A quantum Mirković-Vybornov isomorphism

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Abstract

We present a quantization of an isomorphism of Mirković and Vybornov which relates the intersection of a Slodowy slice and a nilpotent orbit closure in $\mathfrak{gl}_N$, to a slice between spherical Schubert varieties in the affine Grassmannian of PGL_n (with weights encoded by the Jordan types of the nilpotent orbits). A quantization of the former variety is provided by a parabolic W-algebra and of the latter by a truncated shifted Yangian. Building on earlier work of Brundan and Kleshchev, we define an explicit isomorphism between these non-commutative algebras, and show that its classical limit is a variation of the original isomorphism of Mirković and Vybornov. On the way, we prove new results about parabolic W-algebras.

1 Introduction

In [MV07a] Mirković and Vybornov construct an isomorphism between slices to (spherical) Schubert varieties in the affine Grassmannian of PGL_n on the one hand, and Slodowy slices in $\mathfrak{gl}_N$ intersected with nilpotent orbit closures on the other. This isomorphism has important applications in geometric representation theory. To name just a few occurrences, it appears in works on the mathematical definition of the Coulomb branch associated to quiver gauge theories [Nak16], the analog of the geometric Satake isomorphism for affine Kac-Moody groups [BF12], and geometric approaches to knot homologies [CK08, CKL10].

These varieties each have quantizations corresponding to natural Poisson structures on them. The main aim of this paper is to show that the Mirković-Vybornov isomorphism is the classical limit of an isomorphism of these quantizations.

To be more precise, the Slodowy slice $\mathcal{S}_e$ through a nilpotent element $e \in \mathfrak{gl}_N$ is quantized by a finite W-algebra. Finite W-algebras algebras have been extensively studied by Kostant, Lynch, Premet, Gan-Ginzburg, and many others (cf. [GG02] and references therein). The quantization of $\mathcal{S}_e \cap \overline{\mathbb{O}_{e'}}$, the intersection of $\mathcal{S}_e$ with the closure of the nilpotent orbit through another nilpotent e', is given by a parabolic W-algebra [Los12, Web11]. Parabolic W-algebras are quotients of finite W-algebras.

Slices to Schubert varieties in the affine Grassmannian of PGL_n are indexed by pairs μ, λ of dominant coweights of PGL_n, such that $\mu \leq \lambda$ in the dominant coroot ordering. We denote the slice by $\overline{\mathsf{Gr}^{\lambda}_{\mu}}$. In [KWWY14] the present authors, along with Kamnitzer, quantized $\overline{\mathsf{Gr}^{\lambda}_{\mu}}$ using algebras called truncated shifted Yangians.

The Mirković-Vybornov isomorphism is an explicit isomorphism of varieties

$$\mathcal{S}_e \cap \overline{\mathbb{O}_{e'}} \cong \overline{\mathsf{Gr}^{\lambda}_{\mu}}, \tag{1.1}$$

where e, e' and are related to μ, λ by a certain combinatorial correspondence (cf. Sections 1.2 and 4.1). Naturally one expects that (1.1) is the classical limit of an isomorphism between the quantizations of these varieties. That is our main result.

Theorem A (Theorem 4.3, part (c))**.** *Suppose e, e' (respectively μ, λ) is a pair of nilpotent elements (respectively dominant coweights) which are related by the Mirković-Vybornov isomorphism (1.1). Then there is an isomorphism of filtered algebras between the parabolic W-algebra quantizing $\mathcal{S}_e \cap \overline{\mathbb{O}_{e'}}$ and the truncated shifted Yangian quantizing $\overline{\mathsf{Gr}^{\lambda}_{\mu}}$.*

One can immediately conclude from this theorem that (1.1) is an isomorphism of *Poisson* varieties (Corollary 4.4). Moreover, since truncated shifted Yangians are explicitly presented, this theorem provides a presentation of parabolic W-algebras in type A. This generalizes Brundan and Kleshchev's foundational work on presentations of finite W-algebras [BK08]. In fact, Losev has speculated that Brundan and Kleshchev's presentation should be understood as a quantization of the Mirković-Vybornov isomorphism [Los12, Rmk. 5.3.4], and Theorem A makes this precise.

Remark 1.1. In [MV07a], the authors consider a second family of isomorphisms, based on work of Maffei [Maf05], between Slodowy slices and type A quiver varieties. This isomorphism has already been quantized by Losev [Los12, Th. 5.3.3], and our work does not seem to add anything new to the understanding of this perspective.

In order to prove Theorem A, we prove several other results which are interesting in their own right. Brundan and Kleshchev describe the highest weights in category $\mathcal{O}$ of a finite W-algebra in terms of row tableau. We describe those highest weights which descend to the parabolic W-algebra using so-called parabolic-singular elements of the Weyl group (Theorem 3.19). These are elements which are simultaneously longest left coset for a parabolic corresponding to μ and shortest right coset representatives for a parabolic corresponding to λ. This allows us to describe the parabolic W-algebra more explicitly:

Theorem B (Theorem 3.21)**.** *In type A, the parabolic W-algebra is the quotient of the finite W-algebra by the intersection of annihilators of simple modules corresponding to parabolic-singular permutations.*

To prove Theorem A we first prove the desired isomorphism in the case where λ is a multiple of the first fundamental coweight (Theorem 4.7). This is an explicit calculation with the Brundan-Kleshchev isomorphism, comparing different subquotients of the Yangian of $\mathfrak{sl}_n$ on the one hand, and the Yangian of $\mathfrak{gl}_n$ on the other. We then use results about the highest weight theory of the truncated shifted Yangian given by Kamnitzer, Tingley and the authors in [KTW$^+$], and the highest weight theory of the parabolic W-algebra from Section 3.3.3, to deduce the general result from the special case.

In Section 5.1 we introduce general "MV slices", and prove an easy but useful result that any two MV slices are Poisson isomorphic (Theorem 5.5). Recently Cautis and Kamnitzer described a variation on the classical Mirković-Vybornov isomorphism, which uses MV slices that are transposes of those used by Mirković and Vybornov (cf. Section 5.3). This isomorphism is much simpler to express in coordinates, and we prove that it is the classical limit of our quantum isomorphism.

Theorem C (Theorem 4.3, part (d))**.** *The classical limit of the quantum Mirković-Vybornov isomorphism in Theorem A agrees with Cautis and Kamnitzer's version of the classical Mirković-Vybornov isomorphism.*

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1.1 Notation

Unless stated otherwise, throughout this paper we will denote $\mathfrak{g} = \mathfrak{sl}_n$. We denote the nodes of its Dynkin diagram by $I = \{1, \dots, n-1\}$. We write $j \sim i$ to mean j and i are connected in the Dynkin diagram. Since Langlands duality often appears in the context of the affine Grassmannian, we will use dual notation, and denote simple coroots by $\{\alpha_i\}_{i \in I}$ and fundamental coweights by $\{\varpi_i\}_{i \in I}$, and dually, the simple roots $\{\alpha_i^\vee\}_{i \in I}$ and fundamental weights by $\{\varpi_i^\vee\}_{i \in I}$. We will denote the Weyl group by $W \cong S_n$, and the simple reflections by s_i for $i \in I$. All spaces considered are varieties, schemes, or ind-schemes over $\mathbb{C}$.

1.2 Combinatorial data

Consider a pair λ, μ of dominant coweights for $\mathfrak{g}$, such that $\lambda \geq \mu$. Write

$$\lambda = \sum_{i=1}^{n-1} \lambda_i \varpi_{n-i}, \qquad \mu = \sum_{i=1}^{n-1} \mu_i \varpi_{n-i}, \qquad \lambda - \mu = \sum_{i=1}^{n-1} m_i \alpha_{n-i} \tag{1.2}$$

so that $\lambda \geq \mu$ means precisely that all $m_i \in \mathbb{Z}_{\geq 0}$. (Our indexing conventions above are chosen to match those of [KWWY14].) Define

$$N = \sum_{i=1}^{n-1} i \lambda_{n-i} \tag{1.3}$$

Then $N\varpi_1 \geq \lambda \geq \mu$. Write $N\varpi_1 - \mu = \sum_i m'_i \alpha_{n-i}$.

We associate a pair of partitions to the above data as follows: first, the partition $\tau \vdash N$ is defined in exponential notation by

$$\tau = \left(1^{\lambda_{n-1}}, 2^{\lambda_{n-2}}, \ldots, (n-1)^{\lambda_1}\right)^t. \tag{1.4}$$

Second, consider the partition $\pi \vdash N$,

$$\pi = (p_1 \leq \cdots \leq p_n), \tag{1.5}$$

defined by

$$p_1 = m'_1, p_2 = m'_2 - m'_1, ..., p_{n-1} = m'_{n-1} - m'_{n-2}, p_n = N - m'_{n-1}. \tag{1.6}$$

Then $\tau \geq \pi$ with respect to the dominance order on partitions.

Remark 1.2. As a matter of convention, we will write partitions as either non-increasing or non-decreasing as appropriate.

2 The affine Grassmannian side

In this section we recall truncated shifted Yangians, and their connection to slices in the affine Grassmannian. We fix throughout a pair $\lambda \geq \mu$ of dominant coweights, as in Section 1.1.

2.1 Slices in the affine Grassmannian

Consider (spherical) Schubert cells $\mathsf{Gr}^\mu, \mathsf{Gr}^\lambda$ in the affine Grassmannian Gr for PGL_n. Our running hypothesis that $\lambda \geq \mu$ implies that $\mathsf{Gr}^\mu \subset \overline{\mathsf{Gr}^\lambda}$, and we let $\mathsf{Gr}^{\overline{\lambda}}_\mu$ be the slice to Gr^μ in $\overline{\mathsf{Gr}^\lambda}$ at the point $t^{w_0\mu}$. See [KWWY14, Section 2.2] for more details and precise definitions, as well as Section 5.2 below.

$\mathsf{Gr}^{\overline{\lambda}}_\mu$ is an irreducible affine variety of dimension $2\langle\rho^\vee, \lambda - \mu\rangle = 2\sum_i m_i$. It has a $\mathbb{C}^\times$–action by loop rotation, which contracts it to the unique fixed point $t^{w_0\mu}$. $\mathsf{Gr}^{\overline{\lambda}}_\mu$ admits a Poisson structure which is homogeneous of degree -1 with respect to the loop rotation, as described in [KWWY14, Section 2C].

Recall that Gr admits a description in terms of lattices: every point is given by a $\mathbb{C}[[t]]$–lattice in $\mathbb{C}((t))^n$; this is only well-defined up to multiplication by a power of t, but we will consistently choose representatives Λ such that $\Lambda \subset \Lambda_0 = \mathbb{C}[[t]]^n$. Denote $E_\pi = \left\{t^{p_i-1}e_i, \ldots, te_i, e_i : \forall i\right\}$ and $E_p = \left\{t^{p-1}e_i, \ldots, e_i : \forall i\right\}$, where $e_1, ..., e_n$ is the standard basis of $\mathbb{C}^n$. Explicitly, we can identify:

$$\mathsf{Gr}^{\overline{N\varpi_1}}_\mu = \left\{ \Lambda \ : \ \begin{array}{ll} (a) & \Lambda \subset \Lambda_0 \text{ a } \mathbb{C}[[t]]\text{–submodule,} \\ (b) & \text{image of } E_\pi \text{ gives basis of } \Lambda_0/\Lambda, \\ (c) & \forall i, \ t^{p_i}e_i \in \Lambda + E_{p_i} \end{array} \right\} \tag{2.1}$$

Since $N\varpi_1 \geq \lambda$, we have inclusions of closed subvarieties $\overline{\mathsf{Gr}^\lambda} \subset \overline{\mathsf{Gr}^{N\varpi_1}}$ and $\mathsf{Gr}^{\overline{\lambda}}_\mu \subset \mathsf{Gr}^{\overline{N\varpi_1}}_\mu$. Considering multiplication by t as an endomorphism of Λ_0/Λ, we can also identify

$$\mathsf{Gr}^{\overline{\lambda}}_\mu = \left\{\Lambda \in \mathsf{Gr}^{\overline{N\varpi_1}}_\mu \ : \ t \in \mathrm{End}_{\mathbb{C}}\left(\Lambda_0/\Lambda\right) \text{ has Jordan type } \leq \tau\right\} \tag{2.2}$$

2.2 Truncated shifted Yangians

For $\mathfrak{g}$ a simply-laced simple complex Lie algebra, let $Y = Y(\mathfrak{g})$ be the associated Yangian. This is a filtered $\mathbb{C}$-algebra with generators $E_\alpha^{(r)}, F_\alpha^{(r)}, H_i^{(r)}$ for $\alpha \in \Delta^+, i \in I$, $r \in \mathbb{Z}_{>0}$, and filtration defined by $\deg(X^{(r)}) = r$ for any generator X. In fact, Y is generated by the elements $E_i^{(r)} := E_{\alpha_i}^{(r)}, F_i^{(r)} := F_{\alpha_i}^{(r)}$ and $H_i^{(r)}$. For the defining relations see Theorem 3.5 in [KWWY14].

We will frequently work with the formal generating series

$$E_i(u) = \sum_{r>0} E_i^{(r)} u^{-r}, \quad F_i(u) = \sum_{r>0} F_i^{(r)} u^{-r}, \quad H_i(u) = 1 + \sum_{r>0} H_i^{(r)} u^{-r}$$

Definition 2.1 (Definition 3.10, [KWWY14])**.** The **shifted Yangian** $Y_\mu \subset Y$ is the subalgebra generated by $E_i^{(r)}, H_i^{(r)}$ where $r \geq 1$ and $F_i^{(s)}$ where $s > \mu_i$.

Introduce formal variables $R_i^{(j)}$ where $i \in I$ and $j = 1, ..., \lambda_i$, and consider the tensor product of algebras

$$Y_\mu[R_i^{(j)}] := Y_\mu \otimes_{\mathbb{C}} \mathbb{C}[R_i^{(j)} : i \in I, j = 1, ..., \lambda_i]. \tag{2.3}$$

Let $R_i(u) = \sum_{j=0}^{\lambda_i} R_i^{(j)} u^{\lambda_i - j}$, where we denote $R_i^{(0)} = 1$. We define $A_i^{(r)} \in Y_\mu[R_i^{(j)}]$ by

$$H_i(u) = r_i(u) \frac{\prod_{j \sim i} A_j(u - \frac{1}{2})}{A_i(u) A_i(u-1)}, \tag{2.4}$$

where $A_i(u) = 1 + \sum_{r>0} A_i^{(r)} u^{-r}$ and

$$r_i(u) = u^{-\lambda_i} R_i(u) \frac{\prod_{j \sim i} (1 - \frac{1}{2} u^{-1})^{m_j}}{(1 - u^{-1})^{m_i}}. \tag{2.5}$$

See Sections 4.1 in [KWWY14] for details.

Remark 2.2. In some situations, it will be more convenient to adjoin formal roots $\gamma_{i,k}$ for the polynomials $R_i(u) = \prod_{k=1}^{\lambda_i} (u - \frac{1}{2}\gamma_{i,k})$. We denote the resulting algebra $Y_\mu^\lambda(\boldsymbol{\gamma})$

Definition 2.3 (Section 4.4, [KWWY14])**.** Let I_μ^λ be the two-sided ideal of $Y_\mu[R_i^{(j)}]$ generated by $A_i^{(r)}$ for $r > m_i$. The **truncated shifted Yangian** is the quotient

$$Y_\mu^\lambda := Y_\mu[R_i^{(j)}] / I_\mu^\lambda$$

Given a specialization of $R_i^{(j)}$ to complex numbers, let $\mathbf{R}_i$ be the multiset of roots:

$$R_i(u) = \prod_{c \in \mathbf{R}_i} (u - \tfrac{1}{2} c) \tag{2.6}$$

Set $\mathbf{R} = (\mathbf{R}_i)_{i \in I}$. We denote by $Y_\mu^\lambda(\mathbf{R})$ the corresponding specialized algebra:

$$Y_\mu^\lambda(\mathbf{R}) = Y_\mu^\lambda \otimes_{\mathbb{C}[R_i^{(j)}]} \mathbb{C}.$$

We call the tuple $\mathbf{R}$ a **set of parameters** of weight λ. This same algebra arises if we number the elements of $\mathbf{R}_i$, and specialize $\gamma_{i,k}$ to the corresponding values. Thus, no statement about the specializations depends on which version we use, but certain statements about the families will be cleaner for $Y_\mu^\lambda(\boldsymbol{\gamma})$.

2.3 Relationship with functions on slices

We now return to our usual assumption that $\mathfrak{g} = \mathfrak{sl}_n$. The main result of [KMWY] is a proof of [KWWY14, Conjecture 2.20] in the case of $\mathfrak{g} = \mathfrak{sl}_n$. By [KWWY14, Theorem 4.10], it follows that:

Theorem 2.4. *For any choice of* $\mathbf{R}$*, there is an isomorphism*

$$\mathrm{gr}(Y_\mu^\lambda(\mathbf{R})) \cong \mathbb{C}[\overline{\mathsf{Gr}_\mu^\lambda}]$$

of graded Poisson algebras.

This isomorphism is given explicitly in terms of generalized minors, see [KWWY14, Section 2A] as well as Section 5.2 below.

2.4 Highest weights and product monomial crystals

Consider a module M over the algebra $Y_\mu^\lambda(\mathbf{R})$. We call a vector $\mathbf{1} \in M$ a **highest weight vector** if it generates M and

$$H_i^{(r)}\mathbf{1} \in \mathbb{C}\mathbf{1}, \qquad E_i^{(r)}\mathbf{1} = 0, \qquad \forall i \in I, r > 0$$

It follows that the series $H_i(u)$ acts on $\mathbf{1}$ by multiplication by some series

$$J_i(u) = \sum_{r \geq 0} J_i^{(r)} u^{-r} \in 1 + u^{-1}\mathbb{C}[[u^{-1}]]$$

We call the tuple $J = (J_i(u))_{i \in I}$ the **highest weight**.

Conversely, given a tuple $J = (J_i(u))_{i \in I}$ of series as above, there is a universal highest weight module $M(J)$ for $Y_\mu^\lambda(\mathbf{R})$ (also called a Verma or standard module). It is generated by a highest weight vector $\mathbf{1}$ with highest weight J, and has a unique simple quotient $L(J)$. The collection of all tuples J such that $M(J) \neq 0$ (equivalently, $L(J) \neq 0$) is called the **set of highest weights** for $Y_\mu^\lambda(\mathbf{R})$.

2.4.1 The product monomial crystal

The highest weights of $Y_\mu^\lambda(\mathbf{R})$ can be classified in terms of the weight μ^* elements of the **product monomial crystal** $\mathcal{B}(\mathbf{R})$. In this section we briefly overview $\mathcal{B}(\mathbf{R})$ and its relation to highest weights in general. We then give a combinatorial model of $\mathcal{B}(\mathbf{R})$ in type A, using partitions.

Remark 2.5. In this paper we will not make use of the crystal structure on $\mathcal{B}(\mathbf{R})$. Rather, we will focus on its underlying set. We refer the reader to [KTW$^+$, Section 2] for further details regarding the crystal $\mathcal{B}(\mathbf{R})$. Note that in [KTW$^+$] the product monomial crystal is denoted $\mathcal{B}(\lambda, \mathbf{R})$.

$\mathcal{B}(\mathbf{R})$ is a subset of the set Laurent monomials in variables $y_{i,c}$ (the "Nakajima monomial crystal"), where $i \in I, c \in \mathbb{C}$ (although strictly speaking it is only a $\mathfrak{g}$–crystal when the parameters $\mathbf{R}$ are "integral", see Section 2.4.3). To define $\mathcal{B}(\mathbf{R})$, one first defines the fundamental monomial crystals $\mathcal{B}(y_{i,c})$, corresponding a fundamental weight ϖ_i and parameter $c \in \mathbb{C}$. It is generated by the monomial $y_{i,c}$ by applying Kashiwara operators. For any $c \in \mathbb{C}$, $\mathcal{B}(y_{i,c})$ is isomorphic to the fundamental $\mathfrak{g}$–crystal of highest weight ϖ_i.

Next, the general product monomial crystal is defined by multiplying together the elements of various fundamental crystals $\mathcal{B}(y_{i,c})$:

$$\mathcal{B}(\mathbf{R}) = \prod_{i\in I, c\in \mathbf{R}_i} \mathcal{B}(y_{i,c}) := \Big\{ p = \prod_{i\in I, c\in \mathbf{R}_i} p_{i,c} : \forall i, c,\ p_{i,c} \in \mathcal{B}(y_{i,c}) \Big\}. \tag{2.7}$$

Here, the product symbol does not signify Cartesian product, but rather the usual product in $\mathbb{C}[y_{i,c}^{\pm}]$.

Remark 2.6. Note that with our conventions (1.2), $\lambda = \varpi_i$ corresponds to $\lambda_{n-i} = 1$ and $\lambda_j = 0$ for $j \neq n-i$. In particular a corresponding set of parameters $\mathbf{R}$ consists of a singleton, namely $\mathbf{R}_{n-i} = \{c\}$, and $\mathcal{B}(\mathbf{R})$ is isomorphic to the fundamental $\mathfrak{g}$ crystal of highest weight ϖ_{n-i}.

We've chosen to follow the conventions of [KWWY14], which differ from those of [KTW$^+$] by a diagram automorphism. We pay for this choice here, since $\mathcal{B}(\mathbf{R}) \cong \mathcal{B}(\lambda^*, \mathbf{R})$, where $\mathcal{B}(\lambda^*, \mathbf{R})$ is the product monomial crystal as defined in [KTW$^+$]. We'll gain from this choice later on, since the formulation of our main results is cleaner with this convention.

The weight of a monomial is defined as follows:

$$\mathrm{wt}\big(\prod_{i,k} y_{i,k}^{a_{i,k}}\big) = \sum_{i,k} a_{i,k}\varpi_i$$

where $i \in I, k \in \mathbb{C}$, and only finitely many of the multiplicities $a_{i,k} \in \mathbb{Z}$ are non-zero. We denote the elements of weight μ by $\mathcal{B}(\mathbf{R})_\mu$.

For any $i \in I, k \in \mathbb{C}$, define the monomial

$$z_{i,k} = \frac{y_{i,k}y_{i,k+2}}{\prod_{j\sim i} y_{j,k+1}}$$

Any element $p \in \mathcal{B}(\mathbf{R})$ can be written in the form

$$p = y_{\mathbf{R}} z_{\mathbf{S}}^{-1} := \prod_{i\in I, c\in\mathbf{R}_i} y_{i,c} \prod_{i\in I, k\in \mathbf{S}_i} z_{i,k}^{-1}, \tag{2.8}$$

for a unique tuple of multisets $\mathbf{S} = (\mathbf{S}_i)_{i\in I}$ (where products are taken with multiplicity). See Section 2 of [KTW$^+$] for more details.

2.4.2 Connection to highest weights

As described in [KTW$^+$, Section 3.6], elements of $\mathcal{B}(\mathbf{R})_{\mu^*}$ correspond to highest weights for $Y_\mu^\lambda(\mathbf{R})$. More precisely, a monomial $p = \prod_{i,k} y_{i,k}^{a_{i,k}}$ corresponds to the series

$$J_i(u) := u^{-\mu_i} \prod_k (u - \tfrac{1}{2}k)^{a_{i,k}}, \tag{2.9}$$

where the rational function on the right-hand side is expanded as an element of $1+u^{-1}\mathbb{C}[[u^{-1}]]$.

Conjecture 2.7 ([KTW$^+$, Conjecture 3.14]). *The correspondence (2.9) defines a bijection between $\mathcal{B}(\mathbf{R})_{\mu^*}$ and the set of highest weights for $Y_\mu^\lambda(\mathbf{R})$.*

In future work, we'll show that this conjecture holds, and its appropriate generalization for non-symmetric types, using a presentation for the Yangian based on its connection to Coulomb branches, as described in [BFN]; as discussed below, Conjecture 2.7 established in type A in [KTW$^+$, Thm. 1.3].

Given $p \in \mathcal{B}(\mathbf{R})_\mu$, recall that we can write $p = y_{\mathbf{R}} z_{\mathbf{S}}^{-1}$. The tuple of multisets $\mathbf{S} = (\mathbf{S}_i)_{i\in I}$ then encodes the action of the elements $A_i^{(r)} \in Y_\mu^\lambda(\mathbf{R})$ on the highest weight vector:

$$A_i(u)\mathbf{1} = \prod_{k\in\mathbf{S}_i} (1 - \tfrac{1}{2}ku^{-1})\mathbf{1} \tag{2.10}$$

2.4.3 Monomials and partitions

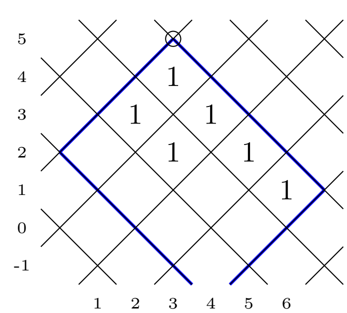
We now focus on the case when $\mathfrak{g} = \mathfrak{sl}_n$. In this case, Conjecture 2.7 holds by [KTW$^+$, Theorem 1.3]. Moreover, there is an alternate description of $\mathcal{B}(\mathbf{R})$ and its combinatorics in terms of tuples of Young diagrams [KTW$^+$, Section 6.2] which we'll now explain. This will be used in Section 4.3.2.

We call a set of parameters $\mathbf{R}$ **integral** if for every i, $\mathbf{R}_i$ consists of integers, and moreover, the parity of the elements in $\mathbf{R}_i$ equals the parity of i. In this case, there is a $\mathfrak{g}$–crystal structure on $\mathcal{B}(\mathbf{R})$. For arbitrary $\mathbf{R}$ we can decompose each $\mathbf{R}_i$ into equivalence classes $\mathbf{R}_i = \bigcup_{\zeta\in\mathbb{C}/2\mathbb{Z}} \mathbf{R}_i(\zeta)$, where $\mathbf{R}_i(\zeta) = \{c \in \mathbf{R}_i \mid c - \zeta \in 2\mathbb{Z} + i\}$. We let $\mathbf{R} = \bigcup_\zeta \mathbf{R}(\zeta)$ be the corresponding decomposition of $\mathbf{R}$.

As sets we have that $\mathcal{B}(\mathbf{R}) \cong \bigotimes_\zeta \mathcal{B}(\mathbf{R}(\zeta))$; we can put a $\mathfrak{g} \oplus \ldots \oplus \mathfrak{g}$–crystal structure here, with a copy of $\mathfrak{g}$ acting independently on each equivalence class $\mathcal{B}(\mathbf{R}(\xi))$. Therefore, to describe $\mathcal{B}(\mathbf{R})$ it suffices to describe each $\mathcal{B}(\mathbf{R}(\zeta))$. Moreover, $\mathcal{B}(\mathbf{R}(\zeta)) \cong \mathcal{B}(\mathbf{R}(\zeta) - \zeta)$, and hence we can confine ourselves to the case where $\mathbf{R}$ is integral.

First let us describe the case of a fundamental crystal $\mathcal{B}(y_{i,c})$, where $c \equiv i \bmod 2$. As a set, it is in bijection with the collection of Young diagrams which fit into an $i \times (n-i)$ box. We picture this by placing the Young diagrams in a skew-grid. The vertices of the skew-grid are labelled by pairs (i, ℓ), where $i \in I$ and $\ell \equiv i \bmod 2$. The $i \times (n-i)$ box is placed in the grid with its top vertex at the point (i, c).

For example if $n = 7, i = 3$, and $c = 5$ and the Young diagram is $(4, 2)$ then we have the following picture. Here we've circled the vertex $(3, 5)$, the $i \times (n-i)$ box is inscribed in blue, and the Young diagram is depicted by placing $1's$ in its boxes:



To associate a monomial to such a picture, we multiply $y_{i,c}$ by $z_{j,\ell}^{-1}$, as (j,ℓ) ranges over the coordinates of the bottom vertices of all the boxes in the partition. For example, the diagram above corresponds to the monomial

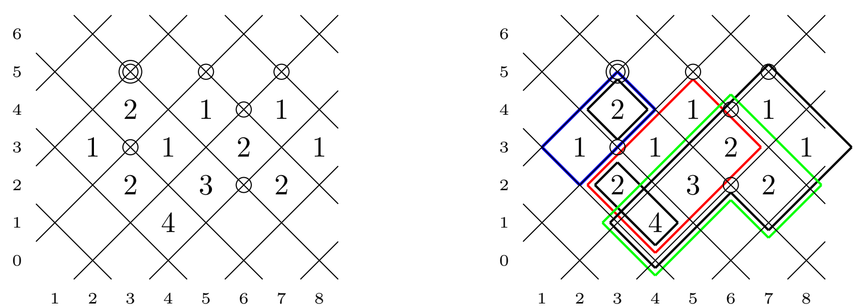
$$y_{3,6} z_{3,4}^{-1} z_{2,3}^{-1} z_{4,3}^{-1} z_{3,2}^{-1} z_{5,2}^{-1} z_{6,1}^{-1} \in \mathcal{B}(y_{3,6})$$

The rest of the elements of $\mathcal{B}(y_{3,6})$ correspond to the other partitions fitting into the blue box.

In general suppose $\mathbf{R}$ is any integral set of parameters. Then elements of $\mathcal{B}(\mathbf{R})$ are identified with diagrams consisting of circled vertices and numbered boxes. The circled vertices correspond to the elements of $\mathbf{R}$: for every $c \in \mathbf{R}_i$ we circle the vertex at (i,c). If $c \in \mathbf{R}_i$ occurs with multiplicity then the vertex is circled multiple times.

Such a diagram corresponds to an element of $\mathcal{B}(\mathbf{R})$ if and only if it can be decomposed into a tuple of overlayed partitions. More precisely, we must be able to place partitions at each circled vertex on the grid, in such a way that the number in a given box counts the times that box appears in a partition. Note that a choice of such partitions may not be unique.

For example, consider the case where $\mathfrak{g} = \mathfrak{sl}_9$ and we take $\mathbf{R}_3 = \{3,5,5\}, \mathbf{R}_5 = \{5\}, \mathbf{R}_6 = \{2,4\}$, and $\mathbf{R}_7 = \{5\}$. The left picture below depicts a candidate element of $\mathcal{B}(\mathbf{R})$. To check that it is an element of $\mathcal{B}(\mathbf{R})$ we must be able to place partitions at the circled vertices so that the number in each box counts the number of partitions that contain it. The right picture depicts such a choice of partitions, verifying that this diagram is indeed in $\mathcal{B}(\mathbf{R})$. Note that since 5 occurs twice in $\mathbf{R}_3$ we are able to place two partitions at $(3,5)$.



To associate a monomial to such a diagram we multiply $y_{\mathbf{R}}$ by $z_{j,l}^{-k}$, where (j,ℓ) ranges over the bottom vertices of the numbered boxes, and k is the number of the box. In the example above, the diagram corresponds to the monomial

$$y_{\mathbf{R}} z_{3,3}^{-2} z_{5,3}^{-1} z_{7,3}^{-1} z_{2,2}^{-1} z_{4,2}^{-1} z_{6,2}^{-2} z_{8,2}^{-1} z_{3,1}^{-2} z_{5,1}^{-3} z_{7,1}^{-2} z_{4,0}^{-4} \in \mathcal{B}(\mathbf{R})$$

We reiterate that the assumption that $\mathbf{R}$ is an integral set of parameters is made only for the sake of convenience. We could set up the same combinatorics for general $\mathbf{R}$, where we depict elements of $\mathcal{B}(\mathbf{R})$ by tuples of such diagrams, one for each $\zeta \in \mathbb{C}/2\mathbb{Z}$ such that $\mathbf{R}(\zeta)$ is nonempty.

2.5 Maps between truncated shifted Yangians

Given $\lambda \geq \mu$, recall that we define $N = \sum_i i\lambda_{n-i}$. Consider a set of parameters $\mathbf{R} = (\mathbf{R}_i)_{i \in I}$ of weight λ, and a set of parameters $\widetilde{\mathbf{R}}$ of weight $N\varpi_1$. Note that the latter is prescribed by the single multiset $\widetilde{\mathbf{R}}_{n-1}$ of size N. For this reason, we will abuse of notation and simply identify $\widetilde{\mathbf{R}} = \widetilde{\mathbf{R}}_{n-1}$.

Our goal is to establish the following commutative diagram:

$$\begin{tikzcd} Y_\mu \arrow[r, "\phi'"] \arrow[dr, "\phi"'] & Y_\mu^{N\varpi_1}(\widetilde{\mathbf{R}}) \arrow[d, dashed, "\phi''"] \\ & Y_\mu^\lambda(\mathbf{R}) \end{tikzcd} \tag{2.11}$$

where ϕ, ϕ' are the (defining) quotient maps.

Theorem 2.8. *A map ϕ'' making the above diagram commute exists iff*

$$\widetilde{\mathbf{R}} = \bigcup_{i=1}^{n-1} \big(\mathbf{R}_i + (n-i-1)\big) \cup \big(\mathbf{R}_i + (n-i-3)\big) \cup \cdots \cup \big(\mathbf{R}_i - (n-i-1)\big) \tag{2.12}$$

as a union of multisets. In this case, ϕ'' quantizes the inclusion $\overline{\mathsf{Gr}^{\lambda}_\mu} \subset \overline{\mathsf{Gr}^{N\varpi_1}_\mu}$ as a closed Poisson subvariety.

The last claim in the theorem simply follows from the form of the identification $\operatorname{gr} Y_\mu^\lambda(\mathbf{R}) \cong \mathbb{C}[\overline{\mathsf{Gr}^{\lambda}_\mu}]$. Indeed as in [KWWY14], for any $\lambda \geq \mu$ the surjection $Y_\mu \to Y_\mu^\lambda(\mathbf{R})$ corresponds to the inclusion $\overline{\mathsf{Gr}^{\lambda}_\mu} \subset \mathsf{Gr}_\mu$ into the opposite cell Gr_μ. So the diagram (2.11) expresses the inclusions $\overline{\mathsf{Gr}^{\lambda}_\mu} \subset \overline{\mathsf{Gr}^{N\varpi_1}_\mu} \subset \mathsf{Gr}_\mu$.

When the map ϕ'' exists, every highest weight module for $Y_\mu^\lambda(\mathbf{R})$ pulls-back to a highest weight module for $Y_\mu^{N\varpi_1}(\widetilde{\mathbf{R}})$. Recall from Section 2.4.2 that an element of the monomial crystal, expressed in the variables $y_{i,k}$, explicitly encodes the action of the series $H_i(u)$ on a highest weight vector. Since $H_i(u) \mapsto H_i(u)$ under $Y_\mu^{N\varpi_1}(\widetilde{\mathbf{R}}) \to Y_\mu^\lambda(\mathbf{R})$, the pull-back of highest weights corresponds to an inclusion of sets $\mathcal{B}(\mathbf{R})_{\mu^*} \subset \mathcal{B}(\widetilde{\mathbf{R}})_{\mu^*}$. Slightly more generally, we have:

Lemma 2.9. *Let $\mathbf{R}, \widetilde{\mathbf{R}}$ satisfy (2.12). Then there is an inclusion of sets*

$$\mathcal{B}(\mathbf{R}) \subset \mathcal{B}(\widetilde{\mathbf{R}})$$

If $\mathbf{R}$ is integral, then this is an inclusion of crystals.

Proof. The case where $\lambda = \varpi_i$ is analogous to [KTW+, Lemma 5.31], and the general case follows by taking products. ☐

Remark 2.10. The above results are analogs of the embedding of $\mathfrak{sl}_n$ representations

$$(\mathbb{C}^n)^{\otimes \lambda_1} \otimes (\wedge^2 \mathbb{C}^n)^{\otimes \lambda_2} \otimes \cdots \otimes (\wedge^{n-1}\mathbb{C}^n)^{\otimes \lambda_{n-1}} \subset (\wedge^{n-1}\mathbb{C}^n)^{\otimes N},$$

and in fact when $\mathbf{R}$ is sufficiently generic Lemma 2.9 can be interpretted as a crystal version of this embedding.

Corollary 2.11. *When $\phi'' : Y_\mu^{N\varpi_1}(\widetilde{\mathbf{R}}) \to Y_\mu^\lambda(\mathbf{R})$ as above exists, we have a containment*

$$\ker \phi'' \subset \bigcap_p \operatorname{Ann} L_p,$$

the intersection being over the simple $Y_\mu^{N\varpi_1}(\widetilde{\mathbf{R}})$–modules L_p with highest weights $p \in \mathcal{B}(\mathbf{R})_{\mu^*} \subset \mathcal{B}(\widetilde{\mathbf{R}})_{\mu^*}$.

Defining this map in the case where we consider $R_i(u)$ as a formal polynomial, rather than specializing to numerical values, is slightly more complicated. Of course, Theorem 2.8 shows that we have a homomorphism $Y_\mu^{N\omega_1} \to Y_\mu^\lambda$ sending

$$\tilde{R}_{n-1}(u) \mapsto \prod_{i=1}^{n-1}\prod_{k=1}^{n-i} R_i(u - \tfrac{n-i-1}{2} + k - 1). \tag{2.13}$$

Unfortunately, this map is not necessarily surjective; it's more convenient to consider the enlarged version where we have a surjective map $Y_\mu^{N\omega_1}(\tilde{\gamma}) \to Y_\mu^\lambda(\boldsymbol{\gamma})$, sending the roots of the LHS of (2.13) to the roots of RHS (by an arbitrary bijection).

2.5.1 Proof of Theorem 2.8

Recall that we set

$$\lambda - \mu = \sum_i m_i \alpha_{n-i}, \quad N\varpi_1 - \mu = \sum_i m'_i \alpha_{n-i} \tag{2.14}$$

In addition denote $N\varpi_1 - \lambda = \sum_i m''_i \alpha_{n-i}$. In particular $m_i = m'_i - m''_i$. We note the following:

Lemma 2.12.

$$N\varpi_1 - \lambda = \sum_{i=2}^{n-1} \lambda_{n-i}\Big((i-1)\alpha_1 + (i-2)\alpha_2 + \ldots + \alpha_{i-1}\Big).$$

Thus, we have that the coefficient $m''_1 = 0$.

Proof. We have $N\varpi_1 - \lambda = \sum_i \lambda_{n-i}(i\varpi_1 - \varpi_i)$. Now observe that

$$i\varpi_1 - \varpi_i = (i-1)\alpha_1 + (i-2)\alpha_2 + \ldots + \alpha_{i-1}. \qquad \square$$

Recall from Section 2.2 that $Y_\mu^\lambda(\mathbf{R}) = Y_\mu/\langle A_i^{(r)} : i \in I, r > m_i\rangle$, where $A_i^{(r)} \in Y_\mu$ are defined by

$$H_i(u) = r_i(u)\frac{A_{i-1}(u-\frac{1}{2})A_{i+1}(u-\frac{1}{2})}{A_i(u)A_i(u-1)}$$

with

$$r_i(u) = \frac{R_i(u)}{u^{\lambda_i}}\frac{(1-\frac{1}{2}u^{-1})^{m_{i-1}+m_{i+1}}}{(1-u^{-1})^{m_i}}$$

Similarly $Y_\mu^{N\varpi_1}(\widetilde{\mathbf{R}}) = Y_\mu/\langle \widetilde{A}_i^{(r)} : i \in I, r > m'_i\rangle$, where $\widetilde{A}_i^{(r)} \in Y_\mu$ are defined by

$$H_i(u) = \widetilde{r}_i(u)\frac{\widetilde{A}_{i-1}(u-\frac{1}{2})\widetilde{A}_{i+1}(u-\frac{1}{2})}{\widetilde{A}_i(u)\widetilde{A}_i(u-1)}$$

where

$$\widetilde{r}_i(u) = \left(\frac{\widetilde{R}(u)}{u^N}\right)^{\delta_{i,n-1}}\frac{(1-\frac{1}{2}u^{-1})^{m'_{i-1}+m'_{i+1}}}{(1-u^{-1})^{m'_i}}$$

From the definitions, for all i we therefore have an equality in Y_μ:

$$r_i(u)\frac{A_{i-1}(u-\frac{1}{2})A_{i+1}(u-\frac{1}{2})}{A_i(u)A_i(u-1)} = \widetilde{r}_i(u)\frac{\widetilde{A}_{i-1}(u-\frac{1}{2})\widetilde{A}_{i+1}(u-\frac{1}{2})}{\widetilde{A}_i(u)\widetilde{A}_i(u-1)}$$

Using the definition of $r_i(u)$ and $\widetilde{r}_i(u)$, for $i=n-1$ we can rewrite this as

$$\frac{\widetilde{A}_{n-2}(u-\frac{1}{2})}{\widetilde{A}_{n-1}(u)\widetilde{A}_{n-1}(u-1)} = \frac{R_{n-1}(u)}{\widetilde{R}(u)}\frac{u^{m''_{n-1}}(u-1)^{m''_{n-1}}}{(u-\frac{1}{2})^{m''_{n-2}}}\frac{A_{n-2}(u-\frac{1}{2})}{A_{n-1}(u)A_{n-1}(u-1)} \tag{2.15}$$

and for $i=1,\ldots,n-2$ as

$$\frac{\widetilde{A}_{i-1}(u-\frac{1}{2})\widetilde{A}_{i+1}(u-\frac{1}{2})}{\widetilde{A}_i(u)\widetilde{A}_i(u-1)} = R_i(u)\frac{u^{m''_i}(u-1)^{m''_i}}{(u-\frac{1}{2})^{m''_{i-1}+m''_{i+1}}}\frac{A_{i-1}(u-\frac{1}{2})A_{i+1}(u-\frac{1}{2})}{A_i(u)A_i(u-1)} \tag{2.16}$$

Corollary 2.13. *There are unique series $f_i(u)\in u^{m''_i}(1+u^{-1}\mathbb{C}[[u^{-1}]])$ such that*

$$\widetilde{A}_i(u) = \frac{f_i(u)}{u^{m''_i}}A_i(u)$$

These satisfy

$$R_{n-1}(u) = \frac{\widetilde{R}(u)f_{n-2}(u-\frac{1}{2})}{f_{n-1}(u)f_{n-1}(u-1)},\quad R_i(u) = \frac{f_{i-1}(u-\frac{1}{2})f_{i+1}(u-\frac{1}{2})}{f_i(u)f_i(u-1)} \tag{2.17}$$

for $i=1,\ldots,n-2$.

Proof. By [GKLO05, Lemma 2.1], $A_i(u)$ and $\widetilde{A}_i(u)$ must differ by multiplication by an element of $1+u^{-1}\mathbb{C}[[u^{-1}]]$. The precise form above follows by rearranging (2.15) and (2.16). ☐

Lemma 2.14. $\ker\phi'\subset\ker\phi$ *if and only if $f_i(u)\in\mathbb{C}[u]$.*

Proof. Assume that

$$\widetilde{A}_i^{(s)}\in\ker\phi = \langle A_i^{(r)} : i\in I, r>m_i\rangle$$

for all $s>m'_i$. Equating coefficients in $u^{m''_i}\widetilde{A}_i(u)=f_i(u)A_i(u)$, we see that $f_i(u)$ cannot contain any negative powers of u. Indeed, if it did then a non-trivial linear combination of elements $\{A_i^{(1)},\ldots,A_i^{(m_i)}\}$ would be zero in $Y_\mu^\lambda(\mathbf{c})$. But these elements are algebraically independent, e.g. by the GKLO representation [KWWY14, Theorem 4.5].

Conversely, if $f_i(u)$ is a polynomial then $\widetilde{A}_i^{(s)}$ is a linear combination of elements from $\ker\phi$. ☐

Theorem 2.8 follows from the next result:

Proposition 2.15. *The map ϕ'' exists iff the following identities hold:*

$$\widetilde{R}(u) = \prod_{i=1}^{n-1}R_i(u+\tfrac{n-i-1}{2})R_i(u+\tfrac{n-i-3}{2})\cdots R_i(u-\tfrac{n-i-1}{2}),$$

$$f_k(u-\tfrac{1}{2}) = \prod_{i=1}^{k-1}R_i(u+\tfrac{k-i-1}{2})R_i(u+\tfrac{k-i-3}{2})\cdots R_i(u-\tfrac{k-i-1}{2})$$

for $k=1,\ldots,n-1$.

Proof. Note that ϕ'' exists if and only if $\ker\phi' \subset \ker\phi$. Hence if ϕ'' exists then $f_i(u)$ is a polynomial by Lemma 2.14, and it is monic of degree m''_i by Corollary 2.13. Since $m''_1 = 0$ by Lemma 2.12, we know that $f_1(u) = 1$. Applying (2.17) with $i = 1$, we then obtain

$$R_1(u) = f_2(u - \tfrac{1}{2})$$

Proceeding by induction on i using (2.17), we get the claimed form of $\widetilde{R}(u)$ and $f_i(u)$.

Conversely, if we define $\widetilde{R}(u)$ and $f_i(u)$ by the claimed form above, then (2.17) holds, and the $f_i(u)$ are monic polynomials of the correct degree. By the previous lemma, it follows that $\ker\phi' \subset \ker\phi$. □

3 Around W-algebras

3.1 Finite W-algebras

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and $e \in \mathfrak{g}$ a nilpotent element. Complete this to an $\mathfrak{sl}_2$-triple $\{f, h, e\}$. The **Slodowy slice** is the affine space $\mathcal{S} = e + \mathfrak{g}^f$, where $\mathfrak{g}^f = \{x \in \mathfrak{g} \mid [x, f] = 0\}$. It naturally inherits a Poisson structure from $\mathfrak{g} \cong \mathfrak{g}^*$ [GG02]. Recall that the symplectic leaves of $\mathfrak{g}$ are the nilpotent orbits $\mathcal{O}$, and $\mathcal{S}$ intersects the symplectic leaves transversally.

We recall now a construction of finite W-algebras which quantize the Slodowy slices. Recall that an $\mathbb{Z}$-grading of $\mathfrak{g}$

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i.$$

is called **good** for a nilpotent e if

1. The operator $\operatorname{ad}(e)$ has degree 2.
2. We have $\mathfrak{g}_i \cap \ker\operatorname{ad}(e) = 0$ for $i \leq -1$.
3. We have $\mathfrak{g}_i \subset \operatorname{image}\operatorname{ad}(e)$ if $i \geq 1$.

Note that by a simple application of $\mathfrak{sl}_2$ representation theory, every nilpotent e has a good grading induced by considering the weights of h.

For any good grading, the space $\mathfrak{g}_{-1}$ is symplectic with the form

$$\langle x, y\rangle = (e, [x, y]) = ([e, x], y) = (x, [y, e]),$$

where $(\cdot,\cdot)$ is the usual Killing form. This follows from the fact that $\operatorname{ad}(e)\colon \mathfrak{g}_{-1} \to \mathfrak{g}_1$ is an isomorphism. Choose a Lagrangian subspace $\mathfrak{l} \subset \mathfrak{g}_{-1}$ and set

$$\mathfrak{m} = \mathfrak{l} \oplus \bigoplus_{i<-1} \mathfrak{g}_i. \tag{3.1}$$

Note that if the grading in question is even (i.e. $\mathfrak{g}_i \neq 0$ implies $i \in 2\mathbb{Z}$) then $\mathfrak{m} = \bigoplus_{i<-1} \mathfrak{g}_i$ and we can avoid the choice. Then $\chi = (e, \cdot) : \mathfrak{m} \to \mathbb{C}$ is a character. Finally, let $\mathfrak{m}_\chi := \operatorname{span}\{a - \chi(a) : a \in \mathfrak{m}\}$.

Define the **finite W-algebra** $W(e) = (U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{m}_\chi)^{\mathfrak{m}}$. By the following theorem, this algebra is a quantization of $\mathcal{S}$.

Theorem 3.1 (Theorem 4.1, [GG02])**.** *There is a filtration on* $U(e,\mathfrak{g})$ *(the Kazhdan filtration) such that* $gr(W(e)) \cong \mathbb{C}[\mathcal{S}]$.

We will be interested in quotients of $W(e)$, called parabolic W-algebras, which quantize the intersection $\mathcal{S} \cap \overline{\mathcal{O}}$.

3.1.1 Conventions

We closely follow the conventions of [BK06, Section 3], [BK05, Section 7], although we do not follow their grading conventions: Brundan and Kleshchev divide their even gradings by two, while we will not. We will also number the boxes of our pyramid differently. Let us briefly outline our conventions here.

For $\pi = (p_1 \le p_2 \le \ldots \le p_n)$ a partition of N, we will consider π as a right-justified pyramid with boxes numbered from right to left, top to bottom. For example, $\pi = (2,3,4)$ will be correspond to

$$\begin{array}{|c|c|c|c|} \cline{3-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 2 & 1 \\ \cline{2-4} \multicolumn{1}{c|}{} & 5 & 4 & 3 \\ \hline 9 & 8 & 7 & 6 \\ \hline \end{array} \tag{3.2}$$

We number the columns of π from left to right, and rows from top to bottom.

Corresponding to the pyramid π, we consider the nilpotent element

$$e_\pi = \sum_{k,\ell} e_{k\ell},$$

summing over pairs $\boxed{k}\boxed{\ell}$ of adjacent boxes in π. The grading on $\mathfrak{g}$ is defined by $\deg(e_{k\ell}) = 2(\mathrm{col}(\ell) - \mathrm{col}(k))$, where $\mathrm{col}(\ell)$ denotes the number of the column containing $\boxed{\ell}$. Finally, the Kazhdan filtration on $U(\mathfrak{g})$ corresponding to π is defined by declaring that

$$\deg(e_{k\ell}) = 2(\mathrm{col}(\ell) - \mathrm{col}(k) + 1) \tag{3.3}$$

Remark 3.2. In [BK06], the authors use the convention of (3.3) in the introduction, but divide by a factor of 2 in [BK06, Section 8], to match the usual filtration on Yangians.

3.2 Brundan and Kleshchev's presentation

3.2.1 Shifted Yangians

In the case where $\mathfrak{g} = \mathfrak{gl}_N$ Brundan and Kleshchev gave a presentation of the W algebra. To describe this result we first recall their definition of the shifted Yangians [BK06]. Here we work with the $\mathfrak{gl}_n$-Yangian Y_n, which is a $\mathbb{C}$-algebra with generators $E_i^{(r)}, F_i^{(r)}$ for $1 \le i < n$, and $D_i^{(r)}$ for $1 \le i \le n$ and $r \ge 1$.

To describe the defining relations of Y_n we follow [BK05, Theorem 5.2] and introduce generating series $D_i(u) = 1 + \sum_{r \ge 1} D_i^{(r)} u^{-r}$, and define $\tilde{D}_i^{(r)}$ via

$$\sum_{r \ge 0} \tilde{D}_i^{(r)} u^{-r} = -D_i(u)^{-1}.$$

The defining relations of Y_n are as follows:

$$
\begin{aligned}
[D_i^{(r)}, D_j^{(s)}] &= 0, \\
[E_i^{(r)}, F_j^{(s)}] &= \delta_{i,j} \sum_{t=0}^{r+s-1} \tilde{D}_i^{(t)} D_{i+1}^{(r+s-1-t)}, \\
[D_i^{(r)}, E_j^{(s)}] &= (\delta_{i,j} - \delta_{i,j+1}) \sum_{t=0}^{r-1} D_i^{(t)} E_j^{(r+s-1-t)}, \\
[D_i^{(r)}, F_j^{(s)}] &= (\delta_{i,j+1} - \delta_{i,j}) \sum_{t=0}^{r-1} F_j^{(r+s-1-t)} D_i^{(t)}, \\
[E_i^{(r)}, E_i^{(s+1)}] - [E_i^{(r+1)}, E_i^{(s)}] &= E_i^{(r)} E_i^{(s)} + E_i^{(s)} E_i^{(r)}, \\
[F_i^{(r+1)}, F_i^{(s)}] - [F_i^{(r)}, F_i^{(s+1)}] &= F_i^{(r)} F_i^{(s)} + F_i^{(s)} F_i^{(r)}, \\
[E_i^{(r)}, E_{i+1}^{(s+1)}] - [E_i^{(r+1)}, E_{i+1}^{(s)}] &= -E_i^{(r)} E_{i+1}^{(s)}, \\
[F_i^{(r+1)}, F_{i+1}^{(s)}] - [F_i^{(r)}, F_{i+1}^{(s+1)}] &= -F_{i+1}^{(s)} F_i^{(r)}, \\
[E_i^{(r)}, E_j^{(s)}] &= 0 \quad \text{if } |i-j| > 1, \\
[F_i^{(r)}, F_j^{(s)}] &= 0 \quad \text{if } |i-j| > 1, \\
[E_i^{(r)}, [E_i^{(s)}, E_j^{(t)}]] + [E_i^{(s)}, [E_i^{(r)}, E_j^{(t)}]] &= 0 \quad \text{if } |i-j| = 1, \\
[F_i^{(r)}, [F_i^{(s)}, F_j^{(t)}]] + [F_i^{(s)}, [F_i^{(r)}, F_j^{(t)}]] &= 0 \quad \text{if } |i-j| = 1.
\end{aligned}
$$

Y_n has a filtration defined as follows [BK06, Section 5]: inductively define elements $E_{i,j}^{(r)}$, for $1 \leq i < j \leq n$ and $r > 0$, by $E_{i,i+1}^{(r)} = E_i^{(r)}$ and $E_{i,j}^{(r)} = [E_{i,j-1}^{(r)}, E_j^{(1)}]$. Similarly define elements $E_{i,j}^{(r)}$ when $i > j$, and also denote $E_{i,i}^{(r)} = D_i^{(r)}$. Then the filtration is defined by declaring the elements $E_{i,j}^{(r)}$ to have degree r; note that Y_n satisfies a PBW theorem in these elements.

Let $\sigma = (s_{i,j})_{1 \leq i,j \leq n}$ be a shift matrix of non-negative integers, meaning that

$$
s_{i,j} + s_{j,k} = s_{i,k}
$$

whenever $|i-j| + |j-k| = |i-k|$.

Definition 3.3 (Section 2, [BK06])**.** The **shifted $\mathfrak{gl}_n$-Yangian** $Y_n(\sigma) \subset Y_n$ is the subalgebra generated by $D_i^{(r)}$ for $r > 0$, $E_i^{(r)}$ for $r > s_{i,i+1}$, and $F_i^{(r)}$ for $r > s_{i+1,i}$, with the induced filtration from Y_n.

There is another family of generators for $Y_n(\sigma)$, denoted $T_{i,j}^{(r)}$ for $1 \leq i, j \leq n$ and $r > s_{i,j}$. See [BK06] for the definition of these generators as well as their relation to the presentation given above. For $1 \leq i \leq n$ we define the principal quantum minor:

$$
Q_i(u) = \sum_{w \in S_i} (-1)^w T_{w(1),1}(u) \cdots T_{w(i),i}(u - i + 1) \tag{3.4}
$$

where $T_{i,j}(u) = \delta_{i,j} + \sum_{r>s_{ij}} T_{i,j}^{(r)} u^{-r}$. For our present purposes, the most important relation involving these new generators is the following equation:

$$D_i(u) = \frac{Q_i(u+i-1)}{Q_{i-1}(u+i-1)} \tag{3.5}$$

Remark 3.4. There is a subtle point here: the identity $D_i(u) = \frac{Q_i(u+i-1)}{Q_{i-1}(u+i-1)}$ is true in the Yangian with no shift. In the shifted Yangian the $T_{ij}^{(r)}$ generators are defined using a Gauss decomposition with shifted generators [BK08, Section 2.2]. Hence the $T_{ij}^{(r)}$ in the shifted Yangian are not the same as the generators with the same name in the full Yangian. However, Brown and Brundan prove that the quantum minors are in fact the same, so the identity is true with the shifted $T_{ij}^{(r)}$ as well [BB09]. More precisely, they prove that $Q_n(u) = Q_n^0(u)$, where $Q_n^0(u)$ is the quantum determinant corresponding to the Yangian with $\sigma = 0$, i.e. the full $\mathfrak{gl}_n$-Yangian. This implies that $Q_i(u) = Q_i^0(u)$ for any $i = 1, .., n$, using the embeddings $Y(\mathfrak{gl}_n) \supset Y(\mathfrak{gl}_{n-1}) \supset \cdots$.

We'll need also the decomposition

$$Y_n(\sigma) \cong SY_n(\sigma) \otimes Z(Y_n(\sigma)), \tag{3.6}$$

where $Z(Y_n(\sigma))$ is the center and $SY_n(\sigma)$ is the subalgebra of $Y_n(\sigma)$ generated by $H_i^{(r)}$ for $r > 0$, $E_i^{(r)}$ for $r > s_{i,i+1}$, and $F_i^{(r)}$ for $r > s_{i+1,i}$. Here $H_i^{(r)}$ are coefficients of $\frac{D_{i+1}(u)}{D_i(u)}$ [BK08, Section 2.6]). The center $Z(Y_n(\sigma))$ is free generated by the coefficients of the series $Q_n(u)$ [BK08, Theorem 2.6].

3.2.2 Brundan and Kleshchev's Theorem

Let $\pi = (p_1 \leq p_2 \leq \ldots \leq p_n)$ be a partition of N, and consider the lower-triangular shift matrix σ where $s_{i,j} = p_j - p_i$ for $i \leq j$. Let $W(\pi)$ be the quotient of $Y_n(\sigma)$ by the two-sided ideal generated by the elements $D_1^{(r)}$ for $r > p_1$,

$$W(\pi) = Y_n(\sigma)/\langle D_1^{(r)} : r > p_1 \rangle \tag{3.7}$$

The algebra $W(\pi)$ inherits a filtration from $Y_n(\sigma)$.

Theorem 3.5 (Theorem 10.1, [BK06])**.** *There is an isomorphism of algebras $W(\pi) \cong W(e_\pi)$. This isomorphism doubles filtered degrees, i.e. $F_{\leq r}W(\pi) \cong F_{\leq 2r}W(e_\pi)$.*

Remark 3.6. Note that the above degree doubling is harmless: the filtration (3.3) on $W(e_\pi)$ is *even*, and so we may safely rescale it removing a factor of two. This is the approach followed by Brundan and Kleshchev, so in their work no such doubling appears. We have elected to maintain the factor of two to match standard conventions on the Kazhdan filtration (e.g. [GG02, Section 4]), while also following usual conventions for filtrations of Yangians.

Let $Row(\pi)$ be the set of row symmetrized π-tableaux, i.e. tableau of shape π with complex entries viewed up to row equivalence. A row tableau $T \in Row(\pi)$ encodes a highest weight of $W(\pi)$ via

$$(u-i+1)^{p_i} D_i(u-i+1) \mapsto \prod_{a \in T_i} (u + \tfrac{1}{2}a - \tfrac{n}{2}), \tag{3.8}$$

where T_i denotes the i-th row of T. Brundan and Kleshchev prove that this describes a bijection between highest weights of $W(\pi)$ and $Row(\pi)$ ([BK06, Section 6]). Given a multiset $\mathbf{R}$ of N complex numbers we let $Row_{\mathbf{R}}(\pi)$ be the set of row tableaux with entries from $\mathbf{R}$ (with the same multiplicities).

3.3 Parabolic W-algebras

We will require some facts about parabolic W-algebras which may be of some independent interest.

Parabolic W-algebras quantize the intersection of a Slodowy slice with the closure of a nilpotent orbit. They arise from Hamiltonian reduction of the primitive quotients of the universal enveloping algebra. These quotients were studied by the first author [Web11, §2] and by Losev [Los12, §5.2].

3.3.1 Differential operators on partial flag varieties

Let G be a reductive complex algebraic group. Given a parabolic P, we can consider the variety G/P, and the universal differential operators on it as a quotient of $U(\mathfrak{g})$. Let $\mathfrak{g} \cong \mathfrak{u}_- \oplus \mathfrak{l} \oplus \mathfrak{u}$ be the decomposition of $\mathfrak{g}$ into a Levi subalgebra, and two complementary radicals, with $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$.

We'll be interested in sheaves of twisted differential operators on G/P. See [BB93, §1-2] for a general discussion of these rings. Since we wish to consider TDOs over more general rings, let us give a complete definition. Fix a $\mathbb{C}$-algebra S.

Definition 3.7. We call a filtered sheaf of algebras

$$\mathscr{D} \supset \cdots \supset \mathscr{D}_{\leq n} \supset \cdots \supset \mathscr{D}_{\leq 0} \supset \mathscr{D}_{\leq -1} = \{0\}$$

over $\mathcal{O}_{G/P} \otimes S$ a **TDO** if there is an isomorphism of Poisson algebras $\operatorname{gr} \mathscr{D} \to \operatorname{Sym}^\bullet(\mathcal{T}(G/P)) \otimes S \cong \pi_* \mathcal{O}_{T*(G/P)} \otimes S$ where $\mathcal{T}(G/P)$ is the tangent sheaf of G/P and $\pi : T^*G/P \to G/P$ is the projection map. The Poisson bracket on $\operatorname{Sym}^\bullet(\mathcal{T}(G/P)) \otimes S$ is the unique S-linear Poisson bracket such that $\{X, Y\}$ is the Lie derivative $\mathcal{L}_X Y$ for X a vector field and Y an arbitrary tensor.

A **homogeneous TDO** is a TDO equipped with a G-equivariant structure, and a Lie algebra map $\mathfrak{g} \to \Gamma(G/P; \mathscr{D}_{\leq 1})$ lifting the action map $\mathfrak{g} \to \Gamma(G/P; \mathcal{T})$.

Generalizing [BB81], we consider the sheaf of $\mathfrak{g}$ valued functions $\mathfrak{g}^0 \cong \mathfrak{g} \otimes \mathcal{O}_{G/P}$. This has a natural subsheaf $\mathfrak{p}^0$ given by local sections of the vector bundle $G \times_P \mathfrak{p}$; this is the kernel of the action map $\mathfrak{g}^0 \to \mathcal{T}(G/P)$ to the tangent sheaf $\mathcal{T}(G/P)$ of G/P. We consider the algebra sheaf $U^0 = U(\mathfrak{g}^0) \otimes_{\mathbb{C}} S$.

Given a character $\gamma \colon \mathfrak{p} \to S$, we can reverse this process. Consider the ideal in U^0 generated by the kernel of the map $U(\mathfrak{p}^0) \otimes_{\mathbb{C}} S \to S$ induced by the character $\gamma - \rho + \rho_P \colon \mathfrak{p}^0 \otimes_{\mathbb{C}} S \to S \otimes \mathcal{O}_{G/P}$. Here ρ is the usual half-sum of positive roots of G, and ρ_P is the half-sum of the positive roots of the Levi subgroup L. We can define a TDO $\mathscr{D}_\gamma$ on G/P by considering the quotient of U^0 by this ideal, with the obvious homogeneous structure.

Proposition 3.8 ([Mil, Thm. 4])**.** *This construction defines a bijection between homogeneous TDOs on G/P and characters $\mathfrak{p} \to S$.*

If we choose $S = \mathrm{Sym}(\mathfrak{p}/[\mathfrak{p},\mathfrak{p}])$, we can take the universal character $\iota\colon \mathfrak{p} \to \mathfrak{p}/[\mathfrak{p},\mathfrak{p}] \subset S$. We can consider the section algebra $A = \Gamma(G/P; \mathscr{D}_\iota)$. In this paper, we'll also consider two other cases: when $S = \mathbb{C}$ and $\gamma\colon \mathfrak{p} \to \mathbb{C}$ is an honest character, and when $S = \widehat{\mathrm{Sym}}(\mathfrak{p}/[\mathfrak{p},\mathfrak{p}])$, the completion of $\mathrm{Sym}(\mathfrak{p}/[\mathfrak{p},\mathfrak{p}])$ at 0, and we consider $\gamma + \iota$ (or Weyl translates of this). We denote the resulting algebras A_γ and $A_{\gamma+\iota}$. We always have that $\mathrm{gr}\, A_\gamma \cong \mathbb{C}[T^*G/P]$ with the grading induced by cotangent scaling. Note that this shows that the algebra A is flat over $\widehat{\mathrm{Sym}}(\mathfrak{p}/[\mathfrak{p},\mathfrak{p}])$, since its fibers have constant character for the grading $\mathbb{C}$ action (so actually every piece of the order filtration is flat). This shows that $A_{\gamma+\iota}$ is flat as well.

Thus, the algebra $A_{\gamma+\iota}$ provides a family over a regular ring which interpolates between the generic behavior around γ, and the specialized behavior at γ. In this case, we let K be the fraction field of S and let $\tilde{\mathscr{D}}_\gamma := \mathscr{D}_{\gamma+\iota} \otimes_S K$ denote the TDO over S associated to $\gamma + \iota$, base changed to K. We let $\tilde{A}_\gamma = A_{\gamma+\iota} \otimes_S K$.

This last algebra is interesting because it satisfies the appropriate analogue of the Beilinson-Bernstein theorem for *all* γ, without any dominance hypothesis.

Theorem 3.9. *The functor*

$$\Gamma(G/P; -)\colon \tilde{\mathscr{D}}_\gamma\text{-mod} \to \tilde{A}_\gamma\text{-mod}$$

is an equivalence.

Proof. In order to show equivalence, we have to show that the natural map $\mathrm{Loc}(\Gamma(G/P \times G/P; (i_\Delta)_*\mathcal{O}_{G/P}) \to \mathcal{O}_{G/P}$ is an isomorphism, as argued in [Kal08, 3.1]. Thus, it's enough to show that we have an equivalence at the generic point of the spectrum of $S = \mathrm{Sym}(\mathfrak{p}/[\mathfrak{p},\mathfrak{p}])$, and thus enough to prove it at any point in this spectrum, that is at $\gamma + \eta$ for η a some character of $\mathfrak{p}$; for simplicity, assume that $\alpha_i^\vee(\eta) \in (0, \epsilon)$ for ϵ a small positive real number. Note that $\alpha_i^\vee(\gamma+\eta-\rho_P) = -1$ if $\alpha_i \in R_P$. If $\alpha_i \notin R_P$, then $\alpha_i^\vee(\gamma+\eta-\rho_P)$ is a fixed complex number $\alpha_i^\vee(\gamma - \rho_P)$ plus $\alpha_i^\vee(\eta) \in (0, \epsilon)$. If we choose ϵ sufficiently small, this will never be a non-negative integer. Thus, this weight is anti-dominant and regular in the sense of [Kit12, §2.6]. Thus, by [Kit12, 2.9], we have equivalence for $\tilde{\mathscr{D}}_\gamma$. □

The algebra A is not quite an analogue of the universal enveloping algebra since even in the case of a Borel $\mathfrak{p} = \mathfrak{b}$, we will not obtain $U(\mathfrak{g})$, but instead the finite extension $U(\mathfrak{g}) \otimes_{Z(\mathfrak{g})} U(\mathfrak{h})$ quantizing the Grothendieck-Springer resolution. When we ultimately compare parabolic W-algebras to Yangians, this algebra matches the larger algebra $Y_\mu^\lambda(\boldsymbol{\gamma})$ where formal roots of R_i are adjoined.

Here, we identify $Z(\mathfrak{g})$ with $U(\mathfrak{h})^W$ via the (shifted) Harish-Chandra homomorphism, so the maximal ideal for the orbit of a weight λ is the ideal of central elements vanishing on the Verma module of highest weight $\lambda + \rho$. Note that in the case of $\mathfrak{gl}_N$, this matches the convention of [BK08, §3.8]: the elements $Z_N^{(r)}$ are sent to the degree r elementary symmetric function in the diagonal elements $e_{i,i}$.

More generally, one can prove that the section algebra A has an action of $N = N(\mathfrak{l})/L$, the normalizer of the Levi $\mathfrak{l}$ of $\mathfrak{p}$ in G modulo the Levi subgroup integrating it. In the case where $\mathfrak{b} = \mathfrak{p}$ discussed above this is the action of $N = W$ on $U(\mathfrak{g}) \otimes_{Z(\mathfrak{g})} U(\mathfrak{h})$ which is trivial on the first tensor factor and is the usual (dot) action on the second. Thus, we can recover $U(\mathfrak{g})$ as the invariants of this action.

Note that we always have a surjective map of S-algebras $U(\mathfrak{g}) \otimes_{\mathbb{C}} S \to A$ as proven by Borho and Brylinski [BB82, 3.8]. Since this is a surjective map, it sends the center $Z(\mathfrak{g}) \otimes_{\mathbb{C}} S$ to the center S of A.

In this case, the map $Z(\mathfrak{g}) \cong U(\mathfrak{h})^W \to S$ is induced by the translation by ρ_P, followed by the obvious projection $\mathfrak{h} \to \mathfrak{p}/[\mathfrak{p}, \mathfrak{p}]$. That is, the induced map on spectra sends a character χ on $\mathfrak{p}$ to the W-orbit of the restriction of $\chi + \rho_P$ to $\mathfrak{h}$.

Definition 3.10. Let $W(0, \mathfrak{p})$ be the invariants of N acting on A.

Note that the natural map $U(\mathfrak{g}) \to A$ factors through $W(0, \mathfrak{p})$. Though the map $W(0, \mathfrak{p}) \hookrightarrow A$ is not surjective, it becomes so after base change to $\mathbb{C}$:

Lemma 3.11. *The algebra A_γ is naturally isomorphic to the quotient of $W(0, \mathfrak{p})$ by the maximal ideal in $Z(\mathfrak{g})$ which corresponds to the weight $\gamma + \rho_P$ under the Harish-Chandra homomorphism.*

Proof. We have a surjective map $U(\mathfrak{g}) \otimes_{\mathbb{C}} S \to A_\gamma$, sending every element of S to a scalar, so $U(\mathfrak{g}) \to A_\gamma$ must be surjective, and of course, this factors through the map $W(0, \mathfrak{p}) \to A_\gamma$. Our calculation above of the map $Z(\mathfrak{g}) \to S$ shows that the maximal ideal for the weight $\gamma + \rho_P$ is indeed killed by this map. That this gives all elements of the ideal is easily checked by considering the associated graded. $\square$

For $\mathfrak{gl}_N$, we have that $\mathfrak{p}$ is block upper triangular matrices for some composition ξ. A character is simply an assignment of a scalar to each block. We define a multiset R_i to be the set of twice the values we assign to a block of length i, and collect these into a set $\mathbf{R}$ of multisets as in Section 2. The factor of 2 is inserted to match the conventions of Section 2.

The vector ρ_P is given by

$$\frac{1}{2}(\xi_1 - 1, \xi_1 - 3, \dots, -\xi_1 + 1, \xi_2 - 1, \xi_2 - 3, \dots, -\xi_2 + 1, \dots)$$

so the weight $\gamma + \rho_P$ is a concatenation of vectors of the form $\frac{1}{2}(r + i - 1, r + i - 3, \dots, r - i + 1)$ for the different $r \in R_i$. The normalizer N acts by permuting these blocks if they have the same size, so after taking invariants for it, we need only remember the multisets $\mathbf{R}$. Thus, we will use $W(0, \mathfrak{p})_{\mathbf{R}}$ to denote this quotient of $W(0, \mathfrak{p})$. Note that if we replace GL_N with SL_N, we simply kill the kernel of the surjective map $U(\mathfrak{gl}_N) \to U(\mathfrak{sl}_N)$, which means that $\mathbf{R}$ would only be well-defined up to simultaneous translation. Alternatively, we can think about this in terms of the unique automorphism of $U(\mathfrak{gl}_N)$ which fixes $U(\mathfrak{sl}_N)$ and sends $Z_N^{(1)} \mapsto Z_N^{(1)} + k$. Thus, we have $W(0, \mathfrak{p})_{\mathbf{R}} \cong W(0, \mathfrak{p})_{\mathbf{R}+k}$ for any $k \in \mathbb{C}$.

Note that if $\mathfrak{p} = \mathfrak{b}$ is a Borel, then all blocks are of size 1 so we only have R_1. We let $U(\mathfrak{g})_{\mathbf{R}} = W(0, \mathfrak{b})_{\mathbf{R}}$. As discussed above, the quotient $U(\mathfrak{g})_{\mathbf{R}}$ can be defined by sending $Z_N^{(s)}$ to the scalar $e_s(R_1)$, that is, by sending the formal polynomial $Z_N(u) \mapsto \prod_{r \in R_1}(u + r/2)$. Our Harish-Chandra homomorphism calculation shows that:

Lemma 3.12. *The surjective map $U(\mathfrak{g}) \to W(0, \mathfrak{p})_{\mathbf{R}}$ factors through $U(\mathfrak{g})_{\widetilde{\mathbf{R}}}$ where $\widetilde{\mathbf{R}}$ satisfies the condition of* (2.12).

Remark 3.13. In this formalism, we can think of the deformation $A_{\gamma + \iota}$ as corresponding to a similar set, where we replace each complex number $r \in R_i$ with a "point" in a formal neighborhood of this point.

3.3.2 Definition of parabolic W-algebras

Now we consider W-algebra analogues of this algebra. Following the notation from Section 3.1, for any module N of a quotient of $U(\mathfrak{g}) \otimes_{\mathbb{C}} S$, we have an induced $\mathfrak{m}$-action where

$$m \cdot n = mn - \chi(m)n \text{ for all } m \in \mathfrak{m}, n \in N$$

where on the RHS, the action is the module structure. Consider the non-commutative Hamiltonian reductions

$$A(e) := \operatorname{Hom}_A\left(A/A\mathfrak{m}_\chi, A/A\mathfrak{m}_\chi\right) = \left(A/A\mathfrak{m}_\chi\right)^{\mathfrak{m}}.$$

$$W(e,\mathfrak{p}) := \operatorname{Hom}_{W(0,\mathfrak{p})}\left(W(0,\mathfrak{p})/W(0,\mathfrak{p})\mathfrak{m}_\chi, W(0,\mathfrak{p})/W(0,\mathfrak{p})\mathfrak{m}_\chi\right) = \left(W(0,\mathfrak{p})/W(0,\mathfrak{p})\mathfrak{m}_\chi\right)^{\mathfrak{m}}.$$

We can also obtain $W(e,\mathfrak{p})_\gamma, A(e)_{\gamma+\iota}$ and $\tilde{A}(e)_\gamma$ over $\mathbb{C}, \widehat{\operatorname{Sym}}(\mathfrak{p}/[\mathfrak{p},\mathfrak{p}]), K$ respectively by tensoring $A(e)$ with the appropriate base ring or by Hamiltonian reduction of the corresponding algebras when $e = 0$. Note that $W(e,\mathfrak{b}) \cong W(e)$, as defined in Section 3.1. In type A, we can use the notation $W(e)_{\mathbf{R}}, W(e,\mathfrak{p})_{\mathbf{R}}$ as in Section 3.3.1; as discussed there, these algebras only depend on $\mathbf{R}$ up to simulateous translation.

The algebra $W(e,\mathfrak{p})_\gamma$ is the sections of a quantum structure sheaf on the S3-variety $\mathfrak{X}^e_{\mathfrak{p}}$, as defined in [BLPW16, §9.2]. As proven in [Web11, Proposition 10] and [Los12, Lemma 5.2.1], the associated graded of this algebra is isomorphic to the algebra of global functions on $\mathfrak{X}^e_{\mathfrak{p}}$.

We can also write $W(e,\mathfrak{p})$ as a quotient of the finite W-algebra $W(e) \to W(e,\mathfrak{p})$ by an ideal $J_{\mathfrak{p}}$. This ideal is constructed by considering the kernel $I_{\mathfrak{p}}$ of the map $U(\mathfrak{g}) \to W(0,\mathfrak{p})$ and then applying Losev's lower dagger operation $J_{\mathfrak{p}} := (I_{\mathfrak{p}})_\dagger$ [Los10]. Note that this ideal must be prime, since $W(e,\mathfrak{p})$ is a domain. Our aim is to ultimately understand this ideal, using the geometry of G/P.

Remark 3.14. We can make a slightly cleaner statement about the classical limit of $W(e,\mathfrak{p})_\gamma$ if the natural map $T^*G/P \to \mathfrak{g}^*$ is generically injective. This is always the case in type A, but for some parabolics in other Lie algebras it fails; for the classical groups, a criterion for this property is given by Hesselink [Hes78, Theorem 7.1]. In this case, the obvious map induces an isomorphism $\mathbb{C}[\mathfrak{X}^e_{\mathfrak{p}}] \cong \mathbb{C}[\mathcal{S}_e \cap (G \cdot \mathfrak{p}^\perp)]$. Thus, in this case, we can think of $W(e,\mathfrak{p})_\gamma$ as a quantization of the latter variety.

As discussed in [Los12, Remark 5.2.2], this also manifests in the natural map $W(e)_\gamma \to W(e,\mathfrak{p})_\gamma$ failing to be surjective on the associated graded for the most obvious filtrations on these algebras.

Let $(\tilde{\mathscr{D}}_\gamma, \mathfrak{m}_\chi)$-mod denote the category of sheaves of $\tilde{\mathscr{D}}_\gamma$ modules on which the module action of $\mathfrak{m}_\chi$ integrates to a group action of the unipotent group M. Let $\mathscr{Q}_\pi = \mathscr{D}_\gamma/\mathscr{D}_\gamma\mathfrak{m}_\chi$. Combining Theorem 3.9 and [Web11, Prop. 10], we obtain the result:

Corollary 3.15. *The functor* $W\Gamma = \operatorname{Hom}(\mathscr{Q}_\pi, -)\colon (\tilde{\mathscr{D}}_\gamma, \mathfrak{m}_\chi)\text{-mod} \to \tilde{A}(e)_\gamma\text{-mod}$ *is an equivalence of categories.*

3.3.3 Highest weights in type A

We now return to the type A setting, and set $\mathfrak{g} = \mathfrak{gl}_N$. In the notation of Section 4.1, we let $e_\pi \in \mathfrak{g}$ be the nilpotent of type π, and let ξ be the composition whose parts are equal to those of λ^t. Fix a character $\gamma \colon \mathfrak{p} \to \mathbb{C}$ and the corresponding $\mathbf{R}$ as defined in the previous section. Take $P \subset SL_N$ to be the parabolic subgroup corresponding to ξ. We set $W(\pi, \mathfrak{p}) = W(e_\pi, \mathfrak{p}), A(\pi) = A(e_\pi), W(\pi, \mathfrak{p})_{\mathbf{R}} = W(e_\pi, \mathfrak{p})_{\mathbf{R}}$, etc.

Assume $w \in W = S_N$ is simultaneously a longest left coset representative of W_π and a shortest right coset representative for W_ξ. We call such a permutation **parabolic-singular**, and let $\mathcal{PS}(\pi, \mathfrak{p})$ be the set of such permutations. Note that such a representative does not exist for every double coset, and in fact $\mathcal{PS}(\pi, \mathfrak{p})$ is non-empty if and only if $\pi \leq \lambda$ in dominance order.

Recall that if A is any algebra with a grading by $\mathbb{Z}$, then we define the B-algebra with respect to this grading to be the quotient $B(A) = A^0 / \sum_{k \in \mathbb{Z}_{>0}} A^{-k} A^k$. Let $W(\pi, \mathfrak{p})$ have the grading induced by eigenvalues of $\rho^\vee - \rho_\pi^\vee$. By [BLPW16, 5.1], this algebra is finite over the center S.

The B-algebra is compatible with base change to an S-algebra. We'll be particularly interested in $B(W(\pi, \mathfrak{p})_{\mathbf{R}})$ and $B(\tilde{A}(\pi)_\gamma)$; these are the base change of $B(A(\pi))$ to the closed point γ and to the generic point of its formal neighborhood respectively. Thus, given a point $\nu \in \operatorname{Spec} B(\tilde{A}(\pi)_\gamma)$, we can takes its Zariski closure in $\operatorname{Spec} B(A(\pi))$ and intersect that with $\operatorname{Spec} B(W(\pi, \mathfrak{p})_{\mathbf{R}})$. Since $\operatorname{Spec} B(A(\pi))$ is finite and thus proper over S, this intersection will be a single point, which we call its **specialization**. For a general finite map, we could have points in $\operatorname{Spec} B(W(\pi, \mathfrak{p})_{\mathbf{R}})$ which are not the specialization of a more generic point, but this will not happen if $B(W(\pi, \mathfrak{p}))$ is free as a module over S (or equivalently, flat over S).

Lemma 3.16. *The B algebra $B(W(\pi, \mathfrak{p}))$ is free of rank $\mathcal{PS}(\pi, \mathfrak{p})$ as an S-module. Thus, a weight for $W(\pi)_{\mathbf{R}}$ is the highest weight of a module over $W(\pi, \mathfrak{p})_{\mathbf{R}}$ if and only if it is the specialization of the highest weight of a module over $\tilde{A}(\pi)_\gamma$.*

Proof. In order to show that an S-algebra is free of a given rank, it suffices to check that it has this rank generically, and that there is no closed point where the rank of the base change is larger.

Let $B := B(A(\pi))$. The base change of B to the generic point $B \otimes_S K$ has dimension equal to $\#\mathcal{PS}(\pi, \mathfrak{p})$ by [BLPW16, 5.3], since the variety $\mathfrak{X}$ has a torus action with fixed points in bijection with $\mathcal{PS}(\pi, \mathfrak{p})$.

On the other hand, as argued in [BLPW16, 5.1], the dimension of $B(W(\pi, \mathfrak{p})_{\mathbf{R}})$ is bounded above by the "commutative B-algebra": the quotient of $\mathbb{C}[\mathfrak{X}]$ by the ideal generated by functions of non-zero weight. By [Hik17, A.1 & 2], this has dimension equal the Euler characteristic of another S3 variety, taking the Slodowy slice to a regular element in the Levi $\mathfrak{l}$ of $\mathfrak{p}$ (which thus has Jordan type η corresponding to the diagonal blocks of $\mathfrak{l}$), and G/Q, where Q is a parabolic with e regular in its Levi (so of type π). Thus $\mathfrak{p}$ and π essentially switch roles. We can obtain a bijection between $\mathcal{PS}(\pi, \mathfrak{p})$ and $\mathcal{PS}(\eta, \mathfrak{q})$ by taking inverse and multiplying by w_0. Note that this requires reversing the order on blocks of η but this order is immaterial, so this presents no issue. Note the appearance of the same reversal in [BLPW16, 10.4-5]. Thus $B(W(\pi, \mathfrak{p})_{\mathbf{R}}) \leq \#\mathcal{PS}(\pi, \mathfrak{p})$

Since the dimension of the fiber is a lower semi-continuous function, this shows that this dimension must be constant, and by the usual argument, B must be a free S-algebra. ∎

Thus, we can use localization to find the highest weights of modules over $\tilde{A}(\pi)_\gamma$, and thus over $W(\pi,\mathfrak{p})_{\mathbf{R}}$. Note that the parabolic-singular permutations are precisely the shortest right coset representatives such that M acts freely on the Schubert cell NwP/P. These are precisely the Schubert cells that carry a (M,χ)-equivariant local system $\mathscr{L}_w$; consider the D-modules on G/P given by $\delta_w = i_!\mathscr{L}_w, \nabla_w = i_*\mathscr{L}_w$.

Lemma 3.17. *The highest weight of $W\Gamma(\nabla_w)$ over the torus $\mathfrak{c}$ is given by*

$$w(\gamma+\rho_P)+\rho = ww_0(\gamma-\rho_P)+\rho.$$

Proof. This is analogous to [HTT08, 12.3.1]. Let $w \in \mathcal{PS}(\pi,\mathfrak{p})$ be a parabolic-singular permutation. The module ∇_w is a pushforward $\iota\colon ww_0Nw_0P/P \hookrightarrow G/P$, and thus, we can just compute the pushforward on this open subvariety. We can identify $wU_-P/P \cong \mathrm{Ad}_w(\mathfrak{u}_-)$, with the subvariety NwP/P sent to $\mathfrak{n}\cap\mathrm{Ad}_w(\mathfrak{u}_-) = \mathfrak{n}\cap\mathrm{Ad}_w(\mathfrak{n}_-)$ since w is a shortest right coset representative. Now we enumerate the roots in $\mathrm{Ad}_w(\mathfrak{u}_-)$ by $\beta_1,\dots,\beta_N$, with the first k roots $\{\beta_1,\dots,\beta_k\}$ being those that are positive. Let x_i denote the corresponding coordinates on $\mathrm{Ad}_w(\mathfrak{u}_-)$ and y_i the dual basis of $\mathrm{Ad}_w(\mathfrak{u}_-)$. We can also assume that $\{\beta_1,\dots,\beta_p\}$ for some $p\leq k$ are the (necessarily simple) weight spaces on which χ is non-zero. The fact that w is a longest left coset representative guarantees that these are any such weight space lies in $\mathrm{Ad}_w(\mathfrak{n}_-)$, so the parabolic-singular property shows that these lie in $\mathrm{Ad}_w(\mathfrak{u}_-)$.

Thus, we can identify the pushforward of $\mathscr{L}_w$ to this affine space with the module over the Weyl algebra $\mathbb{W}=\mathbb{C}[\mathfrak{u}_-]\otimes\mathrm{Sym}(\mathfrak{u}_-)$ of $\mathfrak{u}_-$ which is generated by a single element e^χ with the relations $\frac{\partial}{\partial x_i}e^\chi = \chi(y_i)e^\chi$ for $i=1,\dots,k$, and $x_i\cdot e^\chi = 0$ for $i=k+1,\dots,n$. The function e^χ generates the Whittaker functions under multiplication by functions which are constant on M-orbits and multiplication by constant vector fields. In these coordinates, we have that $\mathfrak{c}$ acts by the Euler operator

$$h\mapsto w(\gamma+\rho-\rho_P)(h)+\sum_{i=1}^{N}\beta_i(h)x_i\frac{\partial}{\partial x_i}. \tag{3.9}$$

Note that since $\mathfrak{c}$ commutes with $\mathfrak{m}$, we have that $\beta_i(h)=0$ if $i\leq p$. On the function e^χ, we have that

$$x_i\frac{\partial}{\partial x_i}e^\chi = \begin{cases}\chi(y_i)x_ie^\chi & i\leq p\\ 0 & p<i\leq k\\ -\beta_i(h) & k<i\end{cases}$$

Thus, equation (3.9) becomes

$$h\cdot e^\chi \mapsto (w(\gamma+\rho-\rho_P)(h)+\sum_{i=k+1}^{n}\beta_i(h))e^\chi. \tag{3.10}$$

Note here that β_i ranges over the roots in $\mathrm{Ad}_w(\mathfrak{u}_-)\cap\mathfrak{n}_- = \mathrm{Ad}_{ww_0^P}(\mathfrak{n}_-)\cap\mathfrak{n}_-$, so the sum is $\rho - ww_0^P\rho = \rho - w\rho + 2w\rho_P$. Thus, we have that the weight of e^χ is $w(\gamma+\rho_P)+\rho = ww_0(\gamma-\rho_P)+\rho$. □

We call $\gamma\in\mathfrak{h}^*\otimes_{\mathbb{C}} S$ **row-sum-distinct** if the restrictions $w\cdot(\gamma+\rho_P)|_{\mathfrak{c}}$ are distinct for $w\in\mathcal{PS}(\pi,\mathfrak{p})$. Note, this is stronger than having stabilizer W_P, as the case of $\pi=(2,2)$ and

$\chi = (4,3,2,1)$ shows: the permutations $(1,4,2,3)$ and $(2,3,1,4)$ have the same restriction. For a fixed P, this is an open condition determined by finitely many hyperplanes.

In particular, the weight $\gamma + \iota$ for $\gamma \in \mathfrak{h}^*$ is always row-sum-distinct, since if $w \cdot (\gamma + \iota + \rho_P)|_{\mathfrak{c}} = w' \cdot (\gamma + \iota + \rho_P)|_{\mathfrak{c}}$ for $w \neq w' \in \mathcal{PS}(\pi, \mathfrak{p})$, then we must have $w \cdot (\gamma + \rho_P)|_{\mathfrak{c}} = w' \cdot (\gamma + \rho_P)|_{\mathfrak{c}}$ and thus $w \cdot \iota|_{\mathfrak{c}} = w' \cdot \iota|_{\mathfrak{c}}$. In this case, w and w' are in the same double coset, and thus must be equal (since $w_\pi^0 w$ is a maximal length representative of the double coset). This actually shows something stronger: the difference $w \cdot (\gamma + \rho_P)|_{\mathfrak{c}} - w' \cdot (\gamma + \rho_P)|_{\mathfrak{c}}$ is never an integral weight (since it is never a complex-valued weight).

Given $\mathbf{R}$, let γ be a vector such that for each element $c \in \mathbf{R}_i$, we have c appears i times, and we let $\mathfrak{p}$ be the corresponding parabolic. For a given $w \in \mathcal{PS}(\pi, \mathfrak{p})$, we consider the weight $w \cdot (\gamma + \rho_P)$; we let

$$\mathsf{T}_w \in Row_{\widetilde{\mathbf{R}}}(\pi) \tag{3.11}$$

be the row-symmetrized tableau of type π which has $w \cdot (\gamma + \rho_P)$ as a row reading (this corresponds to a filling in the alphabet $\widetilde{\mathbf{R}}$). We let L_w be the simple module attached to this tableau by Brundan and Kleshchev. We let $\widetilde{\mathsf{T}}_w$ and $\widetilde{L}_w, \widetilde{\nabla}_w$ be corresponding objects for $\gamma + \iota$, base changed to K.

Lemma 3.18. *We have an isomorphism $\widetilde{L}_w \cong W\Gamma(\widetilde{\nabla}_w)$.*

Proof. By row-sum-distinctness, each of the simples $\widetilde{L}_w$ have distinct highest weights for $\mathfrak{c}$, as do $W\Gamma(\tilde{\nabla}_w)$. In fact, since the weights of different $W\Gamma(\tilde{\nabla}_w)$'s are never congruent modulo integral weights, there are no $\mathfrak{c}$-equivariant maps between them, and thus no $W(\pi, \mathfrak{p})$-equivariant ones. Thus, $W\Gamma(\tilde{\nabla}_w)$ will be simple, and isomorphic to whichever of the modules $\tilde{L}_{w'}$ has the same highest weight for $\mathfrak{c}$. By construction, this is $\tilde{L}_w$. ☐

Let $\gamma \colon \mathfrak{p} \to \mathbb{C}$ be an arbitrary character.

Theorem 3.19. *The highest weights of modules in category $\mathcal{O}$ over $W(\pi, \mathfrak{p})_{\mathbf{R}}$ are given by the tableaux T_w for $w \in \mathcal{PS}(\pi, \mathfrak{p})$.*

Proof. First we check this for $\gamma + \iota$ after base change to K. In this case, the simple modules are given by $\tilde{L}_w$. By Lemma 3.16, the simples at γ have highest weights obtained by specialization, that is, they are L_w. ☐

Lemma 3.20. *The action of $W(\pi, \mathfrak{p})_{\mathbf{R}}$ on $W\Gamma(\nabla_w)$ is faithful.*

Proof. The module ∇_w is a naive pushforward from the open subset $ww_0Nw_0P/P \subset G/P$, so we can show faithfulness on this open subset. On this open subset, ∇_w is the pushforward of the Whittaker functions on an affine subspace, which is faithful. ☐

A standard argument shows that a faithful module of finite length over a domain must have a faithful composition factor. Equivalently, we have that $J_{\mathfrak{p}}$ is the intersection of the annihilators of $W(\pi)_{\widetilde{\mathbf{R}}}$ acting on the composition factors of $W\Gamma(\nabla_w)$. Thus, we have:

Theorem 3.21. *The algebra $W(\pi, \mathfrak{p})_{\mathbf{R}}$ acts faithfully on at least one L_w for $w \in \mathcal{PS}(\pi, \mathfrak{p})$; that is*

$$W(\pi, \mathfrak{p})_{\mathbf{R}} \cong W(\pi)_{\widetilde{\mathbf{R}}} / \bigcap_{w \in \mathcal{PS}(\pi, \mathfrak{p})} \mathrm{Ann}(L_w).$$

4 The quantized Mirković-Vybornov isomorphism

4.1 The main theorem

Recall our notation from 1.2: $\lambda \geq \mu$ are dominant coweights of $\mathfrak{g}$, and $\tau \geq \pi$ are partitions of N.

Recall from (2.2) that for $\Lambda \in \mathsf{Gr}^{\overline{\lambda}}_{\mu}$, we have $\Lambda \subset \Lambda_0$ with $\Lambda_0/\Lambda \cong \mathbb{C}^N$ via the basis E_π. Furthermore the operator on Λ_0/Λ induced by multiplication by t is nilpotent of type $\leq \tau$. Therefore we have a map $\mathsf{Gr}^{\overline{\lambda}}_{\mu} \to \overline{\mathbb{O}_\tau} \subset \mathfrak{gl}_N$.

Theorem 4.1 ([MV07b]). *The map sending $\Lambda \in \mathsf{Gr}^{\overline{\lambda}}_{\mu}$ to the action of t on $\Lambda_0/\Lambda \cong \mathbb{C}^N$ defines an isomorphism of varieties*

$$\mathsf{Gr}^{\overline{\lambda}}_{\mu} \xrightarrow{\sim} \mathcal{T}_\pi \cap \overline{\mathbb{O}_\tau}$$

for a suitable transverse slice $\mathcal{T}_\pi$ to $\mathbb{O}_\pi \subset \overline{\mathbb{O}_\tau}$. Moreover, the following diagram commutes:

$$\begin{tikzcd} \mathsf{Gr}^{\overline{N\varpi_1}}_{\mu} \arrow[r, "\sim"] & T_\pi \cap \mathcal{N}_{\mathfrak{gl}_N} \\ \mathsf{Gr}^{\overline{\lambda}}_{\mu} \arrow[u, hook] \arrow[r, "\sim"] & T_\pi \cap \overline{\mathbb{O}_\tau} \arrow[u, hook] \end{tikzcd}$$

where the vertical arrows are the inclusions of closed subvarieties.

We review this theorem in more detail in Section 5.3 below.

Remark 4.2. In this paper, we use a formulation of the above result above due to Cautis-Kamnitzer [CK08, Section 3.3]. It is somewhat simpler than the original construction of Mirković-Vybornov. It is possible to modify the results of this paper to precisely match the original Mirković-Vybornov isomorphism, however this comes at the cost of less pleasant maps of algebras and associated combinatorics.

We note that although in general $\mathcal{T}_\pi$ differs from the better-known Slodowy slice, these are isomorphic as Poisson varieties (cf. Section 5.1). Therefore the above theorem implies that $\mathcal{S}_\pi \cap \overline{\mathbb{O}_\tau} \cong \mathsf{Gr}^{\overline{\lambda}}_{\mu}$, where $\mathcal{S}_\pi$ is the Slodowy slice.

Now, on the one hand $\mathsf{Gr}^{\overline{\lambda}}_{\mu}$ is quantized by $Y^{\lambda}_{\mu}(\mathbf{R})$ for any set of parameters $\mathbf{R}$. On the other hand, $\mathcal{S}_\pi \cap \overline{\mathbb{O}_\tau}$ is quantized by $W(\pi, \mathfrak{p})_{\mathbf{R}}$. Our main result shows that we can lift the isomorphism of Mirković and Vybornov to the level of quantizations.

Theorem 4.3.

(a) There is an isomorphism of filtered algebras

$$\Phi : Y^{N\varpi_1}_{\mu} \xrightarrow{\sim} W(\pi)$$

compatible with specialization of parameters on both sides.

(b) *Φ induces a bijection between highest weight modules over $Y_\mu^{N\varpi_1}$ and $W(\pi)$, such that*

$$\begin{array}{ccc} \mathcal{B}(\widetilde{\mathbf{R}}) & \xrightarrow{\sim} & Row_{\mathbf{R}}(\pi) \\ \cup\!\shortmid & & \cup\!\shortmid \\ \mathcal{B}(\mathbf{R}) & \xrightarrow{\sim} & Row_{\widetilde{\mathbf{R}}}(\pi)^{\circ} \end{array}$$

when $\widetilde{\mathbf{R}}$ and $\mathbf{R}$ are related as in (2.12). (Here $Row_{\widetilde{\mathbf{R}}}(\pi)^{\circ}$ is a set of row symmetrized π–tableaux parametrizing highest weights for $W(\pi,\mathfrak{p})_{\mathbf{R}}$, see Section 4.3.2)

(c) *Φ induces isomorphisms of filtered algebras $Y_\mu^\lambda(\mathbf{R}) \xrightarrow{\sim} W(\pi,\mathfrak{p})_{\mathbf{R}}$, via the commutative diagram*

$$\begin{array}{ccc} Y_\mu^{N\varpi_i}(\widetilde{\mathbf{R}}) & \xrightarrow{\sim} & W(\pi)_{\widetilde{\mathbf{R}}} \\ \downarrow\!\!\!\downarrow & & \downarrow\!\!\!\downarrow \\ Y_\mu^\lambda(\mathbf{R}) & \xrightarrow{\sim} & W(\pi,\mathfrak{p})_{\mathbf{R}} \end{array}$$

(d) *The classical limit agrees with the MV isomorphism.*

We will split the proof of this theorem into parts, which occupy the remainder of the paper. In Sections 4.2, 4.3 and 5, we will prove parts (a), (b) and (d) of this theorem, respectively. Part (c) follows from parts (a), (b) by a simple argument, as we will show presently.

This linkage uses the quotient maps:

$$Y_\mu^{N\omega_1} \to Y_\mu^\lambda \qquad W(\pi) \to W(\pi,\mathfrak{p}), \tag{4.1}$$

introduced in Sections 2.5 and 3.3.2 respectively. Note that these maps are not surjective, but the sources sit as subalgebras in the quotients

$$Y_\mu^{N\omega_1}(\tilde{\gamma}) \to Y_\mu^\lambda(\gamma) \qquad A(\pi,\mathfrak{b}) \to A(\pi,\mathfrak{p}), \tag{4.2}$$

and generate each of these algebras over its center so we can consider the algebras $Y_\mu^{N\omega_1}(\tilde{\gamma})$ and $A(\pi,\mathfrak{b})$ in their stead. If we show that we have an isomorphism $Y_\mu^\lambda(\gamma) \cong A(\pi,\mathfrak{p})$, we thus obtain the desired map in Theorem 4.3(c).

Proof of Theorem 4.3(c) (assuming parts (a) and (b)). In order to complete the proof, we need only show that the kernels in (4.1) match under Φ after base change at these maximal ideals. By Theorem 3.21, the kernel of the latter map is the intersection of the annihilators of the simple modules over $Y_\mu^\lambda(\gamma)$. By Theorem 4.3(b) and Corollary 2.11, we thus obtain an induced surjective map $Y_\mu^\lambda(\gamma) \to A(\pi,\mathfrak{p})$.

Since each piece of the filtration of $Y_\mu^\lambda(\gamma)$ and $A(\pi,\mathfrak{p})$ is finite dimensional, if the map Φ is not a filtered isomorphism, it will fail to be an isomorphism after specialization at a maximal ideal in the center. Thus, we can consider the quotient $Y_\mu^\lambda(\mathbf{R})$ with $\mathbf{R}$ giving a maximal ideal of the center, and the corresponding quotient of $W(\pi,\mathfrak{p})_{\mathbf{R}}$.

When we take associated graded of both sides, we obtain the functions on $\mathsf{Gr}^{\bar{\lambda}}_\mu$ (Theorem 2.4) and $\mathcal{S}_\pi \cap \overline{\mathbb{O}_\tau}$, respectively. Both are irreducible varieties of the same dimension, thus a surjective ring map from one to the other must be an isomorphism. □

We note an immediate corollary about the original Mirković-Vybornov isomorphism:

Corollary 4.4. *$\mathsf{Gr}^{\overline{\lambda}}_\mu$ and $\mathcal{T}_\pi \cap \overline{\mathbb{O}_\tau}$ are isomorphic as Poisson varieties. This isomorphism intertwines the $\mathbb{C}^\times$–action by loop rotation on $\mathsf{Gr}^{\overline{\lambda}}_\mu$ with the square root of the Kazhdan action on $\mathcal{T}_\pi \cap \overline{\mathbb{O}_\tau}$ (see Remark 3.6).*

4.2 Proof of Theorem 4.3(a): The case of $\lambda = N\varpi_1$

In this section, we will consider the case where $\lambda = N\varpi_1$ and μ is a dominant weight such that $\mu \leq \lambda$. From this data, we have a partition $\pi \vdash N$ as in Section 1.2. We'll describe (Theorem 4.7) an isomorphism between $Y^{N\varpi_1}_\mu$ and $W(\pi)$.

To state the theorem precisely, first we need to define a map

$$\phi : Y_\mu \to SY_n(\sigma) \tag{4.3}$$

by

$$\begin{aligned} H_i(u) &\mapsto \frac{(u+\frac{i-1}{2})^{\mu_i}}{u^{\mu_i}} \frac{D_{i+1}(-u-\frac{i-1}{2})}{D_i(-u-\frac{i-1}{2})} \\ E_i(u) &\mapsto E_i(-u-\tfrac{i-1}{2}) \\ F_{\mu,i}(u) &\mapsto (-1)^{\mu_i} F_{\mu,i}(-u-\tfrac{i-1}{2}) \end{aligned}$$

for $i \in I$. Here $F_{\mu,i}(u) = \sum_{r>0} F_i^{(\mu_i+r)} u^{-r}$, on each side.

Proposition 4.5. *The map ϕ is an isomorphism of filtered algebras.*

For the proof, we will make use of the following lemma regarding "non-standard" embeddings of the shifted Yangian $Y_\mu \hookrightarrow Y$:

Lemma 4.6. *Fix a monic polynomial*

$$Q_i(u) = u^{\mu_i} + Q_i^{(1)} u^{\mu_i - 1} + \ldots + Q_i^{(\mu_i)} \in \mathbb{C}[u]$$

for each $i = 1, \ldots, n-1$. There is a corresponding embedding $Y_\mu \hookrightarrow Y$, defined on the generators by

$$\begin{aligned} &E_i^{(r)} \mapsto E_i^{(r)}, \\ &H_i^{(r)} \mapsto H_i^{(r)} + Q_i^{(1)} H_i^{(r-1)} + \ldots + Q_i^{(\mu_i)} H_i^{(r-\mu_i)}, \\ &F_i^{(s)} \mapsto F_i^{(r)} + Q_i^{(1)} F_i^{(s-1)} + \ldots + Q_i^{(\mu_i)} F_i^{(s-\mu_i)} \end{aligned}$$

for all $r > 0$ and $s > \mu_i$, and where we interpret $H_i^{(0)} = 1$ and $H_i^{(r)} = 0$ for $r < 0$.

Proof. Assuming that this map defines a homomorphism, it is easy to see that it is an embedding: its associated graded agrees with that of the defining embedding $Y_\mu \subset Y$.

To prove that it is a homomorphism, one can verify the relations directly; we give a different argument. By [KTW$^+$, Lemma 3.7], Y_μ is a left coideal of Y with respect to its defining embedding $Y_\mu \subset Y$ (see Definition 2.1). By [KTW$^+$, Proposition 3.8], there is a 1-dimensional module $\mathbb{C}\mathbf{1}_Q$ for Y_μ determined by the polynomials $Q_i(u)$. We can then consider

$$Y_\mu \xrightarrow{\Delta} Y \otimes Y_\mu \longrightarrow Y \otimes \mathrm{End}(\mathbb{C}\mathbf{1}_Q) \cong Y$$

The composition is precisely the claimed homomorphism. $\square$

Proof of Proposition 4.5. When $\mu = 0$ the fact that this map defines an isomorphism $Y \xrightarrow{\sim} SY_n$ follows from [BK05, Remark 5.12], after a minor modification: here we are following Drinfeld's conventions as opposed to the "opposite" presentation of [BK05].

When $\mu \neq 0$, consider the composition

$$Y_\mu \hookrightarrow Y \xrightarrow{\sim} SY_n$$

where the second arrow is the above $\mu = 0$ isomorphism, while the first arrow is the embedding from the previous lemma for the polynomials $Q_i(u) = (u + \frac{i-1}{2})^{\mu_i}$. This map $Y_\mu \hookrightarrow SY_n$ agrees with ϕ on the generators of Y_μ, and its image is precisely $SY_n(\sigma)$.

$\square$

Recall the algebra $Y_\mu[R^{(j)}]$ from (2.3); note that as $\lambda = N\varpi_1$ we only adjoin variables $R^{(j)} := R^{(j)}_{n-1}$ for $j = 1, \ldots, N$. We extend ϕ to an isomorphism $\phi : Y_\mu[R^{(j)}] \to SY_n(\sigma) \otimes \mathbb{C}[Z^{(1)}, ..., Z^{(N)}]$, where the $Z^{(j)}$ are formal variables. On the central generators ϕ is defined by the equation

$$(-1)^N R(-u + \tfrac{n}{2}) \mapsto u^N + Z^{(1)} u^{N-1} + \cdots + Z^{(N)} =: Z_N(u) \tag{4.4}$$

We now consider the following diagram:

$$\begin{tikzcd}
 & Y_n(\sigma) \arrow[d, "\psi"] \arrow[dd, bend left=80, "\kappa"] \\
Y_\mu[R^{(j)}] \arrow[r, "\phi"] \arrow[d, "\tau"] & SY_n(\sigma) \otimes \mathbb{C}[Z^{(1)}, ..., Z^{(N)}] \arrow[d, "\xi"] \\
Y_\mu^{N\varpi_1} \arrow[r, dashed, "\Phi"] & W(\pi)
\end{tikzcd}$$

Here $\tau : Y_\mu[R^{(j)}] \to Y_\mu^{N\varpi_1}$ and $\kappa : Y_n(\sigma) \to W(\pi)$ are the quotient maps. The map ψ is the identity on $SY_n(\sigma)$ and on the center is defined by the equation

$$Z_N(u) = u^{p_1}(u-1)^{p_2} \cdots (u-n+1)^{p_n} \psi\big(Q_n(u)\big). \tag{4.5}$$

The map ξ is equal to κ on $SY_n(\sigma)$ and on the center is defined by the equation

$$\xi(Z_N(u)) = u^{p_1}(u-1)^{p_2} \cdots (u-n+1)^{p_n} \kappa\big(Q_n(u)\big).$$

Note that by [BK08, Lemma 3.7] the right hand side of the above equation is a polynomial in u of degree N, and hence ξ is a well-defined surjection. By construction we have that $\kappa = \xi \circ \psi$.

Theorem 4.7. *We have that $\phi(I_\mu^\lambda) = \ker(\xi)$ and therefore ϕ descends to an isomorphism $\Phi : Y_\mu^{N\varpi_1} \to W(\pi)$ of filtered algebras. The map Φ induces an isomorphism $Y_\mu^{N\varpi_1}(\mathbf{R}) \cong W(\pi)_{\mathbf{R}}$.*

The proof of Theorem 4.7 will be given in the next section. Note that the final claim, relating central quotients, is clear from (4.4) and the above discussion.

4.2.1 Proof of Theorem 4.7

Our first order of business is to determine the image of $A_i^{(\ell)}$ under ϕ. From the identity $D_i(u) = \frac{Q_i(u+i-1)}{Q_{i-1}(u+i-1)}$ of (3.5) we obtain that

$$\frac{D_{i+1}(-u-\frac{i-1}{2})}{D_i(-u-\frac{i-1}{2})} = \frac{Q_{i-1}(-u+\frac{i-1}{2})}{Q_i(-u+\frac{i-1}{2})}\frac{Q_{i+1}(-u+\frac{i+1}{2})}{Q_i(-u+\frac{i+1}{2})}.$$

and hence the image

$$\phi\big(H_i(u)\big) = \frac{(u+\frac{i-1}{2})^{\mu_i}}{u^{\mu_i}}\frac{\psi(Q_{i-1}(-u+\frac{i-1}{2}))}{\psi(Q_i(-u+\frac{i-1}{2}))}\frac{\psi(Q_{i+1}(-u+\frac{i+1}{2}))}{\psi(Q_i(-u+\frac{i+1}{2}))} \tag{4.6}$$

The next result is analogous to Corollary 2.13:

Lemma 4.8. *There exist unique series $s_i(u) \in \mathbb{C}[Z^{(1)},\dots,Z^{(N)}][[u^{-1}]]$ with constant term 1 such that*

$$\phi(A_i(u)) = s_i(u)\psi\big(Q_i(-u+\tfrac{i-1}{2})\big)$$

for $i = 1,\dots,n-1$. These satisfy the equations

$$\phi\big(r_i(u)\big)\frac{s_{i-1}(u-\frac{1}{2})s_{i+1}(u-\frac{1}{2})}{s_i(u)s_i(u-1)} = \frac{(u+\frac{i-1}{2})^{\mu_i}}{u^{\mu_i}} \tag{4.7}$$

for $i = 1,\dots,n-2$, and

$$\phi(r_{n-1}(u)\frac{s_{n-2}(u-\frac{1}{2})}{s_{n-1}(u)s_{n-1}(u-1)} = \frac{(u+\frac{n-2}{2})^{\mu_{n-1}}}{u^{\mu_{n-1}}}\psi\big(Q_n(-u+\tfrac{n}{2})\big) \tag{4.8}$$

Moreover, these equations determine the $s_i(u)$ uniquely.

Proof. For each i, we have two factorizations for $\phi\big(H_i(u)\big)$: one in terms of $\phi\big(r_i(u)\big)$ and the $\phi\big(A_j(u)\big)$ by (2.4), and one in terms of the $\psi\big(Q_j(-u)\big)$ by (4.6) (with appropriate shifts in u in both cases). The claim now follows by applying the uniqueness of such factorizations [GKLO05, Lemma 2.1]. ∎

Lemma 4.9. *For $i = 1,\dots,n-1$,*

$$s_i(u) = \frac{(u-\frac{i-1}{2})^{p_1}(u-\frac{i-3}{2})^{p_2}\cdots(u+\frac{i-1}{2})^{p_i}}{u^{m_i}} \tag{4.9}$$

Proof. Denote the right-hand side of 4.9 by $x_i(u)$. By the previous lemma, it suffices to show that the $x_i(u)$ satisfy the equations (4.7), (4.8).

For the case of equation (4.8), the left-hand side is

$$\phi\big(r_{n-1}(u)\big)\cdot\frac{x_{n-2}(u-\frac{1}{2})}{x_{n-1}(u)x_{n-1}(u-1)}$$

$$=u^{-N}R(u)\frac{(1-\frac{1}{2}u^{-1})^{m_{n-2}}}{(1-u^{-1})^{m_{n-1}}}\cdot\frac{u^{m_{n-1}}(u-1)^{m_{n-1}}}{(u-\frac{1}{2})^{m_{n-2}}(u+\frac{n-2}{2})^{p_{n-1}}\prod_{j=1}^{n-1}(u-\frac{n}{2}+j-1)^{p_j}}$$

after cancelling common factors between $x_{n-2}(u-\frac{1}{2})$ and $x_{n-1}(u)$. This reduces to

$$\frac{R(u)}{u^{p_n-p_{n-1}}}\frac{1}{(u+\frac{n-2}{2})^{p_{n-1}}\prod_{j=1}^{n-1}(u-\frac{n}{2}+j-1)^{p_j}}$$

Now consider the right hand side of (4.8). Applying (4.5) and (4.4), we get

$$\begin{aligned}&\frac{(u+\frac{n-2}{2})^{\mu_{n-1}}}{u^{\mu_{n-1}}}\phi\big(Q_n(-u+\tfrac{n}{2})\big)\\ &=\frac{(u+\frac{n-2}{2})^{p_n-p_{n-1}}}{u^{p_n-p_{n-1}}}\frac{Z_N(-u+\frac{n}{2})}{(-u+\frac{n}{2})^{p_1}(-u+\frac{n-2}{2})^{p_2}\cdots(-u-\frac{n-2}{2})^{p_n}}\\ &=\frac{(u+\frac{n-2}{2})^{p_n-p_{n-1}}}{u^{p_n-p_{n-1}}}\frac{(-1)^N\phi\big(R(u)\big)}{(-1)^N(u-\frac{n}{2})^{p_1}(u-\frac{n-2}{2})^{p_2}\cdots(u+\frac{n-2}{2})^{p_n}}\end{aligned}$$

and we see that the right and left sides agree.

Verifying that the $x_i(u)$ satisfy equation (4.7) for $1\le i<n-1$ is analogous, and is left as an exercise to the reader. □

Lemma 4.10. $\phi(I_\mu^\lambda)\subset\ker(\xi)$

Proof. Combining the two lemmas,

$$\phi\big(A_i(u)\big)=\frac{(u-\frac{i-1}{2})^{p_1}(u-\frac{i-3}{2})^{p_2}\cdots(u+\frac{i-1}{2})^{p_i}}{u^{m_i}}\psi\big(Q_i(-u+\tfrac{i-1}{2})\big).$$

By Theorem 3.5 in [BK08] we have that $\kappa\big(T_{\ell k}^{(r)}\big)=0$ for $r>p_k$. Therefore for $k=1,...,n$

$$\frac{(u-\frac{i-1}{2}+k-1)^{p_k}}{u^{p_k}}\kappa\big(T_{\ell k}(-u+\tfrac{i-1}{2}-k+1)\big)$$

is a polynomial in u^{-1} of degree p_k. Observe by (3.4) that

$$\begin{aligned}&\xi\circ\psi\big(Q_i(-u+\tfrac{i-1}{2})\big)\\ &\qquad=\sum_{w\in S_{n-i}}(-1)^w\kappa\big(T_{w(1),1}(-u+\tfrac{i-1}{2})\big)\cdots\kappa\big(T_{w(i),i}(-u+\tfrac{i-1}{2}-i+1)\big)\end{aligned}$$

Since $p_1+\cdots+p_{n-i}=m_i$, it follows that $\xi\circ\phi\big(A_i(u)\big)$ is a polynomial in u^{-1} of degree m_i. This proves the claim. □

Lemma 4.11. $\phi(I_\mu^\lambda)\supset\ker(\xi)$

Proof. By Lemmas 4.8 and 4.9, we have

$$\phi(A_1(u)) = s_1(u)\psi\big(Q_1(-u)\big) = \psi\big(Q_1(-u)\big)$$

Noting that $D_1(u) = Q_1(u)$, it follows that $\psi\big(D_1^{(p)}\big) = (-1)^r\phi\big(A_1^{(r)}\big)$.

By definition $\ker(\kappa) = \langle D_1^{(r)} : r > p_1\rangle$, so

$$\ker(\xi) = \psi(\ker(\kappa)) = \langle \psi\big(D_1^{(r)}\big) : r > p_1\rangle$$

Since $p_1 = m_1$, the elements $\phi\big(A_1^{(r)}\big) \in \phi\big(I_\mu^\lambda\big)$ for $r > p_1$. So $\ker(\xi) \subset \phi(I_\mu^\lambda)$. □

This completes the proof of Theorem 4.7(a).

4.3 Proof of Theorem 4.3(b): The product monomial crystal and row tableau

Let $\mathbf{R}$ be a set of parameters of weight λ and define $\widetilde{\mathbf{R}}$ to be the corresponding set of parameters of weight $N\varpi_1$, as in Theorem 2.8. We let γ be a W_P-invariant weight such that the values of the weight on blocks of size i are given by the elements of R_i with multiplicity; while this is not unique, its orbit under the Weyl group is. Note that the elements of $\widetilde{\mathbf{R}}$ are just the entries of $\gamma + \rho_P$.

Note that the isomorphism Φ preserves the notion of highest weight vector and highest weight module: it sends E's to E's and H's to D's. In this section we describe how the highest weights of $Y_\mu^\lambda(\mathbf{R})$ and $Y_\mu^{N\varpi_1}(\widetilde{\mathbf{R}})$ (as described in Section 2.4) match up respectively with the highest weights of $W(\pi)_{\widetilde{\mathbf{R}}}$ and $W(\pi,\mathfrak{p})_{\mathbf{R}}$ (as described in [BK08] and Section 3.2).

That is, we will describe the commutative diagram

$$\begin{array}{ccc} \mathcal{B}(\widetilde{\mathbf{R}}) & \xrightarrow{\sim} & Row_{\widetilde{\mathbf{R}}}(\pi) \\ \cup\!\mid & & \cup\!\mid \\ \mathcal{B}(\mathbf{R}) & \xrightarrow{\sim} & Row_{\widetilde{\mathbf{R}}}(\pi)^\circ \end{array} \tag{4.10}$$

as prescribed by Theorem 4.3(b). Both vertical maps are natural inclusions of subsets, and the horizontal maps are bijections induced by Φ.

4.3.1 A bijection for $\lambda = N\varpi_1$

Consider the isomorphism $\Phi : Y_\mu^{N\varpi_1} \to W(\pi)$ from Theorem 4.7. By Equation (4.4), it follows that Φ descends to an isomorphism

$$Y_\mu^{N\varpi_1}(\widetilde{\mathbf{R}}) \cong W(\pi)_{\widetilde{\mathbf{R}}}.$$

On the one hand, the set of highest weights $\mathcal{B}(\widetilde{\mathbf{R}})_\mu$ of $Y_\mu^{N\varpi_1}(\widetilde{\mathbf{R}})$ is in bijection with the set

$$H_\mu^{N\varpi_1}(\widetilde{\mathbf{R}}) = \left\{ (\mathbf{S}_i)_{i\in I} : \begin{array}{l} |\mathbf{S}_i| = m_i \text{ and} \\ \mathbf{S}_1 + n \subset \mathbf{S}_2 + (n-1) \subset \cdots \subset \mathbf{S}_{n-1} + 1 \subset \widetilde{\mathbf{R}} \end{array} \right\} \tag{4.11}$$

As in (2.10), the highest weight corresponding to $(\mathbf{S}_i) \in H_\mu^{N\varpi_1}(\widetilde{\mathbf{R}})$ is given by

$$A_i(u) \mapsto \prod_{s \in \mathbf{S}_i} (1 - \tfrac{1}{2} s u^{-1}) = u^{-m_i} \prod_{s \in \mathbf{S}_i} (u - \tfrac{1}{2} s)$$

On the other hand, recall from Section 3.2.2 that the set of highest weights for $W(\pi)_{\widetilde{\mathbf{R}}}$ is $Row_{\widetilde{\mathbf{R}}}(\pi)$, the set of row symmetrized π–tableaux T on the alphabet $\widetilde{\mathbf{R}}$, and that $T \in Row_{\widetilde{\mathbf{R}}}(\pi)$ encodes a highest weight according to

$$(u-i+1)^{p_i} D_i(u-i+1) \mapsto \prod_{a \in T_i} (u + \tfrac{1}{2} a - \tfrac{n}{2})$$

Proposition 4.12. *Let $\widetilde{\mathbf{R}}$ be a multiset of size N. The isomorphism $\Phi : Y_\mu^{N\varpi_1}(\widetilde{\mathbf{R}}) \to W(\pi)_{\widetilde{\mathbf{R}}}$ induces a bijection*

$$Row_{\widetilde{\mathbf{R}}}(\pi) \to \mathcal{B}(\widetilde{\mathbf{R}})_\mu$$

given by $T \mapsto \mathbf{S} = (\mathbf{S}_i)$, where

$$\mathbf{S}_i = \big(T_1 \cup \cdots \cup T_i\big) - (n-i+1),$$

and T_i denotes the i-th row of T.

Equivalently, the ith row

$$T_i = \big(\mathbf{S}_i + (n-i)\big) \setminus \big(\mathbf{S}_{i-1} + (n-i+1)\big)$$

is the difference between parts of the "flag" of multisets (4.11).

Proof. We begin with the equation

$$(u-i+1)^{p_i} D_i(u-i+1) = (u-i+1)^{p_i} \frac{Q_i(u)}{Q_{i-1}(u)} = \frac{(u - \frac{i-1}{2})^{m_i} \Phi\big(A_i(-u + \frac{i-1}{2})\big)}{(u - \frac{i-1}{2})^{m_{i-1}} \Phi\big(A_{i-1}(-u + \frac{i-2}{2})\big)}$$

The first equality is Equation (3.5), while the second equality follows from Lemmas 4.8, 4.9 after cancelling common factors.

For a highest weight $\mathbf{S} = (\mathbf{S}_i)$ for $Y_\mu^{N\varpi_1}(\widetilde{\mathbf{R}})$, the right-hand side maps to

$$\frac{\prod_{s \in \mathbf{S}_i} (u + \frac{s-i+1}{2})}{\prod_{s \in \mathbf{S}_{i-1}} (u + \frac{s-i+2}{2})}$$

To find the corresponding tableaux $T \in Row_{\widetilde{\mathbf{R}}}(\pi)$, we must write the above as

$$\prod_{a \in T_i} (u + \tfrac{a}{2} - \tfrac{n}{2}),$$

which leads $T_i = \big(\mathbf{S}_i + (n-i+1)\big) \setminus \big(\mathbf{S}_{i-1} + (n-i+2)\big)$. This proves the proposition. $\square$

4.3.2 A bijection for general λ

Next we'll prove that the bijection of Proposition 4.12 induces a bijection between the highest weights of $Y_\mu^\lambda(\mathbf{R})$ and the highest weights of $W(\pi,\mathfrak{p})_{\mathbf{R}}$. We'll do this by first identifying the tableau in $Row(\pi)_{\widetilde{\mathbf{R}}}$ which descend to highest weights of $W(\pi,\mathfrak{p})_{\mathbf{R}}$; we term these "overshadowing tableau". Once this is done, we need only check that these satisfy the same conditions as the subcrystal $\mathcal{B}(\mathbf{R}) \subset \mathcal{B}(\widetilde{\mathbf{R}})$ (cf. Lemma 2.9).

Let $Row(\pi)^\circ_{\widetilde{\mathbf{R}}}$ denote the set of highest weights of $W(\pi,\mathfrak{p})_{\mathbf{R}}$. By Theorem 3.21 there is an inclusion $Row(\pi)^\circ_{\widetilde{\mathbf{R}}} \subset Row(\pi)_{\widetilde{\mathbf{R}}}$. Now suppose $c \in \mathbf{R}_i$. Then in $\widetilde{\mathbf{R}}$ the element c has $n-i$ "descendants", namely the elements

$$\{c+n-i-1, c+n-i-3, ..., c-n+i+1\}.$$

We'll call this set the c-**block** in $\widetilde{\mathbf{R}}$.

Given a row tableau $T \in Row(\pi)_{\widetilde{\mathbf{R}}}$, we can divide the boxes of the tableau into c-blocks. Note that this decomposition will not be unique if $\widetilde{\mathbf{R}}$ contains any element with multiplicity greater than 1. We say that the tableau T is **overshadowing** if this division into c-blocks can be chosen so that for every $c \in \mathbf{R}$ the elements of the c-block occur in strictly decreasing order down the tableau.

Put another way, given $T \in Row(\pi)_{\widetilde{\mathbf{R}}}$, an $\mathbf{R}$-coloring of T is a coloring of the contents of T using $|\mathbf{R}|$ colors, such that for every $c \in \mathbf{R}$ the elements colored c form a c-block, and they are in strictly decreasing order down the rows. Clearly T is overshadowing if and only there exists and $\mathbf{R}$-coloring of T.

Lemma 4.13. *$Row(\pi)^\circ_{\widetilde{\mathbf{R}}}$ is precisely the subset of overshadowing row tableau in $Row(\pi)_{\widetilde{\mathbf{R}}}$.*

Proof. By Theorem 4.13, the set $Row(\pi)^\circ_{\widetilde{\mathbf{R}}}$ is the set of tableaux where the row reading word is of the form $w \cdot (\gamma + \rho_{\mathfrak{p}})$, for $w \in \mathcal{PS}(\pi,\mathfrak{p})$ and γ is a $W_{\mathfrak{p}}$-invariant weight where each element of $\mathbf{R}_i$ appears $n-i$ times. Thus, the coordinates of $\gamma+\rho_{\mathfrak{p}}$ are the concatenations of the c-blocks for the different $c \in \mathbf{R}_i$ for all i, ordered by the value of c. The longest left coset property says that every pair of elements of the same c-block must be reversed in order. That is, they must be in decreasing order in rows (that is, they must satisfy the overshadowing condition) or in the same row. On the other hand, if they are in the same row, the shortest right coset condition assures that they must have remained in the same order, contradicting the longest left coset property. Thus, this tableau must be overshadowing.

Conversely, if a tableau is overshadowing, then the division into c-blocks fixes a unique parabolic-singular permuation which sends $\gamma + \rho_{\mathfrak{p}}$ to a row reading of this tableau which matches the c-blocks of the tableaux c-blocks of $\gamma+\rho_{\mathfrak{p}}$, while ordering each row by the order on c-blocks in $\gamma+\rho_{\mathfrak{p}}$. This makes the shortest right coset property clear, and the longest left coset property follows because overshadowing shows that every c-block is completely reversed. $\square$

Let $\mathbb{B}(\lambda)$ be the crystal associated to an irreducible representation of $\mathfrak{g}$ of highest weight λ. By [KTW+, Prop. 2.9], the crystal $\mathcal{B}(\widetilde{\mathbf{R}})$ is isomorphic to $\mathbb{B}(t_1\varpi_1) \otimes \cdots \otimes \mathbb{B}(t_q\varpi_1)$, where $\widetilde{\mathbf{R}} = \{c_1^{t_1}, ..., c_q^{t_q}\}$.

Now, we shall describe the inclusion $\mathcal{B}(\mathbf{R}) \subset \mathcal{B}(\widetilde{\mathbf{R}})$. First, consider the case when λ is fundamental. The elements of $\mathcal{B}(y_{i,c})$ are in bijection with partitions fitting inside an $i \times n-i$

box, that is, with no more than i parts and $\xi_p \leq n-i$ (cf. Section 2.4.3). We identify a partition with its diagram $\{(a,b) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \mid 1 \leq a \leq \xi_b\}$, and to the partition ξ we associate the monomial

$$y_{\xi,c} = y_{i,c} \cdot \prod_{(a,b)\in\xi} z^{-1}_{i-a+b,c-a-b}.$$

Thus, we wish to factor these into terms corresponding to $\mathcal{B}(y_{1,c+j})$ for $j = -i+1, \dots, i-1$. This is easily done using the formula

$$y_{i,c} = y_{1,-i+1} y_{1,-i+3} \cdots y_{1,i-1} \prod_{k=1}^{i-1} \prod_{j=0}^{k-1} z^{-1}_{i-k,2j-k}.$$

Thus, we have that

$$y_{\xi,c} = \prod_{p=1}^{i} y_{1,c+i-2p+1} \prod_{q=1}^{\xi_p+i-p} z^{-1}_{q,c-q+i-2p} = \prod_{p=1}^{i} y_{(\xi_p+i-p),c+i+1-2p}$$

where we consider $(\xi_p + i - p)$ as a partition with one row. This gives us an element in $\mathcal{B}(y_{1,c+i+1-2p})$, resulting in the inclusion

$$\mathcal{B}(y_{i,c}) \subset \prod_{j=-i+1,\dots,i-1} \mathcal{B}(y_{1,c+j}).$$

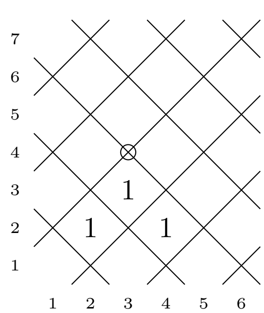
For general λ we take the product over all such inclusions. More precisely, for $p \in \mathcal{B}(\mathbf{R})$ we write $p = \prod_{i\in I, c\in\mathbf{R}_i} y_{\xi^{n-i,c},c}$. Then by the above argument we can view $y_{\xi^{i,c},c} \in \prod_{j=-n+i+1,\dots,n-i-1} \mathcal{B}(y_{1,c+j})$, and hence

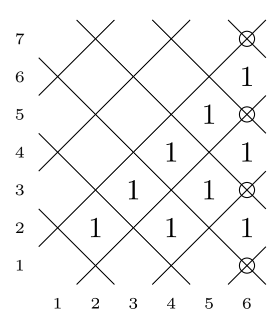
$$p \in \prod_{i\in I, c\in\mathbf{R}_i} \prod_{j=-n+i+1,\dots,n-i-1} \mathcal{B}(y_{1,c+j}) = \mathcal{B}(\widetilde{\mathbf{R}}).$$

This procedure has a nice description in terms of diagrams. Consider a monomial $p \in \mathcal{B}(\mathbf{R})$. Recall that p can be represented diagrammatically as in Section 2.4.3, where here we assume that $\mathbf{R}$ is an integral set of parameters

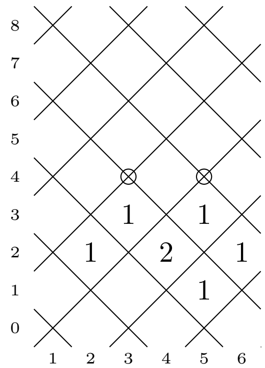
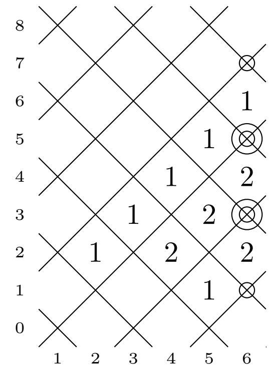
To define the image of p in $\mathcal{B}(\widetilde{\mathbf{R}})$, the idea is to "project" the circled vertices onto the line corresponding to the $n-1$ node of the Dynkin diagram, and fill the squares along this projection with 1s.

For instance, if we work in type A_6, with $\mathbf{R}_3 = \{4\}$, all other $\mathbf{R}_i$ empty, and we attach the partition $(2,1)$ to this vertex then we have the picture on the left; after projecting we obtain the picture on the right:





In general, the inclusion $\mathcal{B}(\mathbf{R}) \to \mathcal{B}(\widetilde{\mathbf{R}})$ is defined by applying this projection to every vertex. For instance, consider the monomial data $p_0 \in \mathcal{B}(\mathbf{R})$ on the left below, where $\mathbf{R}_3 = \mathbf{R}_5 = \{4\}$ and all other $\mathbf{R}_i$ are empty. The corresponding monomial data in $\mathcal{B}(\widetilde{\mathbf{R}})$ is on the right:

Finally we are ready to prove:

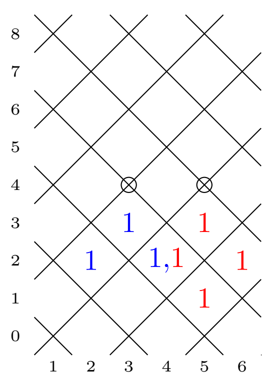
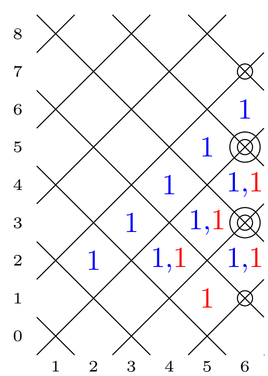
Proposition 4.14. *Under the bijection of Proposition 4.12, $Row(\pi)^\circ_{\widetilde{\mathbf{R}}}$ is identified with $\mathcal{B}(\mathbf{R})$.*

This completes the proof of part (b) of Theorem 4.3.

Proof. By the above discussion, we view $\mathcal{B}(\mathbf{R}) \subset \mathcal{B}(\widetilde{\mathbf{R}})$. Let $\mathbf{S} = (\mathbf{S}_i) \in \mathcal{B}(\mathbf{R})$, and suppose it corresponds to $T \in Row(\pi)_{\widetilde{\mathbf{R}}}$ under the bijection of Proposition 4.12. Denoting the rows of T by T_i, we have that $T_n = \widetilde{\mathbf{R}} \setminus (\mathbf{S}_{n-1} + 1), T_1 = \mathbf{S}_1 + n$ and for $i = 1, ..., n-2$,

$$T_i = (\mathbf{S}_i + (n-i)) \setminus (\mathbf{S}_{i-1} + (n-i+1)).$$

We'll show that $T \in Row(\pi)^\circ_{\widetilde{\mathbf{R}}}$, i.e. T has an $\mathbf{R}$-coloring. We'll first show that it suffices to prove this in the case when p consists of only one vertex (i.e. $|\mathbf{R}| = 1$). Without loss of generality assume that $\mathbf{R}$ is integral. Now color each partition in p. For instance we could have the example on the left below. When we view p as a monomial datum in $\mathcal{B}(N\varpi_1, \widetilde{\mathbf{R}})$ we remember the color of the partitions. In the example we obtain the diagram on the right.

Now, when we apply the bijection, we naturally obtain a row tableau whose entries are colored (we don't know a priori that this is an $\mathbf{R}$-colored tableau - this is what we want to show). Indeed when we look at $(\mathbf{S}_i + (n-i)) \setminus (\mathbf{S}_{i-1} + (n-i+1))$, we preserve the color of the elements that haven't been cancelled (for $c \in \mathbf{S}_i$, the element $c + n - i \in \mathbf{S}_i + (n-i)$ is understood to have the same color as c). Moreover, the last row is given by $\widetilde{\mathbf{R}} \setminus (\mathbf{S}_{n-1} + 1)$, and the elements of $\widetilde{\mathbf{R}}$ are colored the same color as the node which "overshadowed" them. This is much easier with an example: the bijection applied to the above monomial data results in the following colored row tableau (which happens to contain two empty rows):

$$\begin{array}{|c|c|}
\hline
 \\
\hline
7 \\
\hline
 \\
\hline
5 & 5 \\
\hline
3 \\
\hline
3 \\
\hline
1 \\
\hline
\end{array}$$

Note that the red content is precisely the block corresponding to $4 \in \mathbf{S}_5$, and the blue content is the block corresponding to $4 \in \mathbf{S}_3$. Moreover if p consisted of, say, just the red partition then the resulting row tableau is the red part of the above tableau. This shows that it suffices to consider the case where $|\mathbf{R}| = 1$, and show that the resulting row tableau is overshadowing.

To this end, suppose $\mathbf{R}_{n-i} = \{k\}$ (so the other multisets $\mathbf{R}_j$ are empty), and in the monomial data p the partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_i \geq 0)$ corresponds to k. Then for $j = 1, ..., i$, T has content $k + i - 2j + 1$ going down the rows, which is manifestly overshadowing. This proves that $T \in Row(\pi)^\circ_{\widetilde{\mathbf{R}}}$ for any $p \in \mathcal{B}(\mathbf{R})$.

To prove that the bijection $\mathcal{B}(\mathbf{R}) \to Row(\pi)^\circ_{\widetilde{\mathbf{R}}}$ is surjective, given $T \in Row(\pi)^\circ_{\widetilde{\mathbf{R}}}$ choose an $\mathbf{R}$-coloring of T. This partitions the contents of T into c-blocks, and for each such block we can reverse the process above to construct a monomial datum. If we do this for all blocks at once we obtain a datum in $\mathcal{B}(\mathbf{R}) \subset \mathcal{B}(\widetilde{\mathbf{R}})$. $\square$

Remark 4.15. Under this bijection, we obtain a crystal structure on overshadowing tableaux. One can easily work out that this coincides with the one induced by Brundan and Kleshchev's crystal structure on row tableau in [BK08, §4.3].

5 Proof of Theorem 4.3(d): The classical limit

In this section, we will study the classical limit of our isomorphism

$$\Phi : Y^\lambda_\mu(\mathbf{R}) \xrightarrow{\sim} W(\pi, \mathfrak{p})_{\mathbf{R}}$$

Our goal is to establish part (d) of Theorem 4.3, and show that this classical limit agrees with the Mirković-Vybornov isomorphism.

Remark 5.1. We may immediately save ourselves some work with an observation: it suffices to prove the case of $\lambda = N\varpi_1$, as in general both isomorphisms are defined by restricting this case to closed subvarieties.

5.1 More about slices to nilpotent orbits

Locally to this subsection, we let G be an algebraic group over $\mathbb{C}$, with Lie algebra $\mathfrak{g}$. We will fix throughout a nonzero nilpotent element $e \in \mathfrak{g}$, and an $\mathfrak{sl}_2$–triple $\{e, h, f\}$. In this appendix, we slightly generalize some of the results on Slodowy slices from [GG02], showing in particular that the classical Slodowy slice and the transverse slice considered in [MV07a] are Poisson isomorphic. Since these results may be of independent interest, we provide brief proofs.

Definition 5.2. Let $C \subset \mathfrak{g}$ be an ad_h-invariant subspace such that $\mathfrak{g} = [\mathfrak{g}, e] \oplus C$. Then the affine space $\mathcal{M} = e + C$ is called an **MV slice**.

The most natural choice of such a slice is the Slodowy slice, where $C = \mathfrak{g}^f$ (cf. Section 3.1). There are many others however.

Remark 5.3. An MV slice $\mathcal{M} = e + C$ is a transverse slice to the nilpotent orbit $\mathbb{O}_e$ at the point e.

From now on, we assume that $\mathcal{M}$ is an MV slice. Note that the eigenvalues of ad_h acting on C are necessarily non-positive. From our $\mathfrak{sl}_2$–triple we get a homomorphism $SL_2 \to G$, and we will denote by $\gamma(t)$ the image of $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ in G. We consider the $\mathbb{C}^\times$–action (the Kazhdan action) on $\mathfrak{g}$ defined by

$$\rho(t) \cdot x = t^2 \left(\mathrm{Ad}\gamma(t^{-1})\right)(x)$$

Note that ρ preserves $\mathcal{M}$ and contracts it to the unique fixed point e.

Consider the decomposition $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ into ad_h weight spaces. As in Section 3.1 there is a non-degenerate skew-symmetric form $\langle x, y \rangle = (e, [x, y])$ on $\mathfrak{g}_{-1}$. Choose a Lagrangian subspace $\mathfrak{l} \subset \mathfrak{g}_{-1}$ with respect to $\langle \cdot, \cdot \rangle$.

Define the nilpotent Lie subalgebra $\mathfrak{m} = \mathfrak{l} \oplus \bigoplus_{i \leq -2} \mathfrak{g}_i$, and the corresponding unipotent subgroup $M \subset G$. Note that $\mathfrak{m}^\perp = [e, \mathfrak{l}] \oplus \bigoplus_{i \leq 0} \mathfrak{g}_i$ is the orthogonal complement of $\mathfrak{m}$ with respect to the Killing form. The following result is a generalization of Lemma 2.1 in [GG02].

Lemma 5.4. *The adjoint action map $\alpha : M \times \mathcal{M} \to e + \mathfrak{m}^\perp$ is a $\mathbb{C}^\times$-equivariant isomorphism of affine varieties. Here $\mathbb{C}^\times$ acts on $e + \mathfrak{m}^\perp$ by ρ, and on $M \times \mathcal{M}$ by*

$$t \cdot (g, x) = \left(\gamma(t^{-1}) g \gamma(t), \rho(t) \cdot x\right)$$

Proof. Since $\mathcal{M}$ is a MV slice we have $C \subset \bigoplus_{i \leq 0} \mathfrak{g}(i) \subset \mathfrak{m}^\perp$, so indeed the image of the adjoint map $M \times \mathcal{M} \to \mathfrak{g}$ is contained in $e + \mathfrak{m}^\perp$.

Next, since $\mathfrak{g} = [\mathfrak{g}, e] \oplus C$ it follows that $[\mathfrak{m}, e] \cap C = 0$. We also have $\dim \mathrm{Ker}(\mathrm{ad} f) = \dim C$, since both spaces are complementary to $[\mathfrak{g}, e]$ in $\mathfrak{g}$. Since $\mathrm{ad} e : \mathfrak{m} \to [e, \mathfrak{m}]$ is an isomorphism,

$$\begin{aligned} \dim \mathfrak{m}^\perp &= \dim \mathfrak{m} + \dim \mathfrak{g}(0) + \dim \mathfrak{g}(-1) = \dim[\mathfrak{m}, e] + \dim \mathrm{Ker}(\mathrm{ad} f) = \\ &= \dim[\mathfrak{m}, e] + \dim C \end{aligned}$$

So $\mathfrak{m}^\perp = [\mathfrak{m}, e] \oplus C$. The remainder of the proof proceeds as in [GG02]. ∎

Following Section 3.2 in [GG02]: e is a regular value for the moment map $\mu : \mathfrak{g}^* \to \mathfrak{m}^*$, $\mu^{-1}(e) = e + \mathfrak{m}^\perp$ (under $\mathfrak{g}^* \cong \mathfrak{g}$), and it follows from Lemma 5.4 that we have a Hamiltonian reduction of the Poisson structure on $\mathfrak{g}^*$ to $\mathcal{M}$. It is induced from the isomorphisms

$$\mathcal{M} \cong \mu^{-1}(e)/M, \quad \mathbb{C}[\mathcal{M}] \cong \left(\mathbb{C}[\mathfrak{g}]/I(\mu^{-1}(e))\right)^M$$

Theorem 5.5. *There is a $\mathbb{C}^\times$–equivariant isomorphism of affine Poisson varieties between any two MV slices.*

Proof. With $\mathfrak{l}$ and $\mathfrak{m}$ fixed as above, for MV slices $\mathcal{M}_1, \mathcal{M}_2$ we have $\mathbb{C}^\times$–equivariant isomorphisms

$$M \times \mathcal{M}_1 \cong e + \mathfrak{m}^\perp \cong M \times \mathcal{M}_2$$

by Lemma 5.4. The Poisson structures on $\mathcal{M}_1, \mathcal{M}_2$ are both induced by Hamiltonian reduction, giving us the desired Poisson isomorphism. ☐

Remark 5.6. Any MV slice $\mathcal{M}$ also inherits an induced Poisson structure as a subvariety of $\mathfrak{g}$ (with its standard Poisson structure under $\mathfrak{g}^*$ under $\mathfrak{g} \cong \mathfrak{g}^*$). This Poisson structure agrees with that given above via Hamiltonian reduction, cf. [GG02, Section 3.2]

5.2 More about affine Grassmannian slices

Let us briefly recall some aspects of the "loop group" description of the slices $\mathsf{Gr}_\mu^{\overline{\lambda}}$, to supplement the lattice description given in Section 2.1. $\mathsf{Gr}_\mu^{\overline{\lambda}}$ is a transverse slice to $\mathsf{Gr}^\mu \subset \overline{\mathsf{Gr}^\lambda}$, defined as the intersection

$$\mathsf{Gr}_\mu^{\overline{\lambda}} = \mathsf{Gr}_\mu \cap \overline{\mathsf{Gr}^\lambda} \tag{5.1}$$

where $\mathsf{Gr}_\mu = G_1[t^{-1}]t^{w_0\mu}$ is an orbit for the opposite group $G_1[t^{-1}] = \mathrm{Ker}(G[t^{-1}] \overset{t\mapsto\infty}{\to} G)$. Every point in Gr_μ has a unique representative of the form $gt^{w_0\mu}$, where $g = (a_{ij}) \in G_1[t^{-1}]$ satisfies

$$a_{ij} = \delta_{ij} + a_{ij}^{(1)}t^{-1} + a_{ij}^{(2)}t^{-2} + \ldots \in \delta_{ij} + t^{p_i-p_j-1}\mathbb{C}[t^{-1}] \tag{5.2}$$

(Recall that we are denoting $w_0\mu = (p_1, \ldots p_n)$) In this way, $\mathsf{Gr}_\mu^{\overline{\lambda}}$ may be considered as a closed subscheme of $G_1[t^{-1}]$.

Remark 5.7. It is sometimes convenient to work with the group $G_1[[t^{-1}]]$. One advantage is that elements of this group admit Gauss decompositions,

$$G_1[[t^{-1}]] = U_1^-[[t^{-1}]]T_1[[t^{-1}]]U_1^+[[t^{-1}]]$$

where $U^\pm, T \subset SL_n$ are the subgroups of upper/lower triangular and diagonal matrices. The varieties $\mathsf{Gr}_\mu^{\overline{\lambda}}$ may also be considered as closed subschemes of $G_1[[t^{-1}]]$.

In particular, we may describe $\mathsf{Gr}_\mu^{\overline{N\varpi_1}}$ as the variety of matrices $g = (a_{ij})$ of the form (5.2) (with $\det g = 1$), with the additional constraint that $a_{ij}^{(r)} = 0$ for $r > p_j$. Then explicitly g corresponds to the lattice

$$\Lambda = \mathrm{span}_{\mathcal{O}} \left\{ t_j^{p} e_j + \sum_{i,r} a_{ij}^{(r)} t^{p_j - r} e_i : 1 \leq j \leq n \right\} \tag{5.3}$$

allowing us to compare with our previous description (2.1) of $\mathsf{Gr}_\mu^{\overline{N\varpi_1}}$.

Using the above identification of $\mathsf{Gr}^{\overline{\lambda}}_{\mu} \subset G_1[[t^{-1}]]$, we now recall how the classical limit of $Y^{\lambda}_{\mu}(\mathbf{R})$ is identified with functions on $\mathsf{Gr}^{\overline{\lambda}}_{\mu}$ (as was promised in Section 2.3). Following Theorems 3.9, 3.12 and Proposition 4.3 in [KWWY14],

$$
\begin{aligned}
A_i^{(r)} &\mapsto [t^{-r}]\Delta_{\{1,\dots,i\},\{1,\dots,i\}},\\
E_i^{(r)} &\mapsto [t^{-r}]\frac{\Delta_{\{1,\dots,i-1,i+1\},\{1,\dots,i\}}}{\Delta_{\{1,\dots,i\},\{1,\dots,i\}}},\\
F_i^{(s)} &\mapsto [t^{-s}]\frac{\Delta_{\{1,\dots,i\},\{1,\dots,i-1,i+1\}}}{\Delta_{\{1,\dots,i\},\{1,\dots,i\}}}
\end{aligned}
$$

Here, for $I, J \subset \{1,\dots,n\}$ we denote by $\Delta_{I,J}$ the minor with rows I and columns J, thought of as a map $G_1[[t^{-1}]] \to \mathbb{C}[[t^{-1}]]$. Meanwhile, $[t^{-r}]$ extracts the coefficient of t^{-r}. Restricting these functions to the closed subscheme $\mathsf{Gr}^{\overline{\lambda}}_{\mu} \subset G_1[[t^{-1}]]$ gives the desired isomorphism.

The following result is clear from the structure of the GKLO representation [KWWY14, Theorem 4.5]. It is also follows from similar results for Zastava spaces [FM99], since $\mathsf{Gr}^{\overline{N\varpi_1}}_{\mu}$ and $Z^{N\varpi_1-\mu}$ are birational.

Proposition 5.8. *The functions $A_i^{(r)}, E_i^{(r)}$ for $i \in I$, $1 \le r \le m_i$ are birational coordinates on $\mathsf{Gr}^{\overline{\lambda}}_{\mu}$.*

5.3 The MV isomorphism

As per usual, fix now $\lambda \ge \mu$ dominant coweights and associated partitions $\tau \ge \pi$ of N.

Let $e_\pi \in \mathfrak{gl}_N$ be the nilpotent element with *lower triangular* Jordan type $\pi = (p_1 \le \cdots \le p_n) \vdash N$, we will consider the **transpose MV slice**

$$
\mathcal{T}_\pi = \left\{ X = (X_{ij}) \in \mathfrak{gl}_N : \begin{array}{ll} (a) & X_{ij} \text{ has size } p_i \times p_j, \\ (b) & X_{ii} \text{ has 1's below diagonal, entries in final} \\ & \quad\text{column} \\ (c) & X_{ij} \text{ for } i \neq j \text{ has entries in final column,} \\ & \quad\text{but not below row } p_j \end{array} \right\} \tag{5.4}
$$

Recall the description of $\mathsf{Gr}^{\overline{\lambda}}_{\mu}$ from Section 2.1. With this description and the definition of $\mathcal{T}_\pi$ in mind, we can now give slightly more precise formulation of the MV isomorphism 4.1:

For any $\Lambda \in \mathsf{Gr}^{\overline{\lambda}}_{\mu}$, identify $\Lambda_0/\Lambda \cong \mathbb{C}^N$ via the basis E_π; in particular we may identify multiplication by t on Λ_0/Λ with an element $X \in \mathfrak{gl}_N$. Then map taking Λ to $X \in \mathfrak{gl}_N$ defines an isomorphism $\mathsf{Gr}^{\overline{\lambda}}_{\mu} \xrightarrow{\sim} T_\pi \cap \overline{\mathbb{O}_\tau}$. It is compatible with the inclusions of closed subvarieties $\mathsf{Gr}^{\overline{\lambda}}_{\mu} \subset \mathsf{Gr}^{\overline{N\varpi_1}}_{\mu}$ and $T_\pi \cap \overline{\mathbb{O}_\tau} \subset T_\pi \cap \mathcal{N}_{\mathfrak{gl}_N}$.

Following [CK, Section 3.3], it is straightforward to write the above isomorphism down explicitly in coordinates. On the affine Grassmannian side, we identify Λ with $g \in G_1[[t^{-1}]]$ as in the previous section and use the coefficients $a_{ij}^{(r)}$ of the matix entries of g as coordinates. On the nilpotent cone side, the image is a block matrix $X = (X_{ij})$. Then under the above isomorphism, the block X_{ij} has interesting entries only in its final column:

$$X_{ij} = \begin{pmatrix} 0 & 0 & \cdots & 0 & \cdots & 0 & -a_{ij}^{(p_j)} \\ \delta_{ij} & 0 & \cdots & 0 & \ddots & \vdots & \vdots \\ 0 & \delta_{ij} & \ddots & & \ddots & 0 & -a_{ij}^{(1)} \\ 0 & 0 & \ddots & \ddots & & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \delta_{ij} & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & \delta_{ij} & 0 \end{pmatrix}$$

5.4 Completing the proof of Theorem 4.3

To finish the proof of the theorem, we will now compare the classical limit of our isomorphism $\Phi : Y_\mu^{N\varpi_1}(\mathbf{R}) \xrightarrow{\sim} W(\pi)_\mathbf{R}$ with the Mirković-Vybornov isomorphism. We interpret the classical limit of Φ as an isomorphism of coordinate rings

$$\Phi : \mathbb{C}[\overline{\mathsf{Gr}_\mu^{N\varpi_1}}] \xrightarrow{\sim} \mathbb{C}[\mathcal{T}_\pi \cap \mathcal{N}_{\mathfrak{gl}_N}] \tag{5.5}$$

In the notation of the previous section, suppose that $g \in \overline{\mathsf{Gr}_\mu^{N\varpi_1}} \subset G_1[[t^{-1}]]$ maps to $X \in \mathcal{T}_\pi \cap \mathcal{N}_{\mathfrak{gl}_N}$ under the Mirković-Vybornov isomorphism (both being closed points). To complete the proof of the theorem, it is sufficient to prove that

$$f(g) = \Phi(f)(X), \qquad \forall f \in \mathbb{C}[\overline{\mathsf{Gr}_\mu^{N\varpi_1}}] \tag{5.6}$$

Remark 5.9. Both sides are irreducible algebraic varieties, so in fact it is sufficient to prove that this equation holds for f ranging over the birational coordinates described in Proposition 5.8.

The isomorphism $\operatorname{gr} Y_\mu^{\overline{N\varpi_1}}(\mathbf{R}) \cong \mathbb{C}[\overline{\mathsf{Gr}_\mu^{N\varpi_1}}]$ was described explicitly in Section 5.2. We now recall Brundan and Kleshchev's identification of the classical limit of $W(\pi)$ with functions on $\mathcal{T}_\pi$, following [BK06], [BK08, Sections 3.3–3.4]. More precisely, they give an explicit isomorphism

$$W(\pi) \xrightarrow{\sim} U(\mathfrak{p})^\mathfrak{m} \subset U(\mathfrak{gl}_N)$$

In the classical limit, we identify $S(\mathfrak{gl}_N) \cong \mathbb{C}[\mathfrak{gl}_N]$ via the trace pairing, and $S(\mathfrak{p})^\mathfrak{m} \cong \mathbb{C}[e_\pi + \mathfrak{m}^\perp]^M$ (see [BK06, §8] for details). Since $\mathcal{T}_\pi$ is an MV slice, by Lemma 5.4, there is an isomorphism

$$\mathbb{C}[e + \mathfrak{m}^\perp]^M \xrightarrow{\sim} \mathbb{C}[\mathcal{T}_\pi]$$

by restriction. In other words, the isomorphism $\operatorname{gr} W(\pi) \cong \mathbb{C}[\mathcal{T}_\pi]$ comes by the composition

$$\operatorname{gr} W(\pi) \hookrightarrow \mathbb{C}[\mathfrak{gl}_N] \twoheadrightarrow \mathbb{C}[\mathcal{T}_\pi] \tag{5.7}$$

where the first arrow is Brundan-Kleshchev's embedding and the second is restriction.

Brundan-Kleshchev's embedding is defined via explicit elements $T_{ij;0}^{(r)} \in U(\mathfrak{gl}_N)$, defined in [BK06, §9] (see also [BK08, §3.3]). Forming the $n \times n$–matrix $T(u) = (T_{ij}(u))$ whose entries are formal series of functions $T_{ij}(u) = \delta_{ij} + \sum_{r>0} T_{ij;0}^{(r)} u^{-r}$, we take its formal Gauss decomposition

$$T(u) = F(u)D(u)E(u)$$

where $D(u)$ is diagonal and $E(u)$ (resp. $F(u)$) is upper (resp. lower) unitriangular. Denote the diagonal entries of $D(u)$ by $D_i(u) = 1 + \sum_{r>0} D_i^{(r)} u^{-r}$, and the super-diagonal entries of $E(u)$ by $E_i(u) = \sum_{r>0} E_i^{(r)} u^{-r}$. Then the elements $D_i^{(r)}, E_i^{(r)} \in U(\mathfrak{gl}_N)$ are the images of the same-named elements of $W(\pi)$ (and similarly for the $F_i^{(s)}$).

By abuse of notation, let us denote by $T_{ij;0}^{(r)}, D_i^{(r)}$, etc. the corresponding elements of the associated graded algebras.

Lemma 5.10. *Suppose $\overline{\mathsf{Gr}_\mu^{N\varpi_1}} \ni g \mapsto X \in \mathcal{T}_\pi \cap \mathcal{N}_{\mathfrak{gl}_N}$ are closed points corresponding under the Mirković-Vybornov isomorphism. Then*

$$T_{i,j;0}^{(r)}(X) = (-1)^r a_{ji}^{(r)}$$

Proof. Recall our conventions on the pyramid π from Section 3.1.1. By definition,

$$T_{ij;0}^{(r)} = \sum_{s=1}^{r} (-1)^{r-s} \sum_{\substack{k_1,\dots,k_s \\ \ell_1,\dots,\ell_s}} e_{k_1,\ell_1} \cdots e_{k_s,\ell_s}$$

where the sum is over sequences with $1 \le k_t, \ell_t \le N$, satisfying conditions (a), (b), (c), (e) and (f) from [BK08, §3.3]. In the classical limit, this is a function on $\mathfrak{gl}_N$ via the trace pairing. When restricted to $\mathcal{T}_\pi$, conditions (a), (b), (e) and (f) imply that the value at X has the form

$$\sum_{s=1}^{r} (-1)^{r-s} \sum_{\substack{x_2,\dots,x_s \\ r_1+\dots+r_s=r}} (-1)^s a_{jx_2}^{(r_1)} a_{x_2x_3}^{(r_2)} \cdots a_{x_si}^{(r_s)}$$

where the sum is over all sequences where $1 \le x_t \le n$. However, condition (c) implies that only the term with $s = 1$ contributes. This proves the claim. □

Now, from (4.3) and Lemma 4.8 it follows that the classical limit of $\Phi : Y_\mu^{N\varpi_1}(\mathbf{R}) \xrightarrow{\sim} W(\pi)_{\mathbf{R}}$ sends

$$A_i^{(r)} \mapsto (-1)^r Q_i^{(r)}, \qquad E_i^{(r)} \mapsto (-1)^r E_i^{(r)}$$

where $Q_i(u) = D_1(u) \cdots D_i(u)$. Therefore, by Remark 5.9 the following result completes the proof of the main theorem:

Proposition 5.11. *With notation as in the previous lemma, suppose that $g \mapsto X$. Then for $i = 1, \dots, n-1$ we have equality of evaluations*

$$A_i^{(r)}(g) = (-1)^r Q_i^{(r)}(X),$$
$$E_i^{(r)}(g) = (-1)^r E_i^{(r)}(X)$$

for the functions $A_i^{(r)}, E_i^{(r)} \in \mathbb{C}[\overline{\mathsf{Gr}_\mu^{N\varpi_1}}]$ and $D_i^{(r)}, E_i^{(r)} \in \mathbb{C}[\mathcal{T}_\pi \cap \mathcal{N}_{\mathfrak{gl}_N}]$, respectively.

Proof. If we take the Gauss decomposion of $T(u)$ and then evaluate the result at the point X, we will get the same result as first evaluating $T(u)$ at X and then taking Gauss decomposition.

By the previous lemma, if we evaluate $T(u)$ at X we get the matrix $g(-u)^T$ (i.e. the transpose of $g = g(t) \in G_1[[t^{-1}]]$ evaluated at $t = -u$). Using the relation between the minors of a matrix and its transpose, we observe that

$$\begin{aligned} E_i^{(r)}(g) &= [t^{-r}] \frac{\Delta_{\{1,\dots,i-1,i+1\},\{1,\dots,i\}}}{\Delta_{\{1,\dots,i\},\{1,\dots,i\}}}\big(g(t)\big) \\ &= [t^{-r}] \frac{\Delta_{\{1,\dots,i\},\{1,\dots,i-1,i+1\}}}{\Delta_{\{1,\dots,i\},\{1,\dots,i\}}}\big(g(t)^T\big) \\ &= (-1)^r [u^{-r}] \frac{\Delta_{\{1,\dots,i\},\{1,\dots,i-1,i+1\}}}{\Delta_{\{1,\dots,i\},\{1,\dots,i\}}}\big(g(-u)^T\big) \end{aligned}$$

The latter precisely extracts the superdiagonal entries of the "E" part of the Gauss decomposition of $g(-u)^T$. Hence $E_i^{(r)}(g) = (-1)^r E_i^{(r)}(X)$, as claimed. A similar calculation applies to $A_i^{(r)}$. ☐

References

[BB81] Alexander Beilinson and Joseph Bernstein, *Localisation de $\mathfrak{g}$-modules*, C. R. Acad. Sci. Paris Sér. I Math. **292** (1981), no. 1, 15–18. MR MR610137 (82k:14015)

[BB82] Walter Borho and Jean-Luc Brylinski, *Differential operators on homogeneous spaces. I. Irreducibility of the associated variety for annihilators of induced modules*, Invent. Math. **69** (1982), no. 3, 437–476. MR 679767 (84b:17007)

[BB93] A. Beĭlinson and J. Bernstein, *A proof of Jantzen conjectures*, I. M. Gel′fand Seminar, Adv. Soviet Math., vol. 16, Amer. Math. Soc., Providence, RI, 1993, pp. 1–50. MR MR1237825 (95a:22022)

[BB09] Jonathan Brown and Jonathan Brundan, *Elementary invariants for centralizers of nilpotent matrices*, J. Aust. Math. Soc. **86** (2009), no. 1, 1–15. MR 2495993 (2010f:17017)

[BF12] Alexander Braverman and Michael Finkelberg, *Pursuing the double affine Grassmannian II: Convolution*, Adv. Math. **230** (2012), no. 1, 414–432. MR 2900549

[BFN] Alexander Braverman, Michael Finkelberg, and Hiraku Nakajima, *Coulomb branches of 3d $\mathcal{N} = 4$ quiver gauge theories and slices in the affine Grassmannian*, (with appendices by Alexander Braverman, Michael Finkelberg, Joel Kamnitzer, Ryosuke Kodera, Hiraku Nakajima, Ben Webster, and Alex Weekes). `arXiv:1604.03625`.

[BK05] Jonathan Brundan and Alexander Kleshchev, *Parabolic presentations of the Yangian $Y(\mathfrak{gl}_n)$*, Comm. Math. Phys. **254** (2005), no. 1, 191–220. MR 2116743 (2005j:17012)

[BK06] ______, *Shifted Yangians and finite W-algebras*, Adv. Math. **200** (2006), no. 1, 136–195.

[BK08] ______, *Representations of shifted Yangians and finite W-algebras*, Mem. Amer. Math. Soc. **196** (2008), no. 918, viii+107. MR 2456464 (2009i:17020)

[BLPW16] Tom Braden, Anthony Licata, Nicholas Proudfoot, and Ben Webster, *Quantizations of conical symplectic resolutions ii: category $\mathcal{O}$ and symplectic duality. with an appendix by i. losev*, Astérisque (2016), no. 384, 75–179. MR 3594665

[CK] Sabin Cautis and Joel Kamnitzer, *Categorical geometric symmetric howe duality*, `arXiv:1611.02210`.

[CK08] Sabin Cautis and Joel Kamnitzer, *Knot homology via derived categories of coherent sheaves. I. The $\mathfrak{sl}(2)$-case*, Duke Math. J. **142** (2008), no. 3, 511–588.

[CKL10] Sabin Cautis, Joel Kamnitzer, and Anthony Licata, *Coherent sheaves and categorical $\mathfrak{sl}(2)$ actions*, Duke Math. J. **154** (2010), no. 1, 135–179. MR 2668555

[FM99] Michael Finkelberg and Ivan Mirković, *Semi-infinite flags. I. case of global curve $\mathbb{P}^1$*, Amer. Math. Soc. Transl. Ser. 2 **194** (1999), no. 44, 81–112. MR 1729360

[GG02] Wee Liang Gan and Victor Ginzburg, *Quantization of Slodowy slices*, Int. Math. Res. Not. (2002), no. 5, 243–255.

[GKLO05] A. Gerasimov, S. Kharchev, D. Lebedev, and S. Oblezin, *On a class of representations of the Yangian and moduli space of monopoles*, Comm. Math. Phys. **260** (2005), no. 3, 511–525. MR 2182434 (2006k:53038)

[Hes78] Wim H. Hesselink, *Polarizations in the classical groups*, Math. Z. **160** (1978), no. 3, 217–234.

[Hik17] Tatsuyuki Hikita, *An algebro-geometric realization of the cohomology ring of hilbert scheme of points in the affine place*, Int. Math. Res. Not. (2017), no. 8, 2538–2561.

[HTT08] Ryoshi Hotta, Kiyoshi Takeuchi, and Toshiyuki Tanisaki, *D-modules, perverse sheaves, and representation theory*, Progress in Mathematics, vol. 236, Birkhäuser Boston Inc., Boston, MA, 2008, Translated from the 1995 Japanese edition by Takeuchi.

[Kal08] Dmitry Kaledin, *Derived equivalences by quantization*, Geom. Funct. Anal. **17** (2008), no. 6, 1968–2004.

[Kit12] Sarah Kitchen, *Cohomology of standard modules on partial flag varieties*, Represent. Theory (2012), no. 16, 317–344. MR 2945222

[KMWY] Joel Kamnitzer, Dinakar Muthiah, Alex Weekes, and Oded Yacobi, *Reducedness of affine Grassmannian slices in type A*, `arXiv:1611.06775`.

[KTW+] Joel Kamnitzer, Peter Tingley, Ben Webster, Alex Weekes, and Oded Yacobi, *Highest weights for truncated shifted Yangians and product monomial crystals*, arXiv:1511.09131.

[KWWY14] Joel Kamnitzer, Ben Webster, Alex Weekes, and Oded Yacobi, *Yangians and quantizations of slices in the affine Grassmannian*, Algebra Number Theory **8** (2014), no. 4, 857–893. MR 3248988

[Los10] Ivan Losev, *Quantized symplectic actions and W-algebras*, J. Amer. Math. Soc. **23** (2010), no. 1, 35–59.

[Los12] ———, *Isomorphisms of quantizations via quantization of resolutions*, Adv. Math. **231** (2012), no. 3-4, 1216–1270. MR 2964603

[Maf05] Andrea Maffei, *Quiver varieties of type A*, Comment. Math. Helv. **80** (2005), no. 1, 1–27.

[Mil] Dragan Miličić, *Localization and representation theory of reductive lie groups*, http://www.math.utah.edu/~milicic/Eprints/book.pdf.

[MV07a] I. Mirković and K. Vilonen, *Geometric Langlands duality and representations of algebraic groups over commutative rings*, Ann. of Math. (2) **166** (2007), no. 1, 95–143.

[MV07b] I. Mirković and M. Vybornov, *Quiver varieties and Beilinson-Drinfeld Grassmannians of type A*, 2007, arXiv:0712.4160.

[Nak16] Hiraku Nakajima, *Towards a mathematical definition of coulomb branches of* 3*-dimensional* $\mathcal{N} = 4$ *gauge theories, I*, Adv. Theor. Math. Phs. **20** (2016), no. 3, 595–669. MR 3565863

[Web11] Ben Webster, *Singular blocks of parabolic category* $\mathcal{O}$ *and finite W-algebras*, J. Pure Appl. Algebra **215** (2011), no. 12, 2797–2804. MR 2811563