A BASIS FOR THE SYMPLECTIC GROUP BRANCHING ALGEBRA

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ABSTRACT. The symplectic group branching algebra, $\mathcal{B}$, is a graded algebra whose components encode the multiplicities of irreducible representations of $\mathrm{Sp}_{2n-2}(\mathbb{C})$ in each finite-dimensional irreducible representation of $\mathrm{Sp}_{2n}(\mathbb{C})$. By describing on $\mathcal{B}$ an ASL structure, we construct an explicit standard monomial basis of $\mathcal{B}$ consisting of $\mathrm{Sp}_{2n-2}(\mathbb{C})$ highest weight vectors. Moreover, $\mathcal{B}$ is known to carry a canonical action of the n-fold product $\mathrm{SL}_2 \times \cdots \times \mathrm{SL}_2$, and we show that the standard monomial basis is the unique (up to scalar) weight basis associated to this representation. Finally, using the theory of Hibi algebras we describe a deformation of $\mathrm{Spec}(\mathcal{B})$ into an explicitly described toric variety.

1. INTRODUCTION

Let us consider a pair of complex reductive algebraic groups G and H with embedding $H \subset G$, and irreducible representations V_G and V_H of G and H, respectively. A description of the multiplicity of V_H in V_G regarded as a representation of H by restriction is called a *branching rule* for (G, H). By Schur's lemma, the *(branching) multiplicity space*, $\mathrm{Hom}_H(V_H, V_G)$, encodes the branching rule.

In this paper, we study the branching multiplicity spaces for the symplectic group $\mathrm{Sp}_{2n}(\mathbb{C})$ of rank n down to the symplectic group $\mathrm{Sp}_{2n-2}(\mathbb{C})$ of rank $n-1$ by investigating the associated *branching algebra*. The branching algebra is a subalgebra of the ring of regular functions over $\mathrm{Sp}_{2n}(\mathbb{C})$. Moreover, it is graded, having the branching multiplicity spaces as its graded components:

$$\mathcal{B} \cong \bigoplus_{(D,F) \in \Lambda_{n-1,n}} \mathrm{Hom}_{\mathrm{Sp}_{n-1}}(\tau_{n-1}^D, \tau_n^F).$$

Here D and F run over highest weights for $\mathrm{Sp}_{2n-2}(\mathbb{C})$ and $\mathrm{Sp}_{2n}(\mathbb{C})$ respectively, and τ_{2n-2}^D and τ_n^F are the associated finite-dimensional irreducible representations.

Branching rules for $(\mathrm{Sp}_{2n}(\mathbb{C}), \mathrm{Sp}_{2n-2}(\mathbb{C}))$, especially their combinatorial aspects, are well known (e.g., [Le71, Ki71, Ki75, KT87, KT90, Pr94]). The main goal of this paper is to investigate the branching algebra $\mathcal{B}$ which governs the branching of symplectic groups. Our main results are several-fold. Firstly, we describe on $\mathcal{B}$ an algebra with straightening law (ASL) structure which presents $\mathcal{B}$ simply in terms of generators and relations. Secondly, we show that this ASL structure is compatible with a canonical weight basis of $\mathcal{B}$ coming from a "hidden symmetry" of $\mathcal{B}$, namely an action of the n-fold product of SL_2 which acts irreducibly on the graded components of $\mathcal{B}$. Finally, we

unify these algebraic results with previously known combinatorial rules governing the branching of symplectic groups via a toric degeneration of B into an explicitly described toric variety.

In Section 2, we construct the branching algebra $\mathcal{B}$ and review branching rules for $(\mathrm{Sp}_{2n}(\mathbb{C}), \mathrm{Sp}_{2n-2}(\mathbb{C}))$. Then in Section 3, we study $\mathcal{B}$ from the perspective of an ASL over a distributive lattice. Our first result shows that $\mathcal{B}$ has a natural standard monomial basis which satisfies simple straightening relations. As a corollary we obtain a finite presentation of $\mathcal{B}$ in terms of generators and relations.

Then, in Section 4, we recall a theorem from [Yac10] which shows that the natural SL_2 action on $\mathcal{B}$ can be canonically extended to an action of

$$L = \mathrm{SL}_2 \times \cdots \times \mathrm{SL}_2 \ (n \text{ copies})$$

in such a way that each multiplicity space $\mathrm{Hom}_{\mathrm{Sp}_{2n-2}(\mathbb{C})}(\tau_{n-1}^D, \tau_n^F)$ is an irreducible L-module. In particular, this theorem describes a canonical decomposition of $\mathcal{B}$ into one-dimensional spaces. Our second result shows this decomposition is compatible with the ASL structure on $\mathcal{B}$. In other words, the standard monomial basis is the unique (up to scalar) weight basis for the action of L on $\mathcal{B}$.

In Section 5, we show that $\mathcal{B}$ can be flatly deformed into a Hibi algebra, and, as a corollary, that $\mathrm{Spec}(\mathcal{B})$ is a deformation of an explicitly described toric variety. In particular, this connects our enumeration of standard monomials with the more common description of branching rules using diagrams of interlacing weights.

In future work we will apply these results to study properties of the canonical weight basis for irreducible representations of the symplectic group arising from this work.

Acknowledgment. We express our sincere thanks to Roger Howe for drawing our attention to this project. We also thank the two anonymous referees for their careful reading and many helpful comments and suggestions which greatly improved the paper.

2. BRANCHING ALGEBRA FOR $(\mathrm{Sp}_{2n}(\mathbb{C}), \mathrm{Sp}_{2n-2}(\mathbb{C}))$

In this section we introduce our main object of study, the branching algebra for the pair $(\mathrm{Sp}_{2n}(\mathbb{C}), \mathrm{Sp}_{2n-2}(\mathbb{C}))$.

2.1. **Symplectic Groups.** Let $Q_n = (q_{a,b})$ be the $n \times n$ matrix with $q_{a,n+1-a} = 1$ for $1 \le a \le n$ and 0 otherwise. Then we define the symplectic group

$$G_n = \mathrm{Sp}(\mathbb{C}^{2n}, Q_n)$$

of rank n as the subgroup of the general linear group $\mathrm{GL}(2n, \mathbb{C})$ preserving the skew symmetric bilinear form on $\mathbb{C}^{2n}$ induced by

$$\begin{bmatrix} 0 & Q_n \\ -Q_n & 0 \end{bmatrix}.$$

Let $\{e_a\}$ be the elementary basis of $\mathbb{C}^{2n}$, and denote by $\{e_a, e_{\overline{a}}\}$ the isotropic pairs, where $e_{\overline{a}} = e_{2n+1-a}$ for $1 \le a \le n$. With respect to this bilinear form, we can take the subgroup U_n of G_n consisting of upper triangular matrices with 1's on the diagonal as a

maximal unipotent subgroup of G_n. We let U_n^- denote the subgroup of lower triangular matrices with 1's on the diagonal.

Let us identify G_{n-1} with the subgroup of G_n which acts as identity on the isotropic space spanned by $\{e_n, e_{\overline{n}}\}$. Then G_{n-1} can be embedded in G_n via

$$(2.1) \qquad \begin{bmatrix} A & B \\ C & D \end{bmatrix} \mapsto \begin{bmatrix} A & 0 & B \\ 0 & I & 0 \\ C & 0 & D \end{bmatrix}$$

where A, B, C, D are $(n-1) \times (n-1)$ matrices, I is the 2×2 identity matrix, and 0's are the zero matrices of proper sizes.

A *Young diagram* is a finite left-justified array of boxes with weakly decreasing row lengths, such as
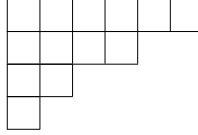
We shall identify a Young diagram F with its sequence of row lengths $(f_1, f_2, \dots)$. By reading column lengths of F, we obtain its associated Young diagram F^t called the *transpose* of F. We write $\ell(F)$ for the number of non-zero entries in F and call it the *length* of F. The Young diagram in the above example is $(6, 4, 2, 1)$ or equivalently $(6, 4, 2, 1, 0, \dots)$ and $\ell(F) = 4$. Its transpose F^t is $(4, 3, 2, 2, 1, 1)$.

Recall that every finite-dimensional irreducible representation of G_n can be uniquely labeled by a Young diagram with less than or equal to n rows by identifying its highest weight μ_F with Young diagram $F = (f_1, \cdots, f_n)$:

$$\mu_F(t) = t_1^{f_1} \cdots t_n^{f_n}.$$

Here t is an element of the maximal torus T_n of G_n

$$T_n = \left\{ \operatorname{diag}(t_1, \cdots, t_n, t_n^{-1}, \cdots, t_1^{-1}) \right\}$$

and $F \in \mathbb{Z}^n$ with $f_1 \geq \cdots \geq f_n \geq 0$. See [GW09, Section 3.2.1] for details. We let τ_n^F denote the irreducible representation of G_n labeled by Young diagram F.

To encode the branching multiplicities of τ_{n-1}^D in τ_n^F for all pairs of Young diagrams (D, F), we will use the following semigroup

$$\Lambda_{n-1,n} = \{(D, F) \in \mathbb{Z}_{\geq 0}^{n-1} \times \mathbb{Z}_{\geq 0}^n : \ell(D) \leq n-1, \ell(F) \leq n\}$$

and construct an algebra graded by $\Lambda_{n-1,n}$. The semigroup structure of $\Lambda_{n-1,n}$ is induced by the natural embedding of $\Lambda_{n-1,n}$ in $\mathbb{Z}_{\geq 0}^{n-1} \times \mathbb{Z}_{\geq 0}^n$ with addition.

2.2. **Branching Algebra $\mathcal{B}$.** On the ring $\mathcal{R}(G_n)$ of regular functions over G_n, we have the natural action of $G_n \times G_n$ given by

$$(2.2) \qquad\qquad ((g_1, g_2) \cdot f)(x) = f(g_1^{-1} x g_2)$$

for $f \in \mathcal{R}(G_n)$ and $(g_1, g_2) \in G_n \times G_n$. With respect to this action, let us consider the affine quotient of G_n by $U_n^- \times 1$.

Lemma 2.1 ([GW09, Theorem 12.1.5]). *As a G_n-module under right translation the $(U_n^- \times 1)$-invariant subalgebra of $\mathcal{R}(G_n)$ contains every irreducible rational representation of G_n with multiplicity one:*

$$(2.3) \qquad\qquad \mathcal{R}(G_n)^{U_n^- \times 1} = \bigoplus_{F \in \Lambda_n} \tau_n^F$$

The algebra $\mathcal{R}(G_n)^{U_n^- \times 1}$ is graded by the semigroup of dominant weights for G_n or equivalently the set Λ_n of Young diagrams of length less than or equal to n.

In this setting, the irreducible representation τ_n^F is the weight space of $\mathcal{R}(G_n)^{U_n^- \times 1}$ under the left action of the maximal torus T_n with weight $\mu_{(-F)}$, i.e.,

$$t \cdot f = \left(t_1^{-f_1} \cdots t_n^{-f_n} \right) f$$

for $f \in \tau_n^F$ and $t \in T_n$. See [GW09, Section 12.1.3] for further details.

By highest weight theory (e.g., [GW09, Section 3.2.1]) the subspace of τ_n^F invariant under the maximal unipotent subgroup of G_{n-1} is spanned by highest weight vectors of G_{n-1}-irreducible representations in τ_n^F. Therefore, the U_{n-1}-invariant subalgebra of $\mathcal{R}(G_n)^{U_n^- \times 1}$ contains the information of the branching multiplicities for (G_n, G_{n-1}). That is,

Proposition 2.2. *The $(U_n^- \times U_{n-1})$-invariant subalgebra of the ring $\mathcal{R}(G_n)$ decomposes as*

$$\mathcal{R}(G_n)^{U_n^- \times U_{n-1}} = \bigoplus_{(D,F) \in \Lambda_{n-1,n}} \mathrm{Hom}_{G_{n-1}}(\tau_{n-1}^D, \tau_n^F) \otimes \left(\tau_{n-1}^D \right)^{U_{n-1}}$$

Note that by Schur's lemma the dimension of $\mathrm{Hom}_{G_{n-1}}(\tau_{n-1}^D, \tau_n^F)$ is equal to the multiplicity of τ_{n-1}^D in τ_n^F.

Definition 2.3. *We call $\mathcal{R}(G_n)^{U_n^- \times U_{n-1}}$ the branching algebra for (G_n, G_{n-1}) and denote it by*

$$\mathcal{B} = \mathcal{R}(G_n)^{U_n^- \times U_{n-1}}$$

The algebra $\mathcal{B}$ has an action of $T_n \times T_{n-1}$, and the weight space corresponding to the T_n weight $(-F)$ and T_{n-1} weight D is precisely the component

$$\mathcal{B}(D, F) := \mathrm{Hom}_{G_{n-1}}(\tau_{n-1}^D, \tau_n^F) \otimes \left(\tau_{n-1}^D \right)^{U_{n-1}}$$

appearing in Proposition 2.2. Since the dimension of $(\tau_{n-1}^D)^{U_{n-1}}$ is one, we can consider $\mathcal{B}(D, F)$ as the branching multiplicity space for (G_n, G_{n-1}):

$$(2.4) \qquad\qquad \mathcal{B}(D, F) \cong \mathrm{Hom}_{G_{n-1}}(\tau_{n-1}^D, \tau_n^F)$$

Therefore the dimension of $\mathcal{B}(D, F)$ is exactly the multiplicity of the irreducible representation τ_{n-1}^D appearing in τ_n^F. Moreover, by keeping track of $T_n \times T_{n-1}$ weights, it is straightforward to check that this defines a $\Lambda_{n-1,n}$-graded algebra structure on $\mathcal{B}$:

$$\mathcal{B} = \bigoplus_{(D,F) \in \Lambda_{n-1,n}} \mathcal{B}(D, F).$$

The dimensions of the graded components of $\mathcal{B}$ are given by the following combinatorial rule. For two Young diagrams $F = (f_1, f_2, \cdots)$ and $D = (d_1, d_2, \cdots)$, we say D *interlaces* F and write $D \sqsubseteq F$, if $f_i \geq d_i \geq f_{i+1}$ for all i.

Lemma 2.4. (1) *(See, e.g., [GW09, Theorem 8.1.5]) For Young diagrams D and F with $(D, F) \in \Lambda_{n-1,n}$, the multiplicity of τ_{n-1}^D in τ_n^F as a G_{n-1} representation is nonzero if and only if*

$$f_j \geq d_j \geq f_{j+2}$$

for $j = 1, 2, \cdots, n-1$. Here we assume $f_{n+1} = 0$.

(2) *(See, e.g., [Pr94, Proposition 10.2]) The multiplicity of τ_{n-1}^D in τ_n^F as a G_{n-1} representation is equal to the number of Young diagrams $E = (e_1, \ldots, e_n)$ satisfying the conditions $D \sqsubseteq E$ and $E \sqsubseteq F$, i.e.,*

$$e_1 \geq d_1 \geq e_2 \geq \cdots \geq e_{n-1} \geq d_{n-1} \geq e_n;$$
$$f_1 \geq e_1 \geq f_2 \geq e_2 \geq \cdots \geq e_{n-1} \geq f_n \geq e_n.$$

If $D \sqsubseteq E$ and $E \sqsubseteq F$ for some E then, we say that the pair (D, F) (or the triple (D, E, F), if E should be specified) satisfies the *doubly interlacing condition*. Note that the branching for (G_n, G_{n-1}) is not multiplicity free, and $D \sqsubseteq E \sqsubseteq F$ does not imply $D \sqsubseteq F$. We also note that the conditions in the second statement can be visualized as, by using the convention of Gelfand-Tsetlin patterns,

$$\begin{array}{ccccccc} f_1 & & f_2 & \cdots & & f_n & & f_n \\ & e_1 & & e_2 & \cdots & & e_{n-1} & & e_n \\ & & d_1 & & \cdots & & d_{n-1} & & d_n \end{array}$$

where the entries are weakly decreasing from left to right along the diagonals.

3. A STANDARD MONOMIAL THEORY FOR $\mathcal{B}$

In this section we show that $\mathcal{B}$ carries a standard monomial theory, in the sense that it has a natural basis which satisfies a straightening algorithm. For the concept of standard monomial theory and its development, we refer to [LR08] and [La03, Mu03].

3.1. **Distributive Lattice for $\mathcal{B}$.** Let $M_{2n} = M_{2n}(\mathbb{C})$ be the space of $2n \times 2n$ complex matrices. For a subset C of $\{1, 2, \cdots, 2n\}$ of cardinality r, let

$$\delta_C : M_{2n} \to \mathbb{C}$$

denote the map assigning a matrix $X \in M_{2n}$ the determinant of the $r \times r$ minor formed by taking rows $\{1, 2, \cdots, r\}$ and columns $\{c_1, c_2, \cdots, c_r\}$:

$$(3.1) \qquad \delta_C(X) = \det \begin{bmatrix} x_{1,c_1} & x_{1,c_2} & \cdots & x_{1,c_r} \\ x_{2,c_1} & x_{2,c_2} & \cdots & x_{2,c_r} \\ \vdots & \vdots & \ddots & \vdots \\ x_{r,c_1} & x_{r,c_2} & \cdots & x_{r,c_r} \end{bmatrix}$$

for $c_1 < c_2 < \cdots < c_r$. We note that δ_C is a weight vector under the left and right multiplication of the diagonal subgroup of $GL_{2n}(\mathbb{C})$, i.e.,

$$(3.2) \qquad (t,s) \cdot \delta_C = \left(t_1^{-1} \cdots t_r^{-1} \right) (s_{c_1} \cdots s_{c_r}) \delta_C$$

In particular, the weight under the left and right actions encode the size of C and the entries of C respectively.

For the branching algebra $\mathcal{B}$, we shall use the following subsets of $\{1, 2, \cdots, n, n+1\}$ for column indexing sets C:
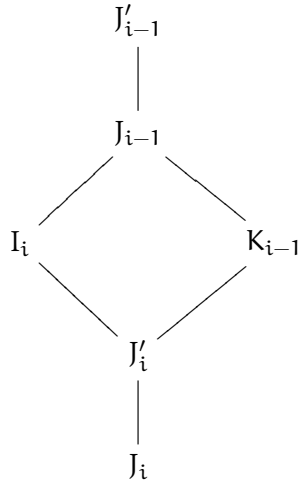
$$
\begin{aligned}
I_i &= \{1, 2, \cdots, i\} \\
J_j &= \{1, 2, \cdots, j, n\} \\
J'_j &= \{1, 2, \cdots, j, n+1\} \\
K_k &= \{1, 2, \cdots, k, n, n+1\}
\end{aligned}
$$

for $1 \le i \le n-1$, $0 \le j \le n-1$ and $0 \le k \le n-2$, with the convention of $J_0 = \{n\}$, $J'_0 = \{n+1\}$, and $K_0 = \{n, n+1\}$.

Definition 3.1. *The **distributive lattice** $\mathcal{L}$ **for** (G_n, G_{n-1}) is the poset consisting of*

$$\left\{ I_i, J_j, J'_j, K_k : 1 \le i \le n-1, 0 \le j \le n-1, 0 \le k \le n-2 \right\}$$

with the following partial order $\preceq$:

for $1 \le i \le n-1$.

Note that the join and meet of incomparable elements can be easily found as

$$(3.3) \qquad \begin{aligned} I_i \wedge K_{i-1} &= J_i' \\ I_i \vee K_{i-1} &= J_{i-1} \end{aligned}$$

for each i. This poset structure is very useful to organize the relations among δ_C for $C \in \mathcal{L}$. The following can be shown by a simple computation, or see [BH98, Lemma 7.2.3].

Proposition 3.2. *For* $1 \le i \le n - 1$, *the following identities hold*

$$\delta_{I_i}\delta_{K_{i-1}} = \delta_{J_i'}\delta_{J_{i-1}} - \delta_{J_i}\delta_{J_{i-1}'}$$

over the space M_{2n} *and therefore over* G_n.

Note that in the above Proposition, on the left hand side, I_i and K_{i-1} are incomparable, while on the right hand side $J_i' \preceq J_{i-1}$ and $J_i \preceq J_{i-1}'$. In other words, by applying these relations, we can express any quadratic monomial in $\{\delta_C : C \in \mathcal{L}\}$ as a linear combination of quadratic monomials whose indices are linearly ordered with respect to $\preceq$. In this sense, we call these relations *straightening relations*, and we can study the branching algebra $\mathcal{B}$ in the context of an algebra with straightening law (ASL) (cf. [BH98, DEP82, Ei80]).

Definition 3.3 ([Ei80])**.** *Let* R *be a ring,* A *an* R-*algebra,* H *a finite partially ordered set contained in* A *which generates* A *as an* R-*algebra. Then* A *is an algebra with straightening law on* H *over* R, *if*

(1) *The algebra* A *is a free* R-*module whose basis is the set of monomials of the form* $\alpha_1 \cdots \alpha_k$ *where* $\alpha_1 \le \cdots \le \alpha_k$ *in* H.
(2) *If* α *and* β *in* H *are incomparable, then*

$$\alpha\beta = \sum_i c_i \gamma_1^{(i)} \ldots \gamma_{k_i}^{(i)}$$

where $\gamma_1^{(i)} \le \cdots \le \gamma_{k_i}^{(i)}$, *and, for* i *such that* $c_i \ne 0$, $r_1^{(i)} \le \alpha, \beta$.

In the following section we show the branching algebra $\mathcal{B}$ is an ASL on $\{\delta_C : C \in \mathcal{L}\}$ over $\mathbb{C}$.

3.2. **Standard monomials for** $\mathcal{B}$. Let us recall that a *Young tableau* is a filling of a Young diagram with positive integers. A Young tableau is called a *semistandard Young tableau*, if its entries in each row are weakly increasing from left to right, and its entries in each column are strictly increasing from top to bottom.

Young tableaux with entries from $\{1, \cdots, m\}$ may be identified with a product of determinants of minors over the space M_m as follows. If the i-th column of a Young tableau T contains the entries

$$t_{1,i} < t_{2,i} < \cdots < t_{r_i,i}$$

for $1 \leq i \leq s$, then the corresponding polynomial in $\mathbb{C}[M_m]$ is

$$(3.4) \qquad \delta_T = \prod_{1 \leq i \leq s} \delta_{\{t_{1,i}, \ldots, t_{r_i, i}\}}$$

where $\delta_{\{t_{1,i}, \ldots, t_{r_i, i}\}}$ is as defined in (3.1). See Example 3.10 below. This type of correspondence between the set of Young tableaux and the set of products of determinants plays an important role in standard monomial theory of Grassmann and flag varieties. For this direction, we refer to, e.g., [Km08, LR08, MS05].

Coming back to our setting, we will be considering monomials of the form $\prod_{i=1}^{s} \delta_{C_i}$ where $C_i \in \mathcal{L}$ (cf. Definition 3.1). These are considered as regular functions on G_n. From (3.2), it is straightforward to see that every such monomial is a weight vector under the left and right actions of the maximal tori of G_n and G_{n-1} respectively. Moreover, by definition of the $U_n^- \times U_{n-1}$ action on $\mathcal{R}(G_n)$, the functions δ_C, and hence their products $\prod_{i=1}^{s} \delta_{C_i}$, are invariant under $U_n^- \times U_{n-1}$. In other words, $\prod_{i=1}^{s} \delta_{C_i} \in \mathcal{B}$. Let $\mathcal{B}' \subset \mathcal{B}$ be the subalgebra generated by δ_C for $C \in \mathcal{L}$. Clearly, $\mathcal{B}'$ is spanned by monomials $\prod_{i=1}^{s} \delta_{C_i}$.

Now, we can form a Young tableau T by concatenating finitely many elements $C_1, \ldots, C_s$ chosen from $\mathcal{L}$ allowing repetition. We further assume that the size of C_i is not smaller than that of C_{i+1} for all i. We note that the weakly increasing condition on the elements along the rows of T can be replaced by the chain condition on $\mathcal{L}$ with respect to the partial order $\preceq$. In other words, the elements $C_1, \ldots, C_s$ of $\mathcal{L}$ are columns of a semistandard tableau T if and only if they are linearly ordered with respect to $\preceq$. This is true in a more general setting (cf. [Km08, Remark 3.3]). With this observation, we define standard monomials for (G_n, G_{n-1}) as follows.

Definition 3.4. *A monomial $\prod \delta_{C_i}$ in $\{\delta_C : C \in \mathcal{L}\}$ is called a standard monomial for (G_n, G_{n-1}) if the column indices C_i form a multiple chain Δ*

$$\Delta = (C_1 \preceq \cdots \preceq C_r)$$

in the poset $\mathcal{L}$. We write δ_Δ for $\prod \delta_{C_i}$.

The observation right after Proposition 3.2 now can be generalized in terms of standard monomials.

Proposition 3.5. *The set of standard monomials for (G_n, G_{n-1}) spans the subalgebra $\mathcal{B}'$ of $\mathcal{B}$ generated by $\{\delta_C : C \in \mathcal{L}\}$.*

Proof. We want to show that every monomial can be expressed as a linear combination of standard monomials. Observe that any monomial $\delta = \prod \delta_{C_i}$ can be expressed as

$$\delta = (\delta_{I_1} \delta_{K_0})^{a_1} \cdots (\delta_{I_{n-1}} \delta_{K_{n-2}})^{a_{n-1}} \delta_\Delta$$

where δ_Δ is not divisible by $I_i K_{i-1}$ for $i = 1, \ldots, n-1$. In particular, δ_Δ is a standard monomial. We prove the claim by induction on $a = \sum a_i$.

If $a = 0$ then $\delta = \delta_\Delta$ is standard, and there is nothing to show. Suppose $a > 0$, and hence some $a_i > 0$. Then by Proposition 3.2,

$$\delta = (\delta_{I_1} \delta_{K_0})^{a_1} \cdots (\delta_{I_i} \delta_{K_{i-1}})^{a_i - 1} \cdots (\delta_{I_{n-1}} \delta_{K_{n-2}})^{a_{n-1}} (\delta_{J_i'} \delta_{J_{i-1}} - \delta_{J_i} \delta_{J_{i-1}'}) \delta_\Delta$$

Since $(\delta_{J_i'}\delta_{J_{i-1}} - \delta_{J_i}\delta_{J_{i-1}'})\delta_\Delta$ is a linear combination of two standard monomials, each of which is not divisible by $I_i K_{i-1}$ for $i = 1, ..., n-1$, the result follows by induction. $\square$

Remark 3.6. *For a quadratic monomial $\delta_C \delta_{C'}$, we can simply apply Proposition 3.2 to express it as a linear combination of quadratic standard monomials*

$$(3.5) \qquad \delta_C \delta_{C'} = \sum_h r_h \delta_{D_h} \delta_{D_h'}$$

Note that for all h, the numbers of entries equal to i in the disjoint union $C \dot\cup C'$ and in the disjoint union $D_h \dot\cup D_h'$ are equal for $1 \le i \le n+1$. Therefore, as tableaux, (D_h, D_h') is obtained from (C, C') just by rearranging the entries of C and C'. In fact the only difference between the tableaux (D_h, D_h') is the position of the entries n and $n+1$.

In general, once we have a linear combination of standard monomials for $\prod_i \delta_{C_i}$

$$\prod_i \delta_{C_i} = \sum_k s_k (\prod_i \delta_{H_{k,i}})$$

then for each k, we have the semistandard tableau H_k formed by $H_{k,i}$'s. Because of the reason explained above, for all k, the Young diagrams of the H_k's are the same, and as tableaux, their only difference is the position of the entries n and $n+1$.

Definition 3.7. *The shape of a standard monomial $\delta_\Delta = \prod \delta_{C_i}$ is F/D with*

$$\begin{aligned} F &= (f_1, \cdots, f_n) \\ D &= (d_1, \cdots, d_{n-1}) \end{aligned}$$

where F is the transpose of the Young diagram $(|C_1|, \cdots, |C_r|)$ and d_k in D is the number of k's in the disjoint union $\dot\cup_{i=1}^r C_i$ for $1 \le k \le n-1$. We write $sh(\delta_\Delta) = F/D$.

The following lemma is an immediate application of (3.2):

Lemma 3.8. *Standard monomials δ_Δ of shape F/D are weight vectors under the action of $T_n \times T_{n-1}$, i.e.,*

$$(t, s) \cdot \delta_\Delta = \left(t_1^{-f_1} \cdots t_n^{-f_n}\right) \left(s_1^{d_1} \cdots s_{n-1}^{d_{n-1}}\right) \delta_\Delta$$

where $t = \mathrm{diag}(t_1, \cdots, t_n, t_n^{-1}, \cdots, t_1^{-1})$ and $s = \mathrm{diag}(s_1, \cdots, s_{n-1}, 1, 1, s_{n-1}^{-1}, \cdots, s_1^{-1})$. In particular, $\delta_\Delta \in \mathcal{B}(D, F)$.

Now let us count the number of standard monomials in $\mathcal{B}(D, F)$.

Proposition 3.9. *There is a bijection*

$$\{\delta_\Delta : sh(\delta_\Delta) = F/D\} \leftrightarrow \{E : D \sqsubseteq E \sqsubseteq F\}$$

In particular, $\dim \mathcal{B}(D, F) = \#\{\delta_\Delta : sh(\delta_\Delta) = F/D\}$.

Proof. The bijection is a variation on the conversion procedure between semistandard Young tableaux and Gelfand-Tsetlin patterns (cf. [GW09, Section 8.1.2]). Let us be more specific about the procedure in our case.

By the definition of $\text{sh}(\delta_\Delta)$, if we erase all the boxes with n and $n+1$ in the semi-standard tableau Δ, then the remaining tableau gives the Young diagram D. As an intermediate step, if we erase only the boxes with $(n+1)$, then it gives a Young diagram E such that $D \sqsubseteq E$ and $E \sqsubseteq F$.

Conversely, given E such that $D \sqsubseteq E \sqsubseteq F$, define a semistandard Young tableau of shape F as follows: label the boxes of F/E by $n+1$, the boxes of E/D by n, and the remaining empty boxes by their row coordinate. The standard monomial corresponding to E is then constructed from this semistandard tableau.

The last statement follows by (2.4) and Lemma 2.4 (2). $\qquad\square$

Example 3.10. *In studying branching multiplicity spaces for* (G_4, G_3)*, the following monomial* δ_Δ

$$\delta_\Delta = \delta_{\{1234\}}\delta_{\{1245\}}\delta_{\{125\}}\delta_{\{14\}}\delta_{\{5\}}$$

as a regular function on G_4 *is a standard monomial. By concatenating its column indices to make the semistandard Young tableau*

1	1	1	1	5
2	2	2	4	
3	4	5		
4	5			

we see the shape of δ_Δ *is* $F/D = (5,4,3,2)/(4,3,1)$*. By erasing all the boxes with 5's, we obtain Young diagram* $E = (4,4,2,1)$*:*

Note that the triple (D, E, F) *satisfies the doubly interlacing condition.*

$$
\begin{array}{ccccccc}
5 & & 4 & & 3 & & 2 \\
& 4 & & 4 & & 2 & & 1 \\
& & 4 & & 3 & & 1 \\
\end{array}
$$

Theorem 3.11. *The branching algebra* $\mathcal{B}$ *is an ASL on* $\{\delta_C : C \in \mathcal{L}\}$ *over* $\mathbb{C}$*. In particular, standard monomials form a* $\mathbb{C}$*-basis of the algebra* $\mathcal{B}$*, and*

$$\mathcal{B}(D, F) = \text{span}\{\delta_\Delta : \text{sh}(\delta_\Delta) = F/D\}.$$

Proof. Let us check the first condition in Definition 3.3. Recall that $\mathcal{B}'$ is the subalgebra of $\mathcal{B}$ generated by $\{\delta_C : C \in \mathcal{L}\}$, and that by Proposition 3.5, $\mathcal{B}'$ is spanned by standard monomials. Now consider the space $\mathcal{B}' \cap \mathcal{B}(D, F)$. By Lemma 3.8, this space contains all the standard monomials of shape F/D, and therefore, it is spanned by standard monomials of shape F/D. By Proposition 3.9, the number of standard monomials of shape F/D is equal to the dimension of the space $\mathcal{B}(D, F)$. This shows that, for all doubly interlacing pairs $(D, F) \in \wedge_{n-1,n}$, the standard monomials of shape F/D are a basis of $\mathcal{B}(D, F)$. Since $\mathcal{B} = \oplus_{(D,F)}\mathcal{B}(D, F)$ and the dimension of $\mathcal{B}(D, F)$ is zero unless (D, F) satisfies the doubly interlacing condition (Lemma 2.4), standard monomials form

a $\mathbb{C}$-basis of $\mathcal{B}$. With Proposition 3.2, which shows that $\mathcal{B}$ satisfies the second condition in Definition 3.3, this shows that the branching algebra $\mathcal{B}$ for (G_n, G_{n-1}) is an ASL on $\{\delta_C : C \in \mathcal{L}\}$ over $\mathbb{C}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Definition 3.12. *The basis $\{\delta_\Delta\}$ from Theorem 3.11 is called the **standard monomial basis** of $\mathcal{B}$.*

4. THE STANDARD MONOMIAL BASIS AS A CANONICAL WEIGHT BASIS

In this section we give an interpretation of the standard monomial basis of the previous section as a canonical weight basis. In section 4.1 we recall a theorem in [Yac10] which shows that the natural SL_2 action on $\mathcal{B}$ can be canonically extended to an action of an n-fold product of SL_2's in such a way that the multiplicity spaces $\mathcal{B}(D, F)$ are irreducible. As a corollary of this theorem we obtain a canonical decomposition of $\mathcal{B}$ into one-dimensional spaces. We then show in section 4.2 that these one-dimensional spaces are exactly the spans of standard monomial basis elements.

4.1. **An irreducible action on the multiplicity spaces .** The branching algebra $\mathcal{B}$ carries a natural algebraic representation of SL_2. Indeed, there is a copy of SL_2 in G_n that commutes with $G_{n-1} \subset G_n$ (cf. (2.1)). This copy of SL_2 acts on the branching multiplicity spaces $\mathcal{B}(D, F)$, i.e. on the graded components of $\mathcal{B}$. This action is described as follows: an element $b \in \mathcal{B}(D, F)$ is a G_{n-1} equivariant map from τ_{n-1}^D to τ_n^F, and given $x \in SL_2$, $x.b$ is another such morphism defined by $(x.b)(v) = x.b(v)$ for any $v \in \tau_{n-1}^D$. Therefore $\mathcal{B}$ is a graded SL_2-algebra. We refer to this action as the "natural" SL_2 action on $\mathcal{B}$.

The branching multiplicity spaces are not irreducible SL_2-modules. Indeed, they are an n-fold tensor product of irreducible SL_2-modules (see Theorem 4.3 below). Nevertheless, the natural SL_2 action can be uniquely extended to an irreducible action of a product of SL_2's. In this section we explain how this is done.

Let $L = SL_2 \times \cdots \times SL_2$ be the n-fold product of SL_2. We want to construct an irreducible action of L on $\mathcal{B}(D, F)$, in such a way that the diagonally embedded $SL_2 \subset L$ recovers the natural action. Notice that in this formulation L is not the product of SL_2's that lives in G_n. Indeed, the latter product of SL_2's does not act on the multiplicity spaces. The existence of this L-action is more subtle, and can only be "seen" by considering all multiplicity spaces together, i.e. by considering the branching algebra.

For two Young diagrams $F = (f_1, f_2, \cdots)$ and $D = (d_1, d_2, \cdots)$, the inequalities in doubly interlacing condition for (D, F) given in Lemma 2.4, i.e.

$$f_i \geq d_i \geq f_{i+2}$$

do not constrain the relation between d_i and f_{i+1}. In other words, we can have either $d_i \geq f_{i+1}$, or $d_i \leq f_{i+1}$, or both. This motivates the following:

Definition 4.1. *An **order type** σ is a word in the alphabet $\{\geq, \leq\}$ of length $n - 1$.*

Suppose $(D, F) \in \Lambda_{n-1,n}$ and $\sigma = (\sigma_1 \cdots \sigma_{n-1})$ is an order type. Then we say (D, F) is of **order type** σ if for $i = 1, ..., n - 1$,

$$\begin{cases} \sigma_i = \text{``} \geq \text{''} \implies d_i \geq f_{i+1} \\ \sigma_i = \text{``} \leq \text{''} \implies d_i \leq f_{i+1} \end{cases}$$

For example, consider the pair (D, F), where $F = (3, 2, 1)$ and $D = (3, 0)$. Since $d_1 \geq f_2$ and $d_2 \leq f_3$, the pair (D, F) is of order type $\sigma = (\geq \leq)$.

It will also be useful to introduce the notion of a **generalized order type**: if $d_i = f_{i+1}$ then we place an "$=$" in the ith position to denote that (D, F) satisfies order types with both $\geq$ and $\leq$ in the ith position. For example, if F is as above and $D = (2, 0)$ then we say (D, F) is of generalized order type $(= \leq)$, since in this case (D, F) satisfies both types $(\geq \leq)$ and $(\leq \leq)$.

Let Σ be the set of order types, and for each $\sigma \in \Sigma$ set

$$\Lambda_{n-1,n}(\sigma) = \{(D, F) \in \Lambda_{n-1,n} : (D, F) \text{ is of order type } \sigma\}.$$

Lemma 4.2. *For $\sigma \in \Sigma$, $\Lambda_{n-1,n}(\sigma)$ is a sub-semigroup of $\Lambda_{n-1,n}$.*

Proof. Suppose $(D, F), (D', F') \in \Lambda_{n-1,n}(\sigma)$, and suppose $\sigma_i = \text{``} \geq \text{''}$. Then $d_i \geq f_{i+1}$ and $d_i' \geq f_{i+1}'$, which of course implies that $d_i + d_i' \geq f_{i+1} + f_{i+1}'$, and hence $(D + D', F + F') \in \Lambda_{n-1,n}(\sigma)$. The argument for $\sigma_i = \text{``} \leq \text{''}$ is entirely analogous. $\qquad \square$

Since $\mathcal{B}$ is $\Lambda_{n-1,n}$-graded, in particular

$$\mathcal{B}(D, F)\mathcal{B}(D', F') \subset \mathcal{B}(D + D', F + F')$$

for $(D, F), (D', F') \in \Lambda_{n-1,n}$. Therefore, by the above lemma,

$$\mathcal{B}(\sigma) = \bigoplus_{(D,F) \in \Lambda_{n-1,n}(\sigma)} \mathcal{B}(D, F)$$

is a subalgebra of $\mathcal{B}$. Note that $\mathcal{B}(\sigma)$ has unit the trivial function on G_n, which is an element of the $(0, 0)$-component.

To each $(D, F) \in \Lambda_{n-1,n}$ we associate an irreducible L-module as follows. Let V_k be the irreducible SL_2-module of dimension $k+1$. Set $D = (d_1, ..., d_{n-1})$ and $F = (f_1, ..., f_n)$, and let $(x_1 \geq y_1 \geq \cdots \geq x_n \geq y_n)$ be the non-increasing rearrangement of the elements $\{d_1, ..., d_{n-1}, f_1, ..., f_n\}$. Define $r_i(D, F) = x_i - y_i$ for $i = 1, ..., n$, and let $\mathcal{A}(D, F)$ be the irreducible L-module

$$\mathcal{A}(D, F) = \bigotimes_{i=1}^{n} V_{r_i(D,F)}.$$

Theorem 4.3 ([Yac10, Theorem 3.5]). *There is a unique representation $(\Phi, \mathcal{B})$ of L satisfying the following two properties:*

(1) *For all $(D, F) \in \Lambda_{n-1,n}$, $\mathcal{B}(D, F)$ is an irreducible L-invariant subspace of $\mathcal{B}$. If $\mathcal{B}(D, F)$ is nonzero, then $\mathcal{B}(D, F)$ is isomorphic to $\mathcal{A}(D, F)$.*

(2) *For all $\sigma \in \Sigma$, L acts as algebra automorphisms on $\mathcal{B}(\sigma)$.*

Moreover, $\text{Res}_{SL_2}^{L}(\Phi)$ recovers the natural action of SL_2 on $\mathcal{B}$.

Let T_{SL_2} be the torus of SL_2 consisting of diagonal matrices. Let $T_L = T_{SL_2} \times \cdots \times T_{SL_2}$ be the diagonal torus of L. The action of T_L on $\mathcal{B}(D, F)$ decomposes it uniquely into weight spaces, which, by the above theorem, are one-dimensional.

Remark 4.4. *The T_L weight spaces of $\mathcal{B}(D, F)$ are also weight spaces for the natural SL_2-action, via the diagonally embedding $T_{SL_2} \subset L$. Moreover, T_L is the unique maximal torus of L containing T_{SL_2}. Therefore, the decomposition we obtain in this way is the unique decomposition of $\mathcal{B}(D, F)$ into spaces which are simultaneously weight spaces for a torus of L and weight spaces for T_{SL_2}. Moreover, the choice of the torus T_{SL_2} is induced by our choice of torus of G_n. In other words, the decomposition of $\mathcal{B}(D, F)$ into one dimensional spaces depends only the choice of torus of G_n.*

We now make this decomposition precise. Suppose $(D, F) \in \Lambda_{n-1,n}$ and (D, F) satisfies the doubly interlacing condition (so that $\mathcal{B}(D, F)$ is nonzero). Then the weight spaces of T_L on $\mathcal{A}(D, F)$, and hence $\mathcal{B}(D, F)$, are indexed by Young diagrams E satisfying the condition $D \sqsubseteq E \sqsubseteq F$. Indeed, the diagram $E = (e_1, e_2, ...)$ corresponds to the weight

$$(4.1) \qquad (t_1, ..., t_n) \in T_L \mapsto \prod_{i=1}^{n} t_i^{2e_i - x_i - y_i}$$

(cf. (Lemma 7.1, [Yac10])). Let $\mathcal{B}(D, E, F)$ denote the one dimensional weight space of $\mathcal{B}(D, F)$ parameterized by E.

Corollary 4.5. *There is a canonical decomposition of $\mathcal{B}$ into one dimensional T_L weight spaces*

$$\mathcal{B} = \bigoplus_{D \sqsubseteq E \sqsubseteq F} \mathcal{B}(D, E, F).$$

In particular, $\mathcal{B}$ has a T_L weight basis which is unique up to scalar.

4.2. **The standard monomial basis is the canonical weight basis .** We now show that the standard monomials basis defined in Theorem 3.11 is compatible with the canonical decomposition appearing in Corollary 4.5. In other words, the standard monomial basis is the unique (up to scalar) T_L weight basis of $\mathcal{B}$.

Suppose $\Delta \subset \mathcal{L}$ is a chain, and the corresponding standard monomial δ_Δ has shape F/D. We say Δ is of order type $\sigma \in \Sigma$ if the pair (D, F) is of type σ. The following lemma shows that order types are intimately connected to the distributive lattice $\mathcal{L}$.

Lemma 4.6. *A set of column indices $\{C_i : i = 1, ..., r\}$ form a chain $\Delta = (C_1 \preceq \cdots \preceq C_r)$ if, and only if, they satisfy a common order type $\sigma \in \Sigma$.*

Proof. First we note that

$$\begin{aligned}
I_i &\quad \text{is of generalized type} &\quad (= \cdots \geq \cdots =) \\
J_j &\quad \text{is of generalized type} &\quad (= \cdots =) \\
J_j' &\quad \text{is of generalized type} &\quad (= \cdots =) \\
K_k &\quad \text{is of generalized type} &\quad (= \cdots \leq \cdots =)
\end{aligned}$$

where in the first line the "$\geq$" sign appears in the ith position, and in the last line the "$\leq$" sign appears in the $k + 1^{th}$ position.

Now suppose a set of column indices $\{C_i : i = 1, ..., r\}$ form a chain in $\mathcal{L}$. Then for all $i \geq 1$ we know that

$$\{I_i, K_{i-1}\} \nsubseteq \{C_1, ..., C_r\}.$$

Define a generalized order type $\sigma = (\sigma_1 \cdots \sigma_{n-1})$ by

$$\sigma_i = \begin{cases} \geq & \text{if } I_i \in \{C_1, ..., C_r\} \\ \leq & \text{if } K_{i-1} \in \{C_1, ..., C_r\} \\ = & \text{otherwise} \end{cases}$$

Clearly the set $\{C_1, ..., C_r\}$ satisfies the type σ.

Conversely, suppose the elements of $\Delta = \{C_1, ..., C_r\}$ satisfy a common order type $\sigma = (\sigma_1 \cdots \sigma_{n-1})$. Then

$$\sigma_i = \text{"} \geq \text{"} \implies I_i \in \Delta, K_{i-1} \notin \Delta$$

$$\sigma_i = \text{"} \leq \text{"} \implies I_i \notin \Delta, K_{i-1} \in \Delta$$

$$\sigma_i = \text{"} = \text{"} \implies I_i \notin \Delta, K_{i-1} \notin \Delta$$

Therefore $\{I_i, K_{i-1}\} \nsubseteq \Delta$ for all i, i.e. Δ is a chain in $\mathcal{L}$. $\qquad\square$

Theorem 4.7. *The standard monomial basis is the unique (up to scalar) T_L weight basis of the representation $(\Phi, \mathcal{B})$ of L.*

Proof. First we prove that if $C \in \mathcal{L}$ then δ_C is a weight vector for the action of T_L on $\mathcal{B}$. Suppose F/D is the shape of δ_C. Since D and F are both columns, by Theorem 4.3(1) $B(D, F)$ is isomorphic either to a trivial L-module, or to

$$V_0 \otimes \cdots V_1 \cdots \otimes V_0$$

where the term V_1 occurs, say, in the ith position.

In the first case, δ_C is clearly a weight vector since it is invariant under L. Consider now the second case, and let $\vec{t} = (t_1, ..., t_n) \in T_L$. Then

$$\begin{aligned} \Phi(\vec{t})(\delta_C) &= \Phi((t_i, ..., t_i))(\delta_C) \\ &= t_i.\delta_C \end{aligned}$$

where the second equality follows since $\text{Res}_{SL_2}^L(\Phi)$ is the natural action of SL_2 on $\mathcal{B}$ (by Theorem 4.3). So now it suffices to see that δ_C is a weight vector under the natural torus action of T_{SL_2} on $\mathcal{B}$. Indeed, for $t \in T_{SL_2}$ we have

$$\begin{aligned} t.\delta_{I_i} &= \delta_{I_i} \\ t.\delta_{J_j} &= t\delta_{J_j} \\ t.\delta_{J_j'} &= t^{-1}\delta_{J_j'} \\ t.\delta_{K_k} &= \delta_{K_k} \end{aligned}$$

where $1 \leq i \leq n-1$, $0 \leq j \leq n-1$ and $0 \leq k \leq n-2$.

We've shown so far that for any $C \in \mathcal{L}$, δ_C is a weight vector for the action of T_L on $\mathcal{B}$. We now show this is true for any standard monomial $\delta_\Delta = \delta_{C_1} \cdots \delta_{C_r}$. Indeed,

by Lemma 4.6, the column indices $\{C_1, ..., C_r\}$ satisfy a common order type. Then by Theorem 4.3(2), for any $l \in L$

$$\Phi(l)(\delta_\Delta) = \Phi(l)(\delta_{C_1}) \cdots \Phi(l)(\delta_{C_r}).$$

Since each δ_{C_i} is a weight vector for the action of T_L, it follows that δ_Δ is also. By Theorem 3.11 and Corollary 4.5 we conclude that the standard monomials are the unique (up to scalar) weight basis of the representation $(\Phi, \mathcal{B})$ of L. $\qquad\square$

We remark that our labeling of standard monomials is compatible with the decomposition of $\mathcal{B}$ appearing in Corollary 4.5. Indeed, suppose δ_Δ is a standard monomial. In section 3.1 we showed how to associate a triple of doubly interlacing Young diagrams $D_\Delta \sqsubseteq E_\Delta \sqsubseteq F_\Delta$ to δ_Δ. Then we have:

$$(4.2) \qquad\qquad \delta_\Delta \in \mathcal{B}(D_\Delta, E_\Delta, F_\Delta).$$

Example 4.8. *Consider δ_Δ as in Example 3.10. Let $\vec{t} = (t_1, ..., t_4) \in T_L$. By the same reasoning as in the proof of Theorem 4.7 we compute that*

$$\Phi(\vec{t})(\delta_\Delta) = t_1^{-1} t_2 t_3^{-1} t_4 \delta_\Delta.$$

On the other hand, using (4.1), it's easy to see that $\mathcal{B}(D_\Delta, E_\Delta, F_\Delta)$ is a weight space of T_L corresponding to the character $\vec{t} \mapsto t_1^{-1} t_2 t_3^{-1} t_4$.

5. Toric Degeneration and Hibi Algebra for (G_n, G_{n-1})

In this section, we show that $\mathcal{B}$ can be flatly deformed into an affine semigroup ring. This will provide another combinatorial description of branching multiplicity spaces.

5.1. **Flat Deformation.** From Theorem 3.11, we can realize the branching algebra $\mathcal{B}$ as an ASL, i.e., the quotient algebra

$$\mathcal{B} \cong \mathbb{C}[z_C : C \in \mathcal{L}]/\mathcal{I}$$

whose defining ideal $\mathcal{I}$ is generated by

$$(5.1) \qquad \left\{ z_{I_i} z_{K_{i-1}} - z_{J_i'} z_{J_{i-1}} + z_{J_i} z_{J_{i-1}'} : 1 \le i \le n-1 \right\}$$

On the other hand, we can define a semigroup ring whose multiplicative structure is compatible with the lattice structure of $\mathcal{L}$ (cf. [Hi87]).

Definition 5.1. *The **Hibi algebra** $\mathcal{H}$ over $\mathcal{L}$ is the quotient of the polynomial ring $\mathbb{C}[z_C : C \in \mathcal{L}]$ by the ideal $\mathcal{I}_0$ generated by*

$$\left\{ z_{I_i} z_{K_{i-1}} - z_{J_i'} z_{J_{i-1}} : 1 \le i \le n-1 \right\}.$$

We note that each generator of $\mathcal{I}_0$ can be written as $z_{I_i} z_{K_{i-1}} - z_{I_i \wedge K_{i-1}} z_{I_i \vee K_{i-1}}$ by (3.3), and also it is the first two terms of the longer relation (5.1) for $\mathcal{B}$. In what follows, by using an analog of the SAGBI degeneration ([CHV96, Theorem 1.2]), we show that $\mathcal{B}$ is a flat deformation of $\mathcal{H}$.

Theorem 5.2. *The branching algebra $\mathcal{B}$ can be flatly deformed into the Hibi algebra $\mathcal{H}$ over $\mathcal{L}$.*

Proof. Our goal is to construct a flat $\mathbb{C}[t]$ module $\mathcal{R}$ whose general fiber is isomorphic to $\mathcal{B}$ and special fiber is isomorphic to the Hibi algebra $\mathcal{H}$. Let us impose a filtration on $\mathcal{B}$ by giving the following weight on each monomial. Fix a large integer N greater than $2n$, and for $C = \{c_1 < \cdots < c_a\} \in \mathcal{L}$ we define its weight as

$$wt(C) = \sum_{r \geq 1} c_r N^{n-r}$$

Also, we define the weight of δ_C as the weight $wt(C)$ of its indexing set C, and the weight of a product $\prod \delta_{C_i}$ as the sum $\sum wt(C_i)$ of the weights of its factors.

Recall that from Theorem 3.11, every element in $\mathcal{B}$ can be expressed uniquely as a linear combination of standard monomials $\delta_\Delta = \prod \delta_{C_i}$ with multiple chains $\Delta = (C_1, \cdots, C_k)$. Set $F_d^{wt}(\mathcal{B})$ to be the space spanned by standard monomials whose weights are not less than d:

$$\{\delta_\Delta : wt(\delta_\Delta) \geq d\}.$$

This filtration $F^{wt} = \{F_d^{wt}\}$ is well defined from the following observation: every product $\prod \delta_{C_i}$ can be expressed as a linear combination of standard monomials with bigger weights, because in the straightening laws (5.1) we have

$$
\begin{aligned}
wt(I_i) + wt(K_{i-1}) &= wt(J_i') + wt(J_{i-1}) \\
&= \sum_{1 \leq r \leq i-1} rN^{n-r} + (n+i)N^{n-i} + (n+1)N^{n-i-1} \\
wt(J_i) + wt(J_{i-1}') &= \sum_{1 \leq r \leq i-1} rN^{n-r} + (n+i+1)N^{n-i} + nN^{n-i-1}
\end{aligned}
$$

and therefore for each i,

$$
\begin{aligned}
wt(I_i) + wt(K_{i-1}) &= wt(J_i') + wt(J_{i-1}) \\
&< wt(J_i) + wt(J_{i-1}')
\end{aligned}
$$

Then we can construct the Rees algebra $\mathcal{R}$ of $\mathcal{B}$ with respect to F^{wt}:

$$\mathcal{R} = \bigoplus_{d \geq 0} F_d^{wt}(\mathcal{B}) t^d$$

and by the general property of the Rees algebras, it is flat over $\mathbb{C}[t]$ with its general fiber isomorphic to $\mathcal{B}$ and special fiber isomorphic to the associated graded ring.

For all incomparable pairs $(A, B) = (I_i, K_{i-1}) \in \mathcal{L}$, since $wt(A) + wt(B) = wt(A \wedge B) + wt(A \vee B)$, $\delta_A \delta_B$ and $\delta_{A \wedge B} \delta_{A \vee B}$ belong to the same associated graded component of $\mathcal{R}$. Therefore, we have $y_A \cdot_{gr} y_B = y_{A \wedge B} \cdot_{gr} y_{A \vee B}$ where y_C are elements corresponding to δ_C in the associated graded ring of $\mathcal{R}$. Then it follows that the associated graded ring of $\mathcal{R}$ is isomorphic to the Hibi algebra $\mathcal{H}$ over $\mathcal{L}$. $\square$

5.2. **Affine Semigroup.** Now we want to realize the Hibi algebra $\mathcal{H}$ over $\mathcal{L}$ as an affine semigroup ring, i.e., a ring generated by a finitely generated semigroup isomorphic to a subsemigroup of $\mathbb{Z}^N$ containing 0 for some N (cf. [BH98, Section 6]). Since $\mathcal{B}$ is a flat deformation of $\mathcal{H}$, we expect that combinatorial properties of branching rules give rise to the affine semigroup structure of $\mathcal{H}$.

Let us define the poset Γ consisting of $t_j^{(i)}$ for $n-1 \leq i \leq n+1$ and $1 \leq j \leq \min(i,n)$ which we shall arrange as

$$
\Gamma = \left\{
\begin{array}{ccccccc}
t_1^{(n+1)} & & t_2^{(n+1)} & & \cdots & & t_n^{(n+1)} \\
& t_1^{(n)} & & t_2^{(n)} & & \cdots & & t_n^{(n)} \\
& & t_1^{(n-1)} & & \cdots & & t_{n-1}^{(n-1)} &
\end{array}
\right\}
$$

with $t_j^{(i+1)} \geq t_j^{(i)} \geq t_{j+1}^{(i+1)}$ for all i and j.

The set $\mathcal{P}(\Gamma)$ of all order preserving maps from Γ to non-negative integers forms a monoid generated by the characteristic functions χ_S on $S = \Gamma/S'$ for order decreasing subsets S' of Γ

$$
\chi_S(t_j^{(i)}) = \left\{
\begin{array}{ll}
1 & \text{if } t_j^{(i)} \in S \\
0 & \text{if } t_j^{(i)} \notin S
\end{array}
\right.
$$

Furthermore, by imposing the following partial order on the set of order decreasing subsets of Γ, we can identify the Hibi algebra $\mathcal{H}$ with the semigroup ring $\mathbb{C}[\mathcal{P}]$ of $\mathcal{P} = \mathcal{P}(\Gamma)$: for two order decreasing subsets S_1' and S_2' of Γ, we say S_1' is bigger than S_2', if $S_2' \subseteq S_1'$ as sets.

Lemma 5.3. *There is an order isomorphism between $\mathcal{L}$ and the set of order decreasing subsets of Γ.*

This is an easy computation similar to [Km08, Theorem 3.8]. Let us specify this isomorphism. For each $C \in \mathcal{L}$, we define the complement S_C of the corresponding order decreasing subset S_C' of Γ as the union of

$$
S_C^{(k)} = \left\{ t_1^{(k)}, t_2^{(k)}, \cdots, t_{m(k)}^{(k)} \right\}
$$

for $n-1 \leq k \leq n+1$ where $m(k)$ is the number of entries in C less than or equal to k. It is straightforward to check that this correspondence gives an order isomorphism. This, in fact, is an example of Birkhoff's representation theorem or the fundamental theorem for finite distributive lattices ([Sta97, Theorem 3.4.1]). See also the example below.

Example 5.4. *Let us consider the following elements from the distributive lattice $\mathcal{L}$ for (G_4, G_3): $A = [1,2,4,5]$, $B = [1,2,5]$, and $C = [1,4]$. Then the corresponding character functions can be visualized as, by identifying them with their values at*

$t_a^{(b)} \in \Gamma$,

$$\chi_{S_A} = \left\{ \begin{matrix} 1 & & 1 & & 1 & & 1 & \\ & 1 & & 1 & & 1 & & 0 \\ & & 1 & & 1 & & 0 & \end{matrix} \right\}$$

$$\chi_{S_B} = \left\{ \begin{matrix} 1 & & 1 & & 1 & & 0 & \\ & 1 & & 1 & & 0 & & 0 \\ & & 1 & & 1 & & 0 & \end{matrix} \right\}$$

$$\chi_{S_C} = \left\{ \begin{matrix} 1 & & 1 & & 0 & & 0 & \\ & 1 & & 1 & & 0 & & 0 \\ & & 1 & & 0 & & 0 & \end{matrix} \right\}$$

Note that for $T = A, B, C$, *the number of* 1's *in the first, second, and third row of* χ_T *is the number of entries less than or equal to 5, 4, and 3 in* T *respectively. The order* $A \preceq B \preceq C$ *can be related to the inclusion order on order decreasing subsets* $S_T' = \chi_{S_T}^{-1}(0)$ *of* Γ:

$$\chi_{S_A}^{-1}(0) \subseteq \chi_{S_B}^{-1}(0) \subseteq \chi_{S_C}^{-1}(0).$$

Proposition 5.5. *There is a bijection between the set of multiple chains in* $\mathcal{L}$ *and the set of order preserving maps from* Γ *to non-negative integers.*

Proof. The bijection in the above lemma provides the bijection between $\mathcal{L}$ and the set of characteristic functions on the complements of order increasing subsets of Γ. This map can be extended to the multiple chains in $\mathcal{L}$ as follows. Let $\Delta = (C_1 \preceq \cdots \preceq C_c)$ be a multiple chain in $\mathcal{L}$ and for each i, let χ_i be the characteristic function on S_{C_i} corresponding to C_i given in the above lemma. Then we can consider the following correspondence:

$$(5.2) \qquad \Delta = (C_1 \preceq \cdots \preceq C_c) \mapsto p(\Delta) = \sum_{r=1}^{c} \chi_i.$$

Recall that starting from Δ, by considering Young diagrams containing entries less than or equal to $n-1, n$, and $n+1$ we obtain Young diagrams D, E, and F respectively with $D \sqsubseteq E \sqsubseteq F$. Now let us consider the following order preserving map $p : \Gamma \to \mathbb{Z}_{\geq 0}$

$$\begin{aligned} p(t_i^{(n+1)}) &= f_i \\ p(t_i^{(n)}) &= e_i \\ p(t_k^{(n-1)}) &= d_k \end{aligned}$$

for $1 \leq i \leq n, 1 \leq k \leq n-1$. Then from the construction of χ_i corresponding to C_i, one can check that $p = p(\Delta)$ and this correspondence gives a bijection. For further details, see [Km08][Hw05]. See also the example below. $\square$

Example 5.6. *Let us consider the chain* $A \preceq B \preceq C$ *given by the elements in Example 5.4. By concatenating them we have the semistandard tableau* Δ *of the shape* $F/D = (3,3,2,1)/(3,2)$:

1	1	1
2	2	4
4	5	
5		

Then the corresponding order preserving map given in the above proof is $\chi_{S_A} + \chi_{S_B} + \chi_{S_C}$:

$$
p(\Delta) = \left\{ \begin{array}{ccccccc} 3 & & 3 & & 2 & & 1 \\ & 3 & & 3 & & 1 & & 0 \\ & & 3 & & 2 & & 0 \end{array} \right\}
$$

Note that the first row $(3,3,2,1)$ *corresponds to the Young diagram* F *of the tableau* Δ. *By erasing the boxes with* 5 *in the tableau, we obtain the Young diagram corresponding to the second row* $E = (3,3,1,0)$, *and finally by erasing the boxes with* 4, *we obtain the Young diagram* D *corresponding to the third row* $(3,2,0)$.

In fact, from the multiplicative structure given in (5.2), this bijection is a semigroup isomorphism. Therefore, we have

Corollary 5.7. *The Hibi algebra* $\mathcal{H}$ *over* $\mathcal{L}$ *is isomorphic to the semigroup ring* $\mathbb{C}[\mathcal{P}]$ *generated by* $\mathcal{P} = \mathcal{P}(\Gamma)$.

Consequently, $\mathrm{Spec}(\mathcal{H})$ is an affine toric variety in the sense of [Stu95, MS05], and Theorem 5.2 shows that $\mathrm{Spec}(\mathcal{B})$ is a toric deformation of $\mathrm{Spec}(\mathcal{H})$.

References

[BH98] W. Bruns and J. Herzog, Cohen-Macaulay rings. Revised edition. Cambridge Studies in Advanced Mathematics, 39. Cambridge University Press, Cambridge, 1998.

[CHV96] A. Conca, J. Herzog, and G. Valla, Sagbi bases with applications to blow-up algebras. J. Reine Angew. Math. 474 (1996), 113–138.

[DEP82] C. De Concini, D. Eisenbud, and C. Procesi, Hodge algebras. Astéisque, 91. Société Mathématique de France, Paris, 1982. 87 pp

[Ei80] D. Eisenbud, Introduction to algebras with straightening laws. Ring theory and algebra, III (Proc. Third Conf., Univ. Oklahoma, Norman, Okla., 1979), pp. 243–268, Lecture Notes in Pure and Appl. Math., 55, Dekker, New York, 1980.

[GW09] R. Goodman and N. R. Wallach, Symmetry, representations, and invariants. Graduate Texts in Mathematics, 255. Springer, Dordrecht, 2009.

[Hi87] T. Hibi, Distributive lattices, affine semigroup rings and algebras with straightening laws. Commutative algebra and combinatorics (Kyoto, 1985), 93–109, Adv. Stud. Pure Math., 11, North-Holland, Amsterdam, 1987.

[Hw05] R. Howe, Weyl Chambers and standard monomial theory for poset lattice cones. Q. J. Pure Appl. Math. 1 (2005), no. 1, 227–239.

[Ki71] R.C. King, Modification rules and products of irreducible representations of the unitary, orthogonal, and symplectic groups. J. Math. Phys. 12 (1971), pp. 1588–1598.

[Ki75] R.C. King, Branching rules for classical Lie groups using tensor and spinor methods. J. Phys. A 8 (1975), 429–449.

[Km08]   S. Kim, Standard monomial theory for flag algebras of $GL(n)$ and $Sp(2n)$. J. Algebra 320 (2008), no. 2, 534–568.

[KT87]   K. Koike and I. Terada, Young-diagrammatic methods for the representation theory of the classical groups of type B_n, C_n, D_n. J. Algebra 107 (1987), no. 2, 466–511.

[KT90]   K. Koike and I. Terada, Young diagrammatic methods for the restriction of representations of complex classical Lie groups to reductive subgroups of maximal rank. Adv. Math. 79 (1990), no. 1, 104–135.

[Le71]   J. Lepowsky, Multiplicity formulas for certain semisimple Lie groups. Bull. Amer. Math. Soc. 77 (1971), 601–605.

[La03]   V. Lakshmibai, The development of standard monomial theory. II. A tribute to C. S. Seshadri (Chennai, 2002), 283–309, Trends Math., Birkhäuser, Basel, 2003.

[LR08]   V. Lakshmibai and K. N. Raghavan, Standard monomial theory. Invariant theoretic approach. Encyclopaedia of Mathematical Sciences, 137. Invariant Theory and Algebraic Transformation Groups, 8. Springer-Verlag, Berlin, 2008.

[MS05]   E. Miller and B. Sturmfels, Combinatorial commutative algebra. Graduate Texts in Mathematics, 227. Springer-Verlag, New York, 2005.

[Mu03]   C. Musili, The development of standard monomial theory. I. A tribute to C. S. Seshadri (Chennai, 2002), 385–420, Trends Math., Birkhäuser, Basel, 2003.

[Pr94]   R. A. Proctor, Young tableaux, Gelfand patterns, and branching rules for classical groups. J. Algebra 164 (1994), no. 2, 299–36.

[Sta97]   R. P. Stanley, Enumerative Combinatorics, Volume I, Cambridge Studies in Advanced Mathematics 49, Cambridge University Press, 1997.

[Stu95]   B. Sturmfels, Gröbner bases and convex polytopes. University Lecture Series, 8. American Mathematical Society, Providence, RI, 1996.

[Yac10]   O. Yacobi, An analysis of the multiplicity spaces in branching of symplectic groups, Selecta Math N.S., Volume 16, Issue 4, (2010)

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