A QUANTUM ANALOGUE OF KOSTANT'S THEOREM FOR THE GENERAL LINEAR GROUP

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ABSTRACT. A fundamental result in representation theory is Kostant's theorem which describes the algebra of polynomials on a reductive Lie algebra as a module over its invariants. We prove a quantum analogue of this theorem for the general linear group, and from this deduce the analogous result for reflection equation algebras.

1. INTRODUCTION

A classical theorem of Kostant's states that the algebra of polynomials $\mathcal{O}(\mathfrak{g})$ on a reductive Lie algebra $\mathfrak{g}$ is free as a module over the invariant polynomials $\mathcal{O}(\mathfrak{g})^G$ (see [K]). This result, which was later generalized by Kostant and Rallis to arbitrary symmetric pairs (see [KR]), is fundamental to representation theory. In particular, it plays an important role in understanding geometric properties of the nilpotent cone, and representation theoretic properties of its ring of regular functions (see e.g. Chapter 6 of [CG]).

In this paper we prove a quantum analog of Kostant's Theorem for the general linear group. Namely, we show that the coordinate ring of quantum matrices $\mathcal{A} = \mathcal{O}(M_q(n))$ is free as a module over $\mathcal{I}$, the subalgebra of invariants under the adjoint coaction of $\mathcal{O}(GL_q(n))$, for q not a root of unity or $q = 1$. At $q = 1$ this is a restatement of Kostant's Theorem for the general linear group.

Several proofs of Kostant's Theorem in the classical case have appeared over the last forty years. Our proof in the quantum case is adapted from the argument in [BL], which is similar to an earlier proof appearing in [W]. In order to explain our approach, we briefly sketch their argument in the case of the general linear group.

Consider the filtration on $\mathcal{O}(\mathfrak{gl}_n(\mathbb{C}))$ given by $deg(x_{ij}) = \delta_{ij}$, where $\{x_{ij}\}$ are the standard coordinates on $\mathfrak{gl}_n(\mathbb{C})$. Now let I be the subalgebra of $\mathcal{O}(\mathfrak{gl}_n(\mathbb{C}))$ consisting of $GL_n(\mathbb{C})$-invariant polynomials, with the induced filtration. Then the fact that $\mathcal{O}(\mathfrak{gl}_n(\mathbb{C}))$ is free over I, follows from the fact that $gr(\mathcal{O}(\mathfrak{gl}_n(\mathbb{C}))$ is free over grI. This, in turn, follows from the standard

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fact that the algebra of polynomials is free as a module over the ring of symmetric polynomials.

While our proof is based on the same idea, the quantum setting presents new complications. Indeed, the above filtration cannot be adapted to the quantum setting in a manner that is compatible with the algebra structure. Therefore we have to use a more subtle approach, whereby we use a succession of filtrations each of which slightly simplifies the relations.

More precisely, we first construct a filtration on $\mathcal{A}$ that is compatible with the algebra structure. We then consider the associated graded algebra $\mathcal{A}' = \mathrm{gr}\mathcal{A}$ as a module over $\mathcal{I}' = \mathrm{gr}\mathcal{I}$. However, our filtration is weak in the sense that the freeness of $\mathcal{A}'$ over $\mathcal{I}'$ does not follow from standard facts. Nevertheless, the algebra $\mathcal{A}'$ has slightly simpler relations than the original algebra. This allows us to define a "stronger" filtration on $\mathcal{A}'$ and again consider it's associated graded algebra $\mathcal{A}''$.

We continue in such a way until we can reduce to the standard fact mentioned above. This argument relies on a theorem of Domokos and Lenagan ([DL]) which gives an explicit presentation of $\mathcal{I}$. We remark that their result has been extended to more general q, and consequently our results extends to these cases as well (see [AZ] and Remark 3.1.2).

Our result implies the analogous statement in the setting of reflection equation algebras (also known as "braided matrices"). The reflection equation algebra $\mathcal{S}$ is another quantization of the coordinate ring of $n \times n$ matrices, which is also endowed with an adjoint coaction of the quantum general linear group. In contrast to $\mathcal{A}$, the reflection equation algebra $\mathcal{S}$ is a comodule-algebra, and moreover its invariants with respect to the adjoint coaction are central. We prove that $\mathcal{S}$ is free as a left $\mathcal{J}$-module, where $\mathcal{J}$ is the algebra of invariants.

As a corollary of our main result, we obtain a (non-canonical) equivariant decomposition of $\mathcal{A}$ as a tensor product of $\mathcal{I}$ and a $\mathcal{G}$-comodule $\mathcal{H}$. We also obtain the analogous result for $\mathcal{S}$; the benefit of this formulation is that the $\mathcal{G}$-comodule corresponding to $\mathcal{H}$ is now an algebra. This algebra can be regarded as a quantization of the algebra of functions on the nilpotent cone (see [D]). It would be interesting to make these decompositions canonical by defining a quantum analogue of the harmonic polynomials.

In the literature there are other quantum analogues of Kostant's Theorem. In [JL] it is proven that the locally finite part of the quantum enveloping algebra $U_q(\mathfrak{g})$ of a semisimple Lie algebra $\mathfrak{g}$ is free over its center. Another analogue appears in [B], where it is shown that the algebra $\mathcal{O}_q(G)$ is free over its invariants with respect to its adjoint coaction for simple simply connected G for generic q, and this is used by Baumann to give a new proof of the Joseph-Letzter Theorem. From our result one can deduce Baumann's theorem for the general linear group.

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2. Preliminaries

2.1. **Filtered algebras.**

We begin by recording some standard notations regarding filtrations. Let (V, F) be a linear space V with a filtration F. All filtrations we consider in this paper will be ascending. For $x \in V$ we denote by $\deg_F(x) = \min\{d : x \in F^d V\}$. The symbol map $\sigma_F^d : F^d V \to \mathrm{gr}_F^d V$ maps an element x to $x + F^{d-1}V$. For any $x \in V$ we let $\sigma_F(x) = \sigma_F^{\deg_F(x)}(x)$.

Lemma 2.1.1. *Let* (A, F) *be a filtered algebra and let* $x, y \in A$. *If* $\deg_F(xy) = \deg_F(x) + \deg_F(y)$ *then* $\sigma_F(xy) = \sigma_F(x)\sigma_F(y)$.

Lemma 2.1.2 (Lemma 4.2, [BL])**.** *Let* (M, F) *be a filtered module over a filtered algebra* (A, F) *and* $\{m_k\}$ *a family of elements of* M. *Suppose that the symbols* $\sigma_F(m_k)$ *form a free basis of the* $\mathrm{gr}_F(A)$*-module* $\mathrm{gr}_F(M)$. *Then* $\{m_k\}$ *is a free basis of the* A*-module* M.

Lemma 2.1.3. *Let* $A = \bigoplus_{d\geq 0} A_d$ *be a unital graded associative algebra, and let* $I = \bigoplus_{d\geq 0} I_d \subset A$ *be a unital graded subalgebra. Suppose* $I_0 = A_0 = \mathrm{span}\{1\}$*, and that* A *is a free left* I*-module. Define* $I_+ = \bigoplus_{d>0} I_d$ *and let* $H \subset A$ *be a graded linear complement to* AI_+*:*

$$H \oplus AI_+ = A.$$

Then the multiplication map

$$H \otimes I \to A$$

is an isomorphism of left I*-modules.*

For a proof of this lemma see the proof of Theorem 6.3.3 in [CG] (p. 319).

2.2. **Quantum groups.**

We recall the definition of the quantum $n \times n$ matrices and the quantum general linear group. Fix $q \in \mathbb{C}^\times$ and let $\mathcal{O}(M_q(n))$ be the bi-algebra of quantum $n \times n$ matrices, i.e. $\mathcal{O}(M_q(n))$ is the $\mathbb{C}$-algebra generated by indeterminates x_{ij} $(i, j = 1, ..., n)$ subject to the following relations:

$$\begin{align} x_{ij}x_{il} &= qx_{il}x_{ij} \tag{1}\\ x_{ij}x_{kj} &= qx_{kj}x_{ij} \tag{2}\\ x_{il}x_{kj} &= x_{kj}x_{il} \tag{3}\\ x_{ij}x_{kl} - x_{kl}x_{ij} &= (q - q^{-1})x_{il}x_{kj} \tag{4}\end{align}$$

where $1 \leq i < k \leq n$ and $1 \leq j < l \leq n$.

We introduce a diagrammatic shorthand to work with these relations. First consider the case $n = 2$. The relations defining $\mathcal{O}(M_q(2))$ are encapsulated in the following diagram:

$$\begin{tikzcd} x_{11} \arrow[r] \arrow[d] \arrow[dr, squiggly] & x_{12} \arrow[d] \arrow[dl, dash] \\ x_{21} \arrow[r] & x_{22} \end{tikzcd}$$

Here, if there is an undirected edge between x_{ij} and x_{kl} then $[x_{ij}, x_{kl}] = 0$. A directed edge from x_{ij} to x_{kl} means $x_{ij}x_{kl} = qx_{kl}x_{ij}$. Finally, a curly directed edge from x_{ij} to x_{kl} means they satisfy the "complicated" relation (4) above.

In general, the relations defining $\mathcal{O}(M_q(n))$ can be expressed as: every 2×2 submatrix of $\mathcal{O}(M_q(n))$ generates a copy of $\mathcal{O}(M_q(2))$. For instance, if $n = 3$ then the submatrix obtained by choosing the first and third row and the second and third columns contributes the following relations:

$$\begin{tikzcd} \bullet & \bullet \arrow[r] \arrow[dd, bend right] \arrow[ddr, squiggly] & \bullet \arrow[dd, bend left] \arrow[ddl, dash] \\ \bullet & \bullet & \bullet \\ \bullet & \bullet \arrow[r] & \bullet \end{tikzcd}$$

Theorem 2.2.1 (Theorem 3.5.1, [PW]). *$\mathcal{O}(M_q(n))$ has a PBW-type basis consisting of monomials $\{x^a : a \in M_n(\mathbb{N})\}$, where $x^a = x_{11}^{a_{11}}x_{12}^{a_{12}} \cdots x_{nn}^{a_{nn}}$.*

Fix $n \in \mathbb{N}$ and let $\mathcal{A} = \mathcal{O}(M_q(n))$. $\mathcal{A}$ has a standard grade defined by setting $\deg(x_{ij}) = 1$ for all i and j. The quantum determinant is a central element of $\mathcal{A}$ given by

$$\det_q = \sum_{w \in S_n} (-q)^{l(w)} x_{1w(1)} \cdots x_{nw(n)}.$$

By adjoining the inverse of $\det_q$ we obtain the quantum general linear group

$$\mathcal{G} = \mathcal{O}(GL_q(n)) = \mathcal{A}[\det_q^{-1}].$$

$\mathcal{G}$ is a Hopf algebra, and we denote the antipode of this algebra by S. We will denote the element x_{ij} by u_{ij} when we are considering it as an element of $\mathcal{G}$. There is an adjoint coaction of $\mathcal{G}$ on $\mathcal{A}$, which, following [DL], we write as a right coaction:

$$\alpha_q : \mathcal{A} \to \mathcal{A} \otimes \mathcal{G}.$$

given by

$$\alpha_q(x_{ij}) = \sum_{m,s=1}^{N} x_{ms} \otimes u_{sj} S(u_{im}).$$

There is a variant of the adjoint coaction, denoted $\beta_q : \mathcal{A} \to \mathcal{A} \otimes \mathcal{G}$, given by the formula

$$\beta_q(x_{ij}) = \sum_{m,s=1}^{N} x_{ms} \otimes S(u_{im}) u_{sj}.$$

This is called the right adjoint coaction.

2.3. **Invariants of the adjoint coaction.** An invariant of the coaction α_q is by definition an element $b \in \mathcal{A}$ such that $\alpha_q(b) = b \otimes 1$. (In [DL] these are referred to as "coinvariants". By Theorem 2.1 in loc. cit. the invariants of the coaction α_q are precisely the cocommutative elements of $\mathcal{A}$.) Let $\mathcal{I}$ denote the set of invariants of $\mathcal{A}$ with respect to the coaction α_q. We let $\mathcal{I}'$ denote the set of invariants of $\mathcal{A}$ with respect to the coaction β_q. Notice that in the classical $(q = 1)$ case, the set $\mathcal{I}$ agrees with the usual invariants of the action of $GL(n)$ on the coordinate ring of its Lie algebra, $\mathcal{O}(M(n))$.

In [DL], Domokos and Lenagan explicitly determine $\mathcal{I}$. Let us describe their result: for $1 \leq d \leq n$ and a subset $I = \{i_1 < \cdots < i_d\} \subset \{1, ..., n\}$, let $\det_{q,I}$ be the principal minor corresponding to I:

$$\det{}_{q,I} = \sum_{w \in S_I} (-q)^{l(w)} x_{i_1 w(i_1)} \cdots x_{i_d w(i_d)}$$

and set

$$\Delta_d = \sum_{|I|=d} \det{}_{q,I}.$$

Similarly set

$$\Delta'_d = \sum_{|I|=d} q^{-2(\sum_{i \in I} i)} \det{}_{q,I}.$$

It's not hard to see that $\Delta_d \in \mathcal{I}$ and $\Delta'_d \in \mathcal{I}'$ for every d ([DL], Propositions 4.1 and 7.2).

Theorem 2.3.1. *([DL], Corollary 6.2 and Theorem 7.3) For $q \in \mathbb{C}^\times$ not a root of unity or $q = 1$, $\mathcal{I}$ is a commutative polynomial algebra on the Δ_d, and similarly $\mathcal{I}'$ is a commutative polynomial algebra on the Δ'_d.*

3. Main results

3.1. **Main Theorem.** We consider $\mathcal{A}$ as a left $\mathcal{I}$-module. Our main result is the following quantum analogue of Kostant's classical theorem:

Theorem 3.1.1. *For* $q \in \mathbb{C}^\times$ *not a root of unity or* $q = 1$*,* $\mathcal{A}$ *is a free graded left* $\mathcal{I}$*-module.*

Remark 3.1.2. *The condition on* q *in the hypothesis of the theorem is needed only for the application of Theorem 2.3.1. In* [AZ]*, Theorem 2.3.1 is extended to include the cases* $q = -1$ *or* q *is a primitive root of unity of odd degree. Therefore our theorem extends to these cases as well. Moreover, Antonov and Zubkov show in* [AZ] *that Theorem 2.3.1 holds over fields of arbitrary characteristic. Theorem 3.1.1, but not its corollary below, also extends to this setting.*

Remark 3.1.3. *The same result and proof hold for* $\mathcal{A}$ *regarded as a right* $\mathcal{I}$*-module, and, moreover, for* $\mathcal{A}$ *regarded as a left and right* $\mathcal{I}'$*-module.*

Before beginning the proof of this theorem we record a corollary. Let $\mathcal{I}_+$ be the augmentation ideal of $\mathcal{I}$, i.e. $\mathcal{I}_+$ equals the elements in $\mathcal{I} = \mathbb{C}[\Delta_1, ..., \Delta_n]$ with zero constant term. Define $\mathcal{I}^{\mathcal{A}}$ to be the left ideal of $\mathcal{A}$ generated by $\mathcal{I}_+$. By ([DL], Lemma 2.2) for $x \in \mathcal{A}$ and $y \in \mathcal{I}$, $\alpha_q(xy) = \alpha_q(x)\alpha_q(y)$. Therefore $\mathcal{I}^{\mathcal{A}}$ is an $\mathcal{G}$-invariant graded subspace of $\mathcal{A}$.

Set $\mathcal{H} = \mathcal{A}/\mathcal{I}^{\mathcal{A}}$. Since q is not a root of unity, the representation theory of $\mathcal{G}$ is semisimple (see e.g. [KS]) and so we can (non-canonically) identify $\mathcal{A}$ with $\mathcal{H} \oplus \mathcal{I}^{\mathcal{A}}$ as graded $\mathcal{G}$-comodules. Now, by Theorem 3.1.1 and Lemma 2.1.3 we conclude:

Corollary 3.1.4. *For* $q \in \mathbb{C}^\times$ *not a root of unity or* $q = 1$*, the multiplication map in* $\mathcal{A}$ *gives an* $\mathcal{G}$*-equivariant isomorphism of graded vector spaces*

$$\mathcal{H} \otimes \mathcal{I} \cong \mathcal{A}.$$

3.2. **Reflection Equation Algebras.** In this section we show that our main theorem has an analogue in the setting of reflection equation algebras.

The reflection equation algebra, denoted $\mathcal{S}$, is another quantization of the coordinate ring of $n \times n$ matrices due to Majid. For a precise definition of $\mathcal{S}$ see [D] and references therein.

For us, the most important properties of $\mathcal{S}$ are the following: $\mathcal{S}$ has an adjoint coaction of $\mathcal{G}$, $\mathcal{S}$ is a comodule-algebra with respect to this action, and there exists a graded $\mathcal{G}$-comodule isomorphism

$$\Phi : \mathcal{A} \to \mathcal{S}$$

intertwining the β-coaction on $\mathcal{A}$ with the adjoint coaction on $\mathcal{S}$. The map Φ is not an algebra homomorphism. Nevertheless, it does satisfy the property

$$\Phi(ab) = \Phi(a)\Phi(b) \tag{5}$$

for $a \in \mathcal{I}'$ and $b \in \mathcal{A}$ (see the proof of Lemma 3.2 in [D]).

Let $\mathcal{J} \subset \mathcal{S}$ be the subalgebra of invariants with respect to the adjoint coaction of $\mathcal{G}$. Since Φ is a comodule isomorphism, $\mathcal{J} = \Phi(\mathcal{I}')$. Since $\mathcal{S}$ is a comodule-algebra, $\mathcal{J}$ is central. Now Theorem 3.1.1 implies the following.

Theorem 3.2.1. *The algebra $\mathcal{S}$ is free as a $\mathcal{J}$-module.*

We also have an analogue of Corollary 3.1.4. Indeed, define $\mathcal{J}^{\mathcal{S}}$ as we did $\mathcal{I}^{\mathcal{A}}$, and let $\mathcal{H}' = \mathcal{S}/\mathcal{J}^{\mathcal{S}}$. In contrast to $\mathcal{H}$, $\mathcal{H}'$ is an algebra which is a quantum deformation of the coordinate ring of the nilpotent cone (see [D]).

Corollary 3.2.2. *For $q \in \mathbb{C}^\times$ not a root of unity or $q = 1$, we have a (non-canonical) $\mathcal{G}$-equivariant isomorphism of graded vector spaces*

$$\mathcal{H}' \otimes \mathcal{J} \cong \mathcal{S}.$$

Note that this is an isomorphism of $\mathcal{J}$-modules, but the map is not an algebra morphism.

4. THE PROOF

4.1. **Sketch of proof.** In this section we sketch the proof of Theorem 3.1.1. Our goal is to reduce the theorem to the following standard fact:

Proposition 4.1.1. *The polynomial algebra $\mathbb{C}[y_1, ..., y_n]$ in n indeterminates is a free module of rank $n!$ over the ring of symmetric polynomials $\mathbb{C}[y_1, ..., y_n]^{S_n}$. Moreover the set*

$$\{y_1^{a_1} \cdots y_n^{a_n} : 0 \le a_i < i \text{ for all } 1 \le i \le n\}$$

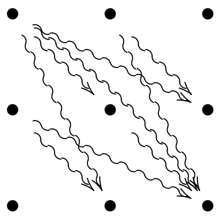
is a basis.

We would like to mimic the proof in [BL] and define a filtration F on $\mathcal{A}$ by setting $F^d\mathcal{A} = span\{x^a : trace(a) \le d\}$, and then appeal to Lemma 2.1.2. The complication is that for $n \ge 3$ this filtration is not compatible with the algebra structure of $\mathcal{A}$. For example $F^0\mathcal{A} \cdot F^0\mathcal{A} \not\subseteq F^0\mathcal{A}$ since

$$x_{23}x_{12} = x_{12}x_{23} - (q - q^{-1})x_{13}x_{22}.$$

To get around this we will use a succession of filtrations, each one of which slightly simplifies the quantum relations.

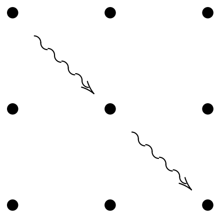
To explain the idea let us consider the case $\mathcal{A} = \mathcal{O}_q(M_3(\mathbb{C}))$. Ignoring the relations of type (1)-(3), the complicated (i.e. "curly") relations in $\mathcal{A}$ are



Let F be the filtration on $\mathcal{A}$ defined by

$$F^d\mathcal{A} = span\{x^a : \sum_{|i-j|<2} a_{ij} \leq d\}.$$

F preserves the algebra structure of $\mathcal{A}$ (cf. Lemma 4.3.1 below) and so we consider the associated graded algebra $\mathcal{A}' = gr_F\mathcal{A}$. In $\mathcal{A}'$ most of the complicated relations disappear and we are left with



Our next step is to introduce a filtration on $\mathcal{A}'$ which will further simplify the relations. $\mathcal{A}'$ has a PBW type basis (cf. Lemma 4.2.2(3) and Lemma 4.3.1(4) below), which by abuse of notation, we can continue to denote as $\{x^a\}$. Define a filtration F' on $\mathcal{A}'$ by

$$F'_d\mathcal{A}' = span\{x^a : \sum_{|i-j|<1} a_{ij} \leq d\}.$$

We show below that F' is compatible with the product in $\mathcal{A}'$, and hence we can consider $\mathcal{A}'' = gr_{F'}\mathcal{A}'$. Now, in $\mathcal{A}''$ all the complicated relations disappear. Moreover, the image, $\mathcal{I}''$, of the subalgebra $\mathcal{I}$ in $\mathcal{A}''$ consists of the symmetric polynomials in the diagonal entries. Therefore it is easy to see that $\mathcal{A}''$ is free over $\mathcal{I}''$ using Proposition 4.1.1. By Lemma 2.1.2 we conclude our result.

4.2. **q-Mutation Systems.** We now introduce the terminology needed to handle successions of filtrations.

Suppose we have an ordered set $\{I, \leq\}$ and a collection of indeterminates $\{x_i\}_{i\in I}$. We would like to discuss an algebra on the $\{x_i\}$ subject to certain commutation relations. Let $\mathcal{F}$ be the free algebra on the $\{x_i\}$. A **standard monomial** $x_{i_1} \cdots x_{i_l} \in \mathcal{F}$ is one such that $i_1 \leq \cdots \leq i_l$.

Definition 4.2.1.

1. *A* **q-*mutation system*** *is a tuple* $S = (\{q_{ij}\}, \{f_{ij}\})$ *where* $i < j \in I$, $q_{ij} \in \mathbb{C}^\times$, *and* $f_{ij} \in \mathcal{F}$. *We denote by* $\mathcal{A}(S)$ *the quotient of* $\mathcal{F}$ *by the two-sided ideal generated by* $x_j x_i - (q_{ij} x_i x_j + f_{ij})$.
2. *Let* $\xi = x_{i_1} \cdots x_{i_l}$ *be a monomial in* $\mathcal{F}$, *and suppose there exists* r *such that* $i_r < i_{r-1}$. *Then an* ***elementary mutation*** *of* ξ *in the* r^{th} *position is the sum of elements*

$$(q_{i_r i_{r-1}} x_{i_1} \cdots x_{i_{r-2}} x_{i_r} x_{i_{r-1}} x_{i_{r+1}} \cdots x_{i_l}) + (x_{i_1} \cdots x_{i_{r-2}} f_{i_r i_{r-1}} x_{i_{r+1}} \cdots x_{i_l}).$$

   A elementary mutation of a polynomial $f \in \mathcal{F}$ *is the polynomial obtained by an elementary mutation of one of its monomials.*
3. *A* q*-mutation system* S *has* ***finite mutation property*** *(FMP) if any monomial* $x_{i_1} \cdots x_{i_l}$ *can be transformed into a linear combination of standard monomials using finitely many elementary mutations.*
4. *The* q*-mutation system* S *satisfies* ***Poincare-Birkhoff-Witt property*** *(PBW) if the images of standard monomials form a basis in* $\mathcal{A}(S)$.

A **weighting** of I is a function $w : I \to \mathbb{Z}_{\geq 0}$. A weighting w defines a filtration F_w of $\mathcal{F}$ by $\deg_{F_w} x_{i_1} \cdots x_{i_l} = \sum w(i_k)$. If a q-mutation system S satisfies the PBW property then a weighting w defines a linear filtration $F_{w,S}$ on $\mathcal{A}(S)$ in a natural way. Precisely, $F^d_{w,S}\mathcal{A}(S)$ is the span of all images of standard monomials ξ such that $\deg_{F_w} \xi \leq d$.

We call a weighting w **compatible** with $S = (\{q_{ij}\}, \{f_{ij}\})$ if for all $i < j$, $\deg_{F_w} f_{ij} \leq w(i) + w(j)$. If a weighting w is compatible with $S = (\{q_{ij}\}, \{f_{ij}\})$ then we define a q-mutation system

$$\sigma_w(S) = (\{q_{ij}\}, \{\sigma^{w(i)+w(j)}_{F_w}(f_{ij})\}).$$

Here we identify the linear spaces $\mathrm{gr}_{F_w}\mathcal{F}$ with $\mathcal{F}$.

Lemma 4.2.2. *Let* $S = (\{q_{ij}\}, \{f_{ij}\})$ *be a* q*-mutation system with a compatible weighting* w. *Suppose* S *satisfies the FMP and PBW properties. Consider the natural projection* $p : \mathcal{F} \to \mathcal{A}(S)$. *Then,*

1. $p(F^d_w \mathcal{F}) = F^d_{w,S}\mathcal{A}(S)$.
2. *The linear filtration* $F_{w,S}$ *is compatible with the algebra structure of* $\mathcal{A}(S)$.
3. *Suppose* $\sigma_w(S)$ *satisfies FMP. Then there is a natural isomorphism* $\mathrm{gr}_{F_{w,S}}\mathcal{A}(S) \cong \mathcal{A}(\sigma_w(S))$.

(4) $\sigma_w(S)$ *satisfies the PBW property.*

Proof. To prove (1) note that by definition of $F_{w,S}$ we have the inclusion $p(F_w^d\mathcal{F}) \supset F_{w,S}^d\mathcal{A}(S)$. Conversely, let $\xi \in F_w^d\mathcal{F}$. Since S satisfies FMP there exist elements $\xi_1 = \xi, \xi_2, ..., \xi_n \in \mathcal{F}$ such that ξ_n is a linear combination of standard monomials, and ξ_{i+1} is an elementary mutation of ξ_i for all i. By the compatibility condition,

$$\deg_{F_w}(\xi_1) \geq \deg_{F_w}(\xi_2) \geq \cdots \geq \deg_{F_w}(\xi_n).$$

Therefore $\xi_n \in F_w^d\mathcal{F}$, and since it's a linear combination of standard monomials $p(\xi_n) \in F_{w,S}^d\mathcal{A}(S)$. Since moreover $p(\xi_1) = \cdots = p(\xi_n)$, we conclude $p(\xi) \in F_{w,S}^d\mathcal{A}(S)$.

Part (2) follows from part (1).

For part (3), the natural surjection from $\mathcal{F}$ to $\mathrm{gr}_{F_{w,S}}\mathcal{A}(S)$ factors through a surjection $f : \mathcal{A}(\sigma_w(S)) \to \mathrm{gr}_{F_{w,S}}\mathcal{A}(S)$. Now note that the weighting w defines a grading on $\mathcal{F}$. Since the relations defining $\mathcal{A}(\sigma_w(S))$ are homogenous, $\mathcal{A}(\sigma_w(S))$ inherits an induced grading, the d^{th} component of which we denote $\mathcal{A}^d(\sigma_w(S))$. The morphism f is clearly graded, and hence $f(\mathcal{A}^d(\sigma_w(S))) = \mathrm{gr}^d_{F_{w,S}}\mathcal{A}(S)$, which implies

$$\dim(\mathcal{A}^d(\sigma_w(S)) \geq \dim(\mathrm{gr}^d_{F_{w,S}}\mathcal{A}(S)).$$

To see that $\dim(\mathcal{A}^d(\sigma_w(S)) \leq \dim(\mathrm{gr}^d_{F_{w,S}}\mathcal{A}(S))$ note that since $\sigma_w(S)$ satisfies FMP, the images of standard monomials of degree d span $\mathcal{A}^d(\sigma_w(S))$. Since S satisfies the PBW property, the standard monomials of degree d form a basis for $\mathrm{gr}^d_{F_{w,S}}\mathcal{A}(S)$.

Finally, part (4) follows from (3). □

4.3. **Proof of main theorem.** We now specialize the terminology introduced above to our case. Let $I = \{(i,j) : i,j = 1, ..., n\}$ be ordered lexicographically, i.e. $(i,j) \leq (k,l)$ if, and only if, $ni + j \leq nk + l$. We introduce a family $\{S_t\}$ of q-mutation systems for $t = 1, ..., n$.

Let us first define the q-mutation system that's naturally associated to $\mathcal{A}$. Let $S_1 = (\{q_{ij,kl}\}, \{f_{ij,kl}\})$ where

$$q_{ij,kl} = \begin{cases} 1 & \text{if } x_{ij} \text{ ——— } x_{kl} \text{ or } x_{ij} \rightsquigarrow x_{kl}\ ; \\ q^{-1} & \text{if } x_{ij} \longrightarrow x_{kl}\ . \end{cases}$$

and

$$f_{ij,kl} = \begin{cases} 0 & \text{if } x_{ij} \text{ ——— } x_{kl} \text{ or } x_{ij} \longrightarrow x_{kl}\ ; \\ (q^{-1} - q)x_{il}x_{kj} & \text{if } x_{ij} \rightsquigarrow x_{kl}\ . \end{cases}$$

It's clear that $\mathcal{A}(S_1) = \mathcal{A}$.

Now let $t \in \{1, ..., n\}$. For $\iota = (i,j) \in I$, let $\epsilon(\iota) = |i-j|$. Let w_t be the weighting defined by $w_t(\iota) = 1$ if $\epsilon(\iota) < n-t$ and zero otherwise. Define $S_t = (\{q_{ij,kl}\}, \{f^{(t)}_{ij,kl}\})$ to be the q-mutation system where the scalars $q_{ij,kl}$ are the same as above and,

$$f^{(t)}_{ij,kl} = \begin{cases} 0 & \text{if } x_{ij} \;\rule[0.5ex]{2em}{0.4pt}\; x_{kl} \text{ or } x_{ij} \longrightarrow x_{kl}\ ; \\ (q^{-1}-q)x_{il}x_{kj}w_{t-1}(i,l)w_{t-1}(k,j) & \text{if } x_{ij} \rightsquigarrow x_{kl}\ . \end{cases}$$

Lemma 4.3.1.

(1) *The weighting* w_t *is compatible with* S_t.
(2) S_t *and* S_{t+1} *are related by* $\sigma_{w_t}(S_t) = S_{t+1}$.
(3) S_t *satisfies that FMP property.*
(4) S_t *satisfies that PBW property.*

Proof. For part (1) suppose $x_{ij} \rightsquigarrow x_{kl}$. Note that then

$$\max\{\epsilon(il), \epsilon(kj)\} > \max\{\epsilon(ij), \epsilon(kl)\}.$$

We want to show that $\deg_{F_{w_t}} f^{(t)}_{ij,kl} \leq \deg_{F_{w_t}} x_{ij}x_{kl}$. The only nontrivial case is when $w_{t-1}(il) = w_{t-1}(kj) = 1$. In this case $\max\{\epsilon(il), \epsilon(kj)\} \leq n-t$, and so $\max\{\epsilon(ij), \epsilon(kl)\} < n-t$ and so $w_t(ij) = w_t(kl) = 1$. Then

$$\deg_{F_{w_t}} x_{ij}x_{kl} = 2 \geq \deg_{F_{w_t}} f^{(t)}_{ij,kl}.$$

To show part (2) we must prove that

$$f^{(t+1)}_{ij,kl} = \sigma^{w(ij)+w(kl)}_{F_{w_t}}(f^{(t)}_{ij,kl})$$

If $f^{(t)}_{ij,kl} = 0$ then $f^{(t+1)}_{ij,kl} = 0$, so the only nontrivial case is when $f^{(t)}_{ij,kl} \neq 0$. As in the previous case, this only happens when $w_t(ij) + w_t(kl) = 2$. Therefore we have to show that $\deg_{F_{w_t}} f^{(t)}_{ij,kl} = 2$ if, and only if, $w_t(il)w_t(kj) = 1$. But this is clear since $\deg_{F_{w_t}} f^{(t)}_{ij,kl} = w_t(il) + w_t(kj)$.

For part (3) we define the descent statistic on an element of $\mathcal{F}$ by first defining

$$des(x_{\iota_1} \cdots x_{\iota_n}) = \sharp\{(k,l) : k < l \text{ and } \iota_k > \iota_l\}.$$

Extend this definition to an arbitrary element in $\mathcal{F}$ by

$$des(\sum \xi_i) = \max\{des(\xi_i)\},$$

where ξ_i are monomials in $\mathcal{F}$. To prove that S_t satisfies FMP it clearly suffices to show that if ξ' is an elementary mutation of ξ then

$$des(\xi') < des(\xi).$$

For this it is enough to show that if $\iota_k < \iota_{k-1}$, then

$$des(x_{\iota_1} \cdots x_{\iota_n}) > des(x_{\iota_1} \cdots x_{\iota_{k-1}} f^{(t)}_{\iota_k \iota_{k-1}} x_{\iota_{k+2}} \cdots x_{\iota_n}).$$

The only nontrivial case is when $x_{\iota_k} \rightsquigarrow x_{\iota_{k-1}}$. This is immediate from our definition of $f^{(t)}_{\iota_k \iota_{k-1}}$ and the definition of lexicographic ordering.

To prove part (4) first note that S_1 satisfies the PBW property by Theorem 2.2.1. Now, by induction, Lemma 4.2.2, and part (2) above, we conclude that S_t satisfies the PBW property.

□

Set $\mathcal{A}_t = \mathcal{A}(S_t)$ to be the algebra associated to S_t. We continue to denote the images of the generators of $\mathcal{F}$ in $\mathcal{A}_t$ by x_{ij}. By Lemma 4.2.2(2) and Lemma 4.3.1(2), we make the identification

$$\mathrm{gr}_{F_{w_t}} \mathcal{A}_t = \mathcal{A}_{t+1}.$$

We now want to use Lemma 2.1.2 to reduce Theorem 3.1.1 to Proposition 4.1.1. In order to do this we first consider the behavior of the algebra $\mathcal{I}$ with respect to the succession of filtrations F_{w_t}.

Definition 4.3.2. *Let $\mathcal{I}_1 = \mathcal{I}$ and define $\mathcal{I}_t \subset \mathcal{A}_t$ by induction to be the associated graded algebra $\mathrm{gr}_{F_{w_{t-1}}} \mathcal{I}_{t-1}$, where $\mathcal{I}_{t-1} \subset \mathcal{A}_{t-1}$ inherits the induced filtration from $F_{w_{t-1}}$.*

Proposition 4.3.3. *The algebra $\mathcal{I}_t$ is generated by $\{\Delta^{(t)}_d : t = 1, ..., n\}$ where*

$$\Delta^{(t)}_d = \sum_{I=\{i_1<\cdots<i_d\}} \sum_{\substack{w\in S_I \\ |i-w(i)|\le n-t}} (-q)^{l(w)} x_{i_1 w(i_1)} \cdots x_{i_d w(i_d)}.$$

Proof. Define $\mathcal{O} \subset \mathcal{F}$ to be the two-sided ideal generated by $\{x_{ij} : i \neq j\}$. Note that $\mathcal{O}$ is invariant under mutations with respect to any system S_t. Let $y_1, ..., y_n$ be indeterminates and set $\mathcal{F}_n$ to be the free algebra on the $\{y_i\}$.

Given an element $h \in \mathcal{F}_n$ we can consider the evaluation $h(x_{11}, ..., x_{nn}) \in \mathcal{F}$. Now, for any $t \in \{1, ..., n\}$ we can preform a sequence of (finitely many) elementary mutations on $h(x_{11}, ..., x_{nn})$ (with respect to S_t) to obtain an element of the form $h'(x_{11}, ..., x_{nn}) + f$. Here $h'(x_{11}, ..., x_{nn})$ is a linear combination of standard monomials and $f \in \mathcal{O}$. Note also that $\deg_{F_{w_t}} h(x_{11}, ..., x_{nn}) = \deg_{F_{w_t}} h'(x_{11}, ..., x_{nn})$, and

$$\deg_{F_{w_t}} f \le \deg_{F_{w_t}} h(x_{11}, ..., x_{nn}).$$

Let $p : \mathcal{F} \to \mathcal{A}_t$ be the natural projection. It follows from the previous assertion that for elements h, f as above,

$$\deg_{F_{w_t}, S_t}(p(h(x_{11}, ..., x_{nn}) + f)) = \deg_{F_{w_t}}(h(x_{11}, ..., x_{nn})). \tag{6}$$

Indeed, if $h(x_{11}, ..., x_{nn})$ and f are both combinations of standard monomials then this is obvious. If only $h(x_{11}, ..., x_{nn})$ is a combination of standard monomials then we can apply mutations to f to reduce to the previous case since the mutations can only decrease the degree of f and leave $f \in \mathcal{O}$. Finally, if neither are a combination of standard monomials then we can apply mutations to $h(x_{11}, ..., x_{nn})$ to reduce to the previous case.

Define a weighting u of $\{1, ..., n\}$ by $u(i) = i$. Then by (6), for $h \in \mathcal{F}_n$,

$$\deg_{F_{w_t,s_t}}(h(\Delta_1^t, ..., \Delta_n^t)) = \deg_{F_u}(h).$$

Lemma 2.1.1 implies that

$$\begin{aligned}\sigma_{F_{w_t,s_t}}(h(\Delta_1^t, ..., \Delta_n^t)) &= \sigma_{F_u}(h)(\sigma_{F_{w_t}}(\Delta_1^{(t)}), ..., \sigma_{F_{w_t}}(\Delta_n^{(t)})) \\ &= \sigma_{F_u}(h)(\Delta_1^{(t+1)}, ..., \Delta_n^{(t+1)})\end{aligned}$$

By induction on t this implies the assertion.

□

We now have all the ingredients to prove Theorem 3.1.1. Notice that the algebra $\mathcal{A}_n$ has no complicated "curly" relations. Indeed, $\mathcal{A}_n$ is generated by the $\{x_{ij}\}$ subject to relations (1)-(3), and, instead of (4), the simple commutativity $x_{ij}x_{kl} = x_{kl}x_{ij}$.

The algebra $\mathcal{A}_n$ is free over the (polynomial) algebra generated by the diagonal variables. Moreover, by Proposition 4.1.1, the latter algebra is free over the symmetric polynomials, and therefore $\mathcal{A}_n$ is free over $\mathcal{I}_n$. More precisely, Proposition 4.1.1 implies that the set of standard monomials

$$\{\prod_{\iota \in I} x_\iota^{r_\iota} : r_{(i,i)} \leq i \text{ for } 1 \leq i \leq n\}$$

is a free basis of $\mathcal{A}_n$ over $\mathcal{I}_n$. Therefore repeated application of Lemma 2.1.2 shows that these monomials form a free basis of $\mathcal{A}$ over $\mathcal{I}$. This completes the proof of the main theorem.

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