[9] ______, *On the structure of the category $\mathcal{O}$ for W-algebras*, preprint, arXiv:0812.1584 (2008).
[10] A. Premet, *Special transverse slices and their enveloping algebras*, Adv. in Math. **170** (2002), 1–55.
[11] ______, *Enveloping algebras of Slodowy slices and the Joseph ideal*, J. Eur. Math. Soc. **9** (2007), 487–543.

Multiplicity Spaces in Symplectic Branching

Oded Yacobi

1. Motivation

The restriction of an irreducible representation of GL_n to GL_{n-1} uniquely decomposes into a direct sum of irreducible representations, i.e. the branching is multiplicity-free. Many combinatorial results about the representation theory of GL_n can be reduced to this fact. Thus, it is natural to ask whether such constructions work for other classical groups. Here we study the symplectic groups.

Let $G_n = Sp_{2n}$ and consider the embedding $G_{n-1} \subset G_n$. In this case the restriction of an irreducible representation of G_n to G_{n-1} is not multiplicity-free. Therefore many techniques that work for the general linear groups cannot be directly applied to this setting.

We use invariant theory to resolve the multiplicities that occur in symplectic branching. Our main result is that the multiplicity spaces that occur in symplectic branching carry a canonical irreducible action of a product of $SL_2's$. We prove an isomorphism of so-called branching algebras, which allows us to reduce questions about symplectic branching to ones about branching from GL_{n+1} to GL_{n-1}.

2. Preliminaries

Let Λ_n be the set of partitions of length n. Λ_n naturally indexes the following sets of irreducible representations:

$$\mathrm{Irr}_{\mathrm{poly}}(GL_n) \longleftrightarrow \Lambda_n \longleftrightarrow \mathrm{Irr}(G_n)$$

Here $\mathrm{Irr}_{\mathrm{poly}}(GL_n)$ is the set of irreducible polynomial representations of GL_n, and $\mathrm{Irr}(G_n)$ is the set of irreducible representations of G_n. For $\lambda \in \Lambda_n$ let V_λ (respectively W_λ) denote the corresponding irreducible representation of GL_n (respectively G_n).

For $\lambda \in \Lambda_n$ we write

$$Res^{GL_n}_{GL_{n-1}} V_\lambda \cong \bigoplus_{\mu \in \Lambda_{n-1}} V_\mu \otimes N^\lambda_\mu,$$

where N^λ_μ is the multiplicity space $Hom_{GL_{n-1}}(V_\mu, V_\lambda)$. Similarly, for $\lambda \in \Lambda_n$ we write

$$Res^{G_n}_{G_{n-1}} W_\lambda \cong \bigoplus_{\mu \in \Lambda_{n-1}} W_\mu \otimes M^\lambda_\mu,$$

where M_μ^λ is the multiplicity space $Hom_{G_{n-1}}(W_\mu, W_\lambda)$. That branching from G_n to G_{n-1} is not multiplicity free is equivalent to the fact that $dim M_\mu^\lambda$ is greater than one for some μ and λ. It's a classical result that:

$$dim M_\mu^\lambda \neq 0 \Leftrightarrow \mu \text{ double interlaces } \lambda.$$

The double interlacing condition here means that $\lambda_i \geq \mu_i \geq \lambda_{i+2}$ for all i.

3. Main Results

Let $\Lambda_{n-1,n} = \Lambda_{n-1} \times \Lambda_n$, and let $(\mu, \lambda) \in \Lambda_{n-1,n}$. Our starting point is the simple observation that M_μ^λ is naturally an SL_2-module. Indeed, there is a natural copy of SL_2 that centralizes $G_{n-1} \subset G_n$. This leads us to ask, what is the SL_2-module structure of M_μ^λ? We answer this by reducing the problem to an analogous one concerning the general linear groups.

Let $U_n \subset G_n$ be the unipotent radical of a Borel subgroup of G_n. Consider the ring of regular functions on G_n which are left-invariant with respect to U_n and right-invariant with respect to U_{n-1}:

$$\mathcal{M} = \mathcal{O}(U_n \backslash G_n / U_{n-1}).$$

By (algebraic) Peter-Weyl theory, $\mathcal{M}$ is $\Lambda_{n-1,n}$-graded:

$$\mathcal{M} = \bigoplus_{(\mu,\lambda) \in \Lambda_{n-1,n}} M_\mu^\lambda.$$

In other words, the graded components are isomorphic to symplectic multiplicity spaces. $\mathcal{M}$ is an example of a branching algebra.

We want to compare $\mathcal{M}$ to a branching algebra corresponding to restriction from GL_{n+1} to GL_{n-1}. Now let $U_n \subset GL_n$ be the upper triangular unipotent matrices and $M_{m,n}$ the $m \times n$ matrices with complex entries. We use (GL_n, GL_{n+1})-duality to construct a branching algebra:

$$\mathcal{N} = \mathcal{O}(U_n \backslash M_{n,n+1} / U_{n-1}) = \bigoplus_{(\mu,\lambda) \in \Lambda_{n-1,n}} N_\mu^{\lambda^+}.$$

Here if $\lambda \in \Lambda_n$ we set $\lambda^+ = (\lambda_1, ..., \lambda_n, 0) \in \Lambda_{n+1}$. Thus $\mathcal{N}$ is also $\Lambda_{n-1} \times \Lambda_n$-graded; its graded components are certain multiplicity spaces that occur in branching from GL_{n+1} to GL_{n-1}.

Notice that $\mathcal{M}$ and $\mathcal{N}$ are both graded by the same semigroup. Moreover, they both carry a natural action of SL_2. Let $f : G_n \to M_{n,n+1}$ be defined by taking g to its principal $n \times (n+1)$ cut-off. Consider the induced map

$$f^* : \mathcal{O}(M_{n,n+1}) \to \mathcal{O}(G_n)$$

on functions. It's not hard to show that $f^*(\mathcal{N}) \subset \mathcal{M}$.

Theorem 1 (Theorem 3.1, [2]). *$f^* : \mathcal{N} \to \mathcal{M}$ is an isomorphism of $\Lambda_{n-1} \times \Lambda_n$-graded SL_2-algebras.*

This theorem allows us to reduce questions about the branching of the symplectic groups to ones about branching from GL_{n+1} to GL_{n-1}. For example, if $(\mu, \lambda) \in \Lambda_{n-1,n}$, then by the above theorem M_μ^λ is isomorphic to $N_\mu^{\lambda^+}$ as SL_2-modules. Moreover, to determine the SL_2-module structure of $N_\mu^{\lambda^+}$ is easy since, by factoring through GL_n, we can simply write down the character. Let F_k be the $k+1$ dimensional irreducible representation of SL_2.

Proposition 2 (cf. [1]). *Suppose μ double interlaces λ. Then as SL_2-modules*

$$M_\mu^\lambda \cong \bigotimes_{i=1}^{n} F_{r_i(\mu,\lambda)},$$

where SL_2 acts by the tensor product representation on the right hand side, and

$$r_i(\mu,\lambda) = \min(\mu_{i-1}, \lambda_i) - \max(\mu_i, \lambda_{i+1})$$

with $\lambda_{n+1} = \mu_0 = 0$.

This answers the question posed above, but it also suggests a deeper question. For $(\mu, \lambda) \in \Lambda_{n-1,n}$ consider the irreducible $L = \prod_{i=1}^n SL_2$-module $A_\mu^\lambda = \bigotimes_{i=1}^n F_{r_i(\mu,\lambda)}$. The above proposition states that $M_\mu^\lambda \cong Res_{SL_2}^L A_\mu^\lambda$, where $SL_2 \subset L$ is diagonally embedded. We therefore ask, is there a natural action of L on M_μ^λ such that $M_\mu^\lambda \cong A_\mu^\lambda$ as L-modules?

To answer this we investigate the double interlacing condition that characterizes symplectic branching. An **order type** σ is a word in the alphabet $\{\geq, \leq\}$ of length $n-1$. Suppose $(\mu, \lambda) \in \Lambda_{n-1,n}$ and $\sigma = (\sigma_1 \cdots \sigma_{n-1})$ is an order type. Then we say (μ, λ) is **of order type** σ if for $i = 1, ..., n-1$,

$$\begin{cases} \sigma_i = \text{“} \geq \text{”} \Longrightarrow \mu_i \geq \lambda_{i+1} \\ \sigma_i = \text{“} \leq \text{”} \Longrightarrow \mu_i \leq \lambda_{i+1} \end{cases}$$

Let Σ be the set of order types, and for each $\sigma \in \Sigma$ let $\Lambda_{n-1,n}(\sigma)$ be the pairs (μ, λ) of order type σ. It's easy to check that $\Lambda_{n-1,n}(\sigma)$ is a semigroup, and therefore

$$\mathcal{M}(\sigma) = \bigoplus_{(\mu,\lambda)\in\Lambda_{n-1,n}(\sigma)} M_\mu^\lambda$$

is a subalgebra of $\mathcal{M}$. Moreover, $\mathcal{M}(\sigma)$ is SL_2 invariant.

Our second result says that the natural SL_2 action on M_μ^λ can be extended canonically to an irreducible action of L, thereby resolving the multiplicities that occur in symplectic branching. We warn the reader here that the group L is not the n-fold product of $SL_2's$ which naturally embeds in G_n. Indeed, this latter n-fold product does not act on M_μ^λ.

Theorem 3 (Theorem 3.8, [2]). *There is a unique representation $(\Phi, \mathcal{M})$ of L satisfying the following two properties:*

1. *for all μ, λ, M_μ^λ is an irreducible L-invariant subspace of $\mathcal{M}$ isomorphic to A_μ^λ, and*
2. *for all $\sigma \in \Sigma$, L acts as algebra automorphisms on $\mathcal{M}(\sigma)$.*

Moreover, $Res^L_{SL_2}(\Phi)$ is the natural action of SL_2 on $\mathcal{M}$.

References

[1] Wallach, Nolan; Yacobi, Oded, A multiplicity formula for tensor products of SL_2 modules and an explicit Sp_{2n} to $Sp_{2n-2} \times Sp_2$ branching formula, *Contemp. Math.*, American Mathematical Society, Providence, R.I., 2009/

[2] Yacobi, Oded, An analysis of the multiplicity spaces in classical symplectic branching, preprint arXiv:0907.324 [math.RT]

Pieces of nilpotent cones for classical groups

Anthony Henderson

(joint work with Pramod Achar, Eric Sommers)

Any complex reductive group G acts with finitely many orbits in its *nilpotent cone*

$$\mathcal{N}(\mathfrak{g}) = \{x \in \mathfrak{g} = \mathrm{Lie}(G) \,|\, x \text{ is nilpotent}\}.$$

For example, it is well known that the GL_n-orbits in $\mathcal{N}(\mathfrak{gl}_n)$ are in bijection with $\mathcal{P}_n$, the set of partitions of n: for $\lambda \in \mathcal{P}_n$, the corresponding orbit $\mathcal{O}^A_\lambda$ consists of those $x \in \mathcal{N}(\mathfrak{gl}_n)$ whose Jordan form has blocks of sizes $\lambda_1, \lambda_2, \cdots$. Moreover, the closure ordering on orbits corresponds to the dominance order on partitions: for $\pi, \lambda \in \mathcal{P}_n$, $\mathcal{O}^A_\pi \subseteq \overline{\mathcal{O}^A_\lambda}$ if and only if λ dominates π.

For general reductive G, the *Springer correspondence* gives an injective map

$$G \setminus \mathcal{N}(\mathfrak{g}) \hookrightarrow \mathrm{Irr}(W),$$

where $\mathrm{Irr}(W)$ denotes the set of isomorphism classes of irreducible representations of the Weyl group W of G. If $\mathcal{O}$ is a nilpotent orbit on the left-hand side and $x \in \mathcal{O}$, the associated irreducible representation of W can be realized as $H^{\mathrm{top}}(\mathcal{B}_x)^{G_x}$, where $\mathcal{B}_x$ is the Springer fibre and we take invariants for the stabilizer G_x. The special property of $G = GL_n$ which makes this injective map bijective is that all these stabilizers are connected and hence act trivially on $H^*(\mathcal{B}_x)$. (In general, to construct all of $\mathrm{Irr}(W)$ one needs to consider not just G_x-invariants but other isotypic components for the action of G_x/G_x°.)

The groups SO_{2n+1} (of type B_n) and Sp_{2n} (of type C_n) are dual to each other and have the same Weyl group $W = \{\pm 1\} \wr S_n$. However, the relationship between their nilpotent orbits is not as simple as one might suppose. We identify $\mathrm{Irr}(W)$ in the usual way with $\mathcal{Q}_n$, the set of bipartitions of n. Shoji in [6] showed that the Springer parameters for the nilpotent orbits are as follows:

$$\begin{aligned} SO_{2n+1} \setminus \mathcal{N}(\mathfrak{so}_{2n+1}) &\longleftrightarrow \mathcal{Q}^B_n := \{(\mu;\nu) \,|\, \mu_i \geq \nu_i - 2,\, \nu_i \geq \mu_{i+1}\}, \\ Sp_{2n} \setminus \mathcal{N}(\mathfrak{sp}_{2n}) &\longleftrightarrow \mathcal{Q}^C_n := \{(\mu;\nu) \,|\, \mu_i \geq \nu_i - 1,\, \nu_i \geq \mu_{i+1} - 1\}. \end{aligned}$$

We write $\mathcal{O}^B_{\mu;\nu}$ for the orbit in $\mathcal{N}(\mathfrak{so}_{2n+1})$ corresponding to $(\mu;\nu) \in \mathcal{Q}^B_n$, and $\mathcal{O}^C_{\mu;\nu}$ for the orbit in $\mathcal{N}(\mathfrak{sp}_{2n})$ corresponding to $(\mu;\nu) \in \mathcal{Q}^C_n$.

To compare the two collections of nilpotent orbits, we consider

$$\mathcal{Q}^\circ_n = \mathcal{Q}^B_n \cap \mathcal{Q}^C_n = \{(\mu;\nu) \,|\, \mu_i \geq \nu_i - 1,\, \nu_i \geq \mu_{i+1}\},$$