad u(1) = 1$$

as an infinite-dimensional matrix operating on the coefficients.

Recall that multiplication by an analytic function $a(z)$ in the unit circle in coefficient space is

$$T[a] = \begin{pmatrix} \hat{a}_0 & & & \\ \hat{a}_1 & \hat{a}_0 & & \\ \hat{a}_2 & \hat{a}_1 & \hat{a}_0 & \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

Problem 2.1 [7 points] If a is a d-degree polynomial, show that $T[a]$ is banded. Use this to prove that

$$\|T[a]\|_{\ell^1} < \infty.$$

Problem 2.2 [5 points extra credit] Show that

$$\|T[a]\|_{\ell^1} < \infty$$

when a is only smooth enough on the boundary so that $\mathcal{F}a \in \ell_2^\infty$.

Problem 2.3 [6 points] Using the previous problems or otherwise, show that

$$T[a]Q$$

is compact when a is a polynomial for

$$Q = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \frac{1}{2} & & \\ & & & \frac{1}{3} & \\ & & & & \ddots \end{pmatrix}.$$

Problem 2.4 [7 points] Consider the ODE

$$u' + au = f \qquad \text{and} \qquad u(0) = c,$$

where a is a polynomial and u is represented by its Taylor series. Assume that the operator

$$\begin{pmatrix} (1, 0, 0, \ldots) \\ \mathcal{D} + T[a] \end{pmatrix}$$

is invertible in ℓ^2 for

$$\mathcal{D} = \begin{pmatrix} 0 & 1 & & & \\ & & 2 & & \\ & & & 3 & \\ & & & & \ddots \end{pmatrix}.$$

Using the previous problems, show that

$$\begin{pmatrix} (1, 0, \cdots, 0) \\ \mathcal{P}_{n-1}(\mathcal{D} + T[a])\mathcal{P}_n^\top \end{pmatrix} \boldsymbol{u}_n = \begin{pmatrix} c \\ \hat{\boldsymbol{f}}_n \end{pmatrix}$$

is nonsingular for large n, and that $\boldsymbol{u}_n$ converges in norm to

$$\hat{\boldsymbol{u}} = \begin{pmatrix} \hat{u}_0 \\ \hat{u}_1 \\ \vdots \end{pmatrix},$$

i.e., the Taylor coefficients of u.

Here

$$\hat{\boldsymbol{f}}_n = \mathcal{P}_n \hat{\boldsymbol{f}} = \begin{pmatrix} \hat{f}_0 \\ \vdots \\ \hat{f}_{n-1} \end{pmatrix}$$

and assume that $\begin{pmatrix} \hat{f}_0 \\ \hat{f}_1 \\ \vdots \end{pmatrix} \in \ell^2$. Feel free to use the convention $\mathcal{P}_n^\top = \mathcal{P}_n$ when $\mathbb{C}^n$ is viewed as a subset of ℓ^2.
