AN ANALYSIS OF THE MULTIPLICITY SPACES IN BRANCHING OF SYMPLECTIC GROUPS

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Abstract. Branching of symplectic groups is not multiplicity-free. We describe a new approach to resolving these multiplicities that is based on studying the associated branching algebra $\mathcal{B}$. The algebra $\mathcal{B}$ is a graded algebra whose components encode the multiplicities of irreducible representations of Sp_{2n-2} in irreducible representations of Sp_{2n}. Our first theorem states that the map taking an element of Sp_{2n} to its principal $n \times (n+1)$ submatrix induces an isomorphism of $\mathcal{B}$ to a different branching algebra $\mathcal{B}'$. The algebra $\mathcal{B}'$ encodes multiplicities of irreducible representations of GL_{n-1} in certain irreducible representations of GL_{n+1}. Our second theorem is that each multiplicity space that arises in the restriction of an irreducible representation of Sp_{2n} to Sp_{2n-2} is canonically an irreducible module for the n-fold product of SL_2. In particular, this induces a canonical decomposition of the multiplicity spaces into one dimensional spaces, thereby resolving the multiplicities.

Contents

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1. Introduction

The purpose of this paper is to give a new interpretation of symplectic branching which, unlike the branching for the other towers of classical groups, is not multiplicity free. In other words, an irreducible representation of Sp_{2n} does not decompose uniquely into irreducible representations of Sp_{2n-2} (embedded as the subgroup fixing pointwise a two-dimensional non-isotropic subspace). We resolve this ambiguity by analyzing the algebraic structure of the associated multiplicity spaces.

Our main object of study is an algebra $\mathcal{B}$ associated to the pair (Sp_{2n-2}, Sp_{2n}). This "branching algebra" is a graded algebra whose components are all the multiplicity spaces that appear in the restriction of irreducible representations of Sp_{2n} to Sp_{2n-2}. By definition $\mathcal{B}$ is a certain subalgebra of the ring of regular functions, $\mathcal{O}(Sp_{2n})$, on Sp_{2n}:

$$\mathcal{B} = \mathcal{O}(\overline{U}_{C_n} \setminus Sp_{2n} \;/\; U_{C_{n-1}}).$$

(See Section 2.2 for notation.) From this realization it follows that $\mathcal{B}$ has a natural action of SL_2 by right translation.

Our first result relates $\mathcal{B}$ to a different branching algebra associated to the branching pair (GL_{n-1}, GL_{n+1}). Using (GL_{n-1}, GL_{n+1})-duality we define the algebra

$$\mathcal{B}' = \mathcal{O}(\overline{U}_n \setminus M_{n,n+1} \;/\; U_{n-1}).$$

This algebra is a "restricted" branching algebra, as it is isomorphic to a direct sum of only certain multiplicity spaces that occur in branching from GL_{n+1} to GL_{n-1}. Note that $\mathcal{B}'$ is also a graded SL_2-algebra. We consider the function $\psi : Sp_{2n} \to M_{n,n+1}$ which maps an element of Sp_{2n} to its $n \times (n+1)$ principal submatrix. The theorem is that the induced map on functions $\psi^* : \mathcal{O}(M_{n,n+1}) \to \mathcal{O}(Sp_{2n})$ restricts to give an isomorphism $\psi^* : \mathcal{B}' \to \mathcal{B}$ of graded SL_2-algebras.

This theorem allows us to reduce questions about branching from Sp_{2n} to Sp_{2n-2} to analogous questions concerning branching from GL_{n+1} to GL_{n-1}. The latter are easier, as they can be "factored" through GL_n. We will illustrate this reduction technique several times, most notably in order to prove our second theorem.

To describe our second theorem we introduce a family of subalgebras of $\mathcal{B}$ indexed by a finite set, Σ, of so-called order types. We prove that each subalgebra $\mathcal{B}_\sigma$ is isomorphic to the algebra, $\mathcal{O}(V)$, of polynomials on a vector space V. This isomorphism is unique up to scalars. Moreover, V can be given the structure of an $L = \prod_{i=1}^{n} SL_2$-module. Therefore, via this isomorphism, we obtain a canonical action of L on $\mathcal{B}_\sigma$ by algebra automorphisms. The action of L is well-defined on the intersections of these subalgebras, allowing us to glue the modules together to obtain a representation, Φ, of L on $\mathcal{B}$.

The representation $(\Phi, \mathcal{B})$ of L satisfies some remarkable properties. First and foremost, it identifies each multiplicity space as an explicit irreducible L-module. Secondly, the restriction of Φ to the diagonal subgroup of L recovers the natural SL_2 action on

$\mathcal{B}$. Finally, it is the unique such representation acting by algebra automorphisms on the subalgebras $\mathcal{B}_\sigma$.

As a corollary of this theorem we obtain resolution of the multiplicities that occur in branching of symplectic groups. Indeed, irreducible L-modules have one dimensional weight spaces. Therefore, via Φ, we obtain a decomposition of the multiplicity spaces into one dimensional spaces. This decomposition is canonical, i.e. depends only the choice of torus of Sp_{2n} that is fixed throughout. The other known approach to this problem uses quantum groups, where these multiplicities are resolved using an infinite dimensional Hopf algebra called the twisted Yangian ([Mol99]).

The basis of $\mathcal{B}$ which we obtain from the action of L is unique up to scalar. In [KY10] we study properties of this basis, and, in particular, show that it is a standard monomial basis, i.e. it satisfies a straightening algorithm. By induction this basis can be used to obtain a basis for irreducible representations of Sp_{2n}. The resulting basis is a partial analogue of the Gelfand-Zetlin basis ([GZ50]); to be properly called a "Gelfand-Zetlin" basis one must also compute the action of Chevalley generators. The only such basis known is the Gelfand-Zetlin-Molev basis arising from the Yangian theory mentioned above ([Mol99]). In a future work we will compare the basis resulting from our work to the Gelfand-Zetlin-Molev basis.

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2. Preliminaries

Our main object of study, $\mathcal{B}$, is an example of a branching algebra. In section 2.1 we define branching algebras and their associated branching semigroups. In section 2.2 we fix some notation that will be used throughout.

2.1. **Branching algebras.** Let G be a connected classical group with identity $e \in G$. Fix a maximal torus T_G, Borel subgroup B_G, and unipotent radical $U_G \subset B_G$ so that $T_G U_G = B_G$. Let $\overline{U}_G$ be the unipotent group opposite U_G. When convenient, we work in the setting of Lie algebras. We denote the complex Lie algebra of a complex Lie group by the corresponding lower-case fraktur letter.

Fix the choice of positive roots giving U_G: $\Phi_G^+ = \Phi(\mathfrak{b}_G, \mathfrak{t}_G)$. Let $\Lambda_G \subset \mathfrak{t}^*$ be the corresponding semigroup of dominant integral weights. Denote by F_G^λ the finite-dimensional irreducible representation of G of highest weight $\lambda \in \Lambda_G$.

For an affine algebraic variety X, let $\mathcal{O}(X)$ denote the algebra of regular functions on X. The group G has the structure of an affine algebraic variety, and $\mathcal{O}(G)$ is a $G \times G$-module under left and right translation. Let $\mathcal{R}_G = \mathcal{O}(\overline{U}_G \backslash G)$ be the left $\overline{U}_G$-invariant functions on G:

$$\mathcal{R}_G = \{f \in \mathcal{O}(G) : f(\overline{u}g) = f(g) \text{ for all } \overline{u} \in \overline{U}_G \text{ and } g \in G\}.$$

Under the right action of G (see Theorem 4.2.7. ,[GW09]):

$$\mathcal{R}_G \cong \bigoplus_{\lambda\in\Lambda_G} ((F_G^\lambda)^*)^{\overline{U}_G} \otimes F_G^\lambda \cong \bigoplus_{\lambda\in\Lambda_G} F_G^\lambda.$$

Henceforth identify F_G^λ with its image in $\mathcal{R}_G$. Let $f_G^\lambda \in F_G^\lambda$ be the unique highest weight vector such that $f_G^\lambda(e) = 1$. We call f_G^λ the **canonical highest weight vector** of F_G^λ. Now let $\lambda, \lambda' \in \Lambda_G$. Then $f_G^\lambda f_G^{\lambda'}$ is U-invariant of weight $\lambda + \lambda'$. Since also $f_G^\lambda f_G^{\lambda'}(e) = 1$, it follows that $f_G^\lambda f_G^{\lambda'} = f_G^{\lambda+\lambda'}$, and therefore $F_G^\lambda F_G^{\lambda'} = F_G^{\lambda+\lambda'}$.

The **Cartan product**, $\pi_{\lambda,\lambda'} : F_G^\lambda \otimes F_G^{\lambda'} \to F_G^{\lambda+\lambda'}$, is defined by $\pi_{\lambda,\lambda'}(v \otimes v') = vv'$. The **Cartan embedding**, $j_{\lambda,\lambda'} : F_G^{\lambda+\lambda'} \to F_G^\lambda \otimes F_G^{\lambda'}$, is given by setting $j_{\lambda,\lambda'}(f_G^{\lambda+\lambda'}) = f_G^\lambda \otimes f_G^{\lambda'}$, and extending by G-linearity.

Suppose now that $H \subset G$ is a connected Lie subgroup, and we have chosen its distinguished subgroups so that $\overline{U}_H \subset \overline{U}_G$, $T_H \subset T_G$, and $U_H \subset U_G$. Consider the subalgebra of bi-invariants

$$\mathcal{B}(H, G) = \mathcal{O}(\overline{U}_G \setminus G \ / \ U_H) \subset \mathcal{R}_G.$$

In other words, $\mathcal{B}(H, G)$ consists of the functions $f \in \mathcal{R}_G$ such that $f(gu) = f(g)$ for all $g \in G$ and $u \in U_H$.

Since T_H normalizes U_H there is an action of T_H on $\mathcal{B}(H, G)$ by right translation. The decomposition of $\mathcal{B}(H, G)$ into T_H weight spaces is:

$$\mathcal{B}(H, G) = \bigoplus_{(\mu,\lambda)\in\Lambda_H\times\Lambda_G} F^{\lambda/\mu}.$$

The weight space $F^{\lambda/\mu}$ is defined as

$$F^{\lambda/\mu} = \{f \in (F_G^\lambda)^{U_H} : f(gt) = t^\mu f(g) \text{ for all } g \in G \text{ and } t \in T_H\}. \tag{1}$$

Equivalently, $F^{\lambda/\mu}$ is the $T_G \times T_H$ weight space of $\mathcal{B}$ corresponding to the weight $(-\lambda)$ for T_G and μ for T_H.

Note that the centralizer $Z_G(H)$ acts on $F^{\lambda/\mu}$ by right translation. As a $Z_G(H)$-module there is a canonical isomorphism

$$F^{\lambda/\mu} \cong \mathrm{Hom}_H(F_H^\mu, F_G^\lambda).$$

(The isomorphism maps $\phi \in \mathrm{Hom}_H(F_H^\mu, F_G^\lambda)$ to $\phi(f_H^\mu) \in F_{\lambda/\mu}$.) In particular, the dimension of $F^{\lambda/\mu}$ counts the branching multiplicity of F_H^μ in the restriction to H of F_G^λ. For this reason $\mathcal{B}(H, G)$ is termed the **branching algebra** for the pair (H, G) (cf. [HTW08], [Zh73] and references therein).

Lemma 2.1. *Let* $(\mu, \lambda), (\mu', \lambda') \in \Lambda_H \times \Lambda_G$. *Then*

$$\pi_{\lambda,\lambda'}(F^{\lambda/\mu} \otimes F^{\lambda'/\mu'}) \subset F^{\lambda+\lambda'/\mu+\mu'}.$$

Proof. Since $\pi_{\lambda,\lambda'}$ is a G-module morphism, in particular it is a U_H-module morphism. Therefore, $\pi_{\lambda,\lambda'}((F_G^\lambda)^{U_H} \otimes (F_G^{\lambda'})^{U_H}) \subset (F_G^{\lambda+\lambda'})^{U_H}$. Now note that $\pi_{\lambda,\lambda'}$ intertwines the T_H action on $(F_G^\lambda)^{U_H} \otimes (F^{\lambda'})_G^{U_H}$ and $(F_G^{\lambda+\lambda'})^{U_H}$. Let $((F_G^\lambda)^{U_H} \otimes (F_G^{\lambda'})^{U_H}))(\mu + \mu')$ be the $\mu + \mu'$ weight space of T_H on $(F_G^\lambda)^{U_H} \otimes (F_G^{\lambda'})^{U_H})$. Then $\pi_{\lambda,\lambda'}$ maps $((F_G^\lambda)^{U_H} \otimes (F_G^{\lambda'})^{U_H}))(\mu + \mu')$

to the weight space $F^{\lambda+\lambda'/\mu+\mu'}$. Since $F^{\lambda/\mu}\otimes F^{\lambda'/\mu'}\subset((F_G^{\lambda})^{U_H}\otimes(F_G^{\lambda'})^{U_H})(\mu+\mu')$ the proof is complete. ☐

By the lemma $\mathcal{B}(H,G)$ is a $\Lambda_H\times\Lambda_G$-graded algebra. Abusing notation a bit, we denote the restriction of $\pi_{\lambda,\lambda'}$ to $F^{\lambda/\mu}\otimes F^{\lambda'/\mu'}$ also by $\pi_{\lambda,\lambda'}$. This will cause no confusion since we will explicitly write

$$F^{\lambda/\mu}\otimes F^{\lambda'/\mu'}\overset{\pi_{\lambda,\lambda'}}{\to}F^{\lambda+\lambda'/\mu+\mu'}$$

when referring to this map, which we call the **Cartan product of multiplicity spaces**. In general the Cartan product of multiplicity spaces is not surjective (see section Proposition 3.6). This observation will be critical.

We can associate to the pair (H,G) the following set:

$$\Lambda_{\mathcal{B}(H,G)}=\{(\mu,\lambda)\in\Lambda_H\times\Lambda_G:F^{\lambda/\mu}\neq\{0\}\}.$$

Lemma 2.2. *The set $\Lambda_{\mathcal{B}(H,G)}$ is a semigroup under entry-wise addition.*

Proof. Suppose $(\mu,\lambda),(\mu',\lambda')\in\Lambda_{\mathcal{B}(H,G)}$. Choose $0\neq x\in F^{\lambda/\mu}$ and $0\neq x'\in F^{\lambda'/\mu'}$. Since $\mathcal{B}(H,G)$ has no zero divisors, $xx'\neq 0$. By the above lemma this implies that $F^{\lambda+\lambda'/\mu+\mu'}\neq 0$, i.e. $(\mu+\mu',\lambda+\lambda')\in\Lambda_{\mathcal{B}(H,G)}$. The other semigroup axioms are trivial to check. ☐

Thus the $\Lambda_H\times\Lambda_G$-graded algebra $\mathcal{B}(H,G)$ can also be regarded as a $\Lambda_{\mathcal{B}(H,G)}$-graded algebra, and we call $\Lambda_{\mathcal{B}(H,G)}$ a **branching semigroup**.

2.2. **Notation.** Henceforth fix an integer $n>1$. Let Λ_n be the semigroup of weakly decreasing sequences of length n consisting of non-negative integers. For $m\geq 1$, let $\Lambda_{m,n}=\Lambda_m\times\Lambda_n$.

The set of dominant weights for irreducible polynomial representations of $GL_n=GL(n,\mathbb{C})$ is identified with Λ_n in the usual way (see e.g. Theorem 5.5.22., [GW09]). For $\lambda\in\Lambda_n$, let $V^\lambda=F^\lambda_{GL_n}$ be the irreducible representation of GL_n with highest weight λ, which we realize in $\mathcal{R}_{GL_n}$ as described in Section 2.1. Let $v_\lambda\in V^\lambda$ be the canonical highest weight vector.

Let T_n be the subgroup of diagonal matrices in GL_n, U_n the subgroup of upper-triangular unipotent matrices, and $\overline{U}_n$ be the subgroup of lower-triangular unipotent matrices. Suppose $1\leq m<n$. We embed GL_m in GL_n as the subgroup:

$$\left\{\begin{bmatrix} g & \\ & I_{n-m}\end{bmatrix}:g\in GL_m\right\}.$$

For $(\mu,\lambda)\in\Lambda_{m,n}$, define the multiplicity space $V^{\lambda/\mu}$ as in (1) with $H=GL_m$ and $G=GL_n$. In the case $m=n-1$ these spaces are classically known (see e.g. Theorem 8.1.1., [GW09]): for $(\mu,\lambda)\in\Lambda_{n-1,n}$,

$$\dim V^{\lambda/\mu}\quad\leq\quad 1 \tag{2}$$

$$V^{\lambda/\mu}\neq\{0\}\quad\Leftrightarrow\quad\mu\ \textbf{interlaces}\ \lambda. \tag{3}$$

The interlacing condition, written $\mu<\lambda$, means that $\lambda_i\geq\mu_i\geq\lambda_{i+1}$ for $i=1,...,n-1$, where $\mu=(\mu_1,...,\mu_{n-1})$ and $\lambda=(\lambda_1,...,\lambda_n)$.

Let $M_{m,n}$ denote the space of $m \times n$ matrices with complex entries. By (GL_n, GL_{n+1}) duality, the space of polynomials $\mathcal{O}(M_{n,n+1})$ decomposes multiplicity free as $GL_n \times GL_{n+1}$ module (see e.g. Theorem 5.6.7.,[GW09]):

$$\mathcal{O}(M_{n,n+1}) \cong \bigoplus_{\lambda \in \Lambda_n} V^{\lambda^*} \otimes V^{\lambda^+}.$$

Here $\lambda^+ \in \Lambda_{n+1}$ is obtained from λ by adding a zero. Notice that V^{λ^*} is an irreducible representation of GL_n, while V^{λ^+} is an irreducible representation of GL_{n+1}.

We will study the algebra of bi-invariants

$$\mathcal{B}' = \mathcal{O}(\overline{U}_n \setminus M_{n,n+1} \ / \ U_{n-1}) \cong \bigoplus_{(\mu,\lambda) \in \Lambda_{n-1,n}} V^{\lambda^+/\mu}.$$

This algebra is a sort of "restricted" branching algebra as it includes only certain multiplicity spaces that occur in branching from GL_{n+1} to GL_{n-1}.

The algebra $\mathcal{B}'$ is an $\Lambda_{n-1,n}$-graded algebra; the (μ, λ) component is $V^{\lambda^+/\mu}$. Moreover, it is naturally an SL_2-algebra. Indeed, there is an obvious $SL_2 \subset GL_{n+1}$ that commutes with GL_{n-1}, and therefore acts on $\mathcal{B}'$ by right translation. This action clearly preserves the graded components of $\mathcal{B}'$, i.e. the multiplicity spaces $V^{\lambda^+/\mu}$ are naturally SL_2-modules.

By Lemma 2.2, the algebra $\mathcal{B}'$ is also graded by the semigroup

$$\Lambda_{\mathcal{B}'} = \{(\mu, \lambda) \in \Lambda_{n-1,n} : V^{\lambda^+/\mu} \neq \{0\}\}.$$

It will be useful for us to have a more concrete realization of this semigroup. Suppose $\mu = (\mu_1, ..., \mu_{n-1}) \in \Lambda_{n-1}$ and $\lambda = (\lambda_1, ..., \lambda_{n+1}) \in \Lambda_{n+1}$. We say μ **double interlaces** λ, written $\mu \ll \lambda$, if for $i = 1, ..., n-1$,

$$\lambda_i \geq \mu_i \geq \lambda_{i+2}. \tag{4}$$

If $(\mu, \lambda) \in \Lambda_{n-1,n}$, then μ double interlaces λ, also written $\mu \ll \lambda$, if $\mu \ll \lambda^+$.

Suppose $(\mu, \lambda) \in \Lambda_{n-1,n}$. It follows that $\mu \ll \lambda$ if, and only if, there exists $\gamma \in \Lambda_n$ such that $\mu < \gamma < \lambda^+$ (see Lemma 5.1). Therefore by (3),

$$\Lambda_{\mathcal{B}'} = \{(\mu, \lambda) \in \Lambda_{n-1,n} : \mu \ll \lambda\}. \tag{5}$$

Next we consider the symplectic groups. Label a basis for $\mathbb{C}^{2n}$ as $e_{\pm 1}, ..., e_{\pm n}$ where $e_{-i} = e_{2n+1-i}$. Denote by s_n the $n \times n$ matrix with one's on the anti-diagonal and zeros everywhere else. Set

$$J_n = \begin{bmatrix} 0 & s_n \\ -s_n & 0 \end{bmatrix}$$

and consider the skew-symmetric bilinear form $\Omega_n(x, y) = x^t J_n y$ on $\mathbb{C}^{2n}$. Define the symplectic group relative to this form: $Sp_{2n} = Sp(\mathbb{C}^{2n}, \Omega)$. In this realization we can take as a maximal torus $T_{C_n} = T_{2n} \cap Sp_{2n}$, a maximal unipotent subgroup $U_{C_n} = U_{2n} \cap Sp_{2n}$, and its opposite $\overline{U}_{C_n} = \overline{U}_{2n} \cap Sp_{2n}$

Embed Sp_{2n-2} in Sp_{2n} as the subgroup fixing the vectors $e_{\pm n}$. Notice that

$$\{g \in Sp_{2n} : ge_{\pm i} = e_{\pm i} \text{ for } i = 1, ..., n-1\} \tag{6}$$

is a subgroup isomorphic to $SL_2 = Sp_2$ commuting with Sp_{2n-2}.

The set of dominant weights for Sp_{2n} is identified with Λ_n (see e.g. Theorem 3.1.20., [GW09]). For $\lambda \in \Lambda_n$, let $W^\lambda = F^\lambda_{Sp_{2n}}$ be the irreducible representation of Sp_{2n} with highest weight λ, which we realize in $\mathcal{R}_{Sp_{2n}}$ as described in section 2.1. Let $w_\lambda \in W^\lambda$ be the canonical highest weight vector.

For $(\mu, \lambda) \in \Lambda_{n-1,n}$ we define the multiplicity space $W^{\lambda/\mu}$ as in (1) with $H = Sp_{2n-2}$ and $G = Sp_{2n}$. Therefore the dimension of $W^{\lambda/\mu}$ is the multiplicity of the irreducible representation W^μ of Sp_{2n-2} in the representation W^λ of Sp_{2n}. In contrast to the general linear groups, the branching of the symplectic groups is not multiplicity-free, i.e. $\dim W^{\lambda/\mu} > 1$ for generic $(\mu, \lambda) \in \Lambda_{n-1,n}$ (see Corollary 3.3).

Our main object of study is the following branching algebra:

$$\mathcal{B} = \mathcal{B}(Sp_{2n-2}, Sp_{2n}).$$

By Lemma 2.1, $\mathcal{B}$ is graded by $\Lambda_{n-1,n}$:

$$\mathcal{B} = \bigoplus_{(\mu,\lambda) \in \Lambda_{n-1,n}} W^{\lambda/\mu}.$$

By Lemma 2.2, we also consider $\mathcal{B}$ as graded over the branching semigroup

$$\Lambda_{\mathcal{B}} = \Lambda(Sp_{2n-2}, Sp_{2n}).$$

The branching algebra $\mathcal{B}$ is naturally an SL_2-module for the copy of SL_2 appearing in (6). Indeed, this copy SL_2 acts on $\mathcal{B}$ by right translation leaving the graded components $W^{\lambda/\mu}$ invariant. In other words, the multiplicity spaces $W^{\lambda/\mu}$ are naturally SL_2-modules. We refer to this action as the "natural" SL_2 action, and denote it simply by $x.b$ for $x \in SL_2$ and $b \in \mathcal{B}$.

Finally, we consider the group $SL_2 = SL(2, \mathbb{C})$. Let $F^k = \mathcal{O}^k(\mathbb{C}^2)$ be the $(k+1)^{\text{th}}$-dimensional irreducible representation of SL_2, realized as the polynomials on $\mathbb{C}^2$ of homogeneous degree k. The SL_2 action is by right translation. For $k, k' \geq 0$ let $\pi_{k,k'} : F^k \otimes F^{k'} \to F^{k+k'}$ be the usual multiplication of functions, and define the embedding of SL_2-modules, $j_{k,k'} : F^{k+k'} \to F^k \otimes F^{k'}$, by setting $j_{k,k'}(x_1^{k+k'}) = x_1^k \otimes x_1^{k'}$ and extending by SL_2-linearity. Here x_1 is the first coordinate function on $\mathbb{C}^2$.

Remark 2.3. *We use the symbols $\pi_{\lambda,\lambda'}$ and $j_{\lambda,\lambda'}$ to denote the Cartan maps for any of the given groups above. It will be clear from context which group we have in mind.*

3. Main Results

In this section we describe in detail the two main theorems of this paper.

3.1. **An isomorphism of branching algebras.** Our first theorem describes an isomorphism of the branching algebras $\mathcal{B}$ and $\mathcal{B}'$. This usefulness of this theorem will become clear, as we use it repeatedly to reduce questions about branching of the symplectic groups to analogous questions about branching of the general linear groups.

Define the map $\psi : Sp_{2n} \to M_{n,n+1}$, which assigns an element $g \in Sp_{2n}$ its principal $n \times (n+1)$ submatrix. Consider the induced map on functions

$$\psi^* : \mathcal{O}(M_{n,n+1}) \to \mathcal{O}(Sp_{2n}). \tag{7}$$

In Lemma 4.6 we show that $\psi^*(\mathcal{B}') \subset \mathcal{B}$, and, moreover, that $\psi^* : \mathcal{B}' \to \mathcal{B}$ is a map of $\Lambda_{n-1,n}$-graded, SL_2 algebras. In fact, much more is true:

Theorem 3.1. *The map $\psi^* : \mathcal{B}' \to \mathcal{B}$ is an isomorphism of $\Lambda_{n-1,n}$-graded, SL_2-algebras.*

We now describe some applications.

For our first application let $(\mu, \lambda) \in \Lambda_{n-1,n}$. By the theorem,

$$W^{\lambda/\mu} \neq \{0\} \Leftrightarrow V^{\lambda^+/\mu} \neq \{0\}.$$

Combining this with (5), we recover a classical result about symplectic branching (see e.g. Theorem 8.1.5., [GW09]):

$$W^{\lambda/\mu} \neq \{0\} \Leftrightarrow \mu \ll \lambda,$$

i.e. $\Lambda_{\mathcal{B}} = \Lambda_{\mathcal{B}'}$.

To describe our second application of this theorem, recall first that the multiplicity spaces $W^{\lambda/\mu}$ are each SL_2-modules. Naturally, one would like to describe the SL_2-module structure of these multiplicity spaces. By Theorem 3.1, $W^{\lambda/\mu} \cong V^{\lambda^+/\mu}$ as SL_2-modules, so it suffices to answer the analogous question for the general linear groups. But this is not too difficult, since branching from GL_{n+1} to GL_{n-1} factors through GL_n.

Given $(\mu, \lambda) \in \Lambda_{n-1,n+1}$ let $(x_1 \geq y_1 \geq \cdots \geq x_n \geq y_n)$ be the non-increasing rearrangement of $(\mu_1, ..., \mu_{n-1}, \lambda_1, ..., \lambda_{n+1})$. Set

$$r_i(\mu, \lambda) = x_i - y_i.$$

Proposition 3.2. *Suppose $(\mu, \lambda) \in \Lambda_{n-1,n+1}$, and $\mu \ll \lambda$. Then as SL_2-modules*

$$V^{\lambda/\mu} \cong \bigotimes_{i=1}^{n} F^{r_i(\mu,\lambda)}$$

where SL_2 acts by the tensor product representation on the right hand side.

As a corollary of Theorem 3.1 and Proposition 3.2 we can describe the SL_2-module structure of the multiplicity spaces $W^{\lambda/\mu}$.

Corollary 3.3. *Suppose $(\mu, \lambda) \in \Lambda_{\mathcal{B}}$. Then as SL_2-modules*

$$W^{\lambda/\mu} \cong \bigotimes_{i=1}^{n} F^{r_i(\mu,\lambda^+)},$$

where SL_2 acts by the tensor product representation on the right hand side.

The above corollary first appeared explicitly as Theorem 3.3, [WY], where it was proved using the combinatorics of partition functions. It can also be obtained using Theorem 5.2, [Mol99], where it is shown that $W^{\lambda/\mu}$ carries an irreducible action of a certain infinite dimensional Hopf algebra called the twisted Yangian.

3.2. **A resolution of multiplicities.** We now describe the second theorem of this paper. Define $L = \prod_{i=1}^{n} SL_2$. For $(\mu, \lambda) \in \Lambda_{n-1,n}$ consider the irreducible L-module

$$A^{\lambda/\mu} = \bigotimes_{i=1}^{n} F^{r_i(\mu,\lambda^+)}.$$

Corollary 3.3 states that for all $(\mu, \lambda) \in \Lambda_{\mathcal{B}}$, $W^{\lambda/\mu} \cong \mathrm{Res}^{L}_{SL_2} A^{\lambda/\mu}$, where $SL_2 \subset L$ is the diagonal subgroup. We therefore ask, is there a canonical action of L on $W^{\lambda/\mu}$ such that $W^{\lambda/\mu} \cong A^{\lambda/\mu}$ as L-modules?

Remarkably, the answer to this question is yes! We will construct a canonical action of L on $\mathcal{B}$ that will be uniquely determined by two properties, the first of which is that each multiplicity space $W^{\lambda/\mu}$ is isomorphic to $A^{\lambda/\mu}$ as an L-module. Moreover, the restriction of this action to the diagonally embedded $SL_2 \subset L$ recovers the natural action of SL_2 on $\mathcal{B}$.

We warn the reader that L is not the product of SL_2's that lives in Sp_{2n}. Indeed, the latter product of SL_2's does not act on the multiplicity spaces. The existence of this L-action is more subtle, and can only be "seen" by considering all multiplicity spaces together, i.e. by considering the branching algebra.

To describe the action of L on $\mathcal{B}$ precisely we investigate the double interlacing condition that characterizes branching of the symplectic groups. Notice that the inequality

$$\lambda_i \geq \mu_i \geq \lambda_{i+2}$$

does not constrain the relation between μ_i and λ_{i+1}. In other words, we can have either $\mu_i \geq \lambda_{i+1}$, or $\mu_i \leq \lambda_{i+1}$, or both. This motivates the following:

Definition 3.4. *An order type* σ *is a word in the alphabet* $\{\geq, \leq\}$ *of length* $n-1$.

Suppose $(\mu, \lambda) \in \Lambda_{\mathcal{B}}$ and $\sigma = (\sigma_1 \cdots \sigma_{n-1})$ is an order type. Then we say (μ, λ) is **of order type** σ if for $i = 1, ..., n-1$,

$$\begin{cases} \sigma_i = \text{“} \geq \text{”} \implies \mu_i \geq \lambda_{i+1} \\ \sigma_i = \text{“} \leq \text{”} \implies \mu_i \leq \lambda_{i+1} \end{cases}$$

For example, consider the double interlacing pair (μ, λ), where $\lambda = (3, 2, 1)$ and $\mu = (3, 0)$. Since $\mu_1 \geq \lambda_2$ and $\mu_2 \leq \lambda_3$, the pair (μ, λ) is of order type $\sigma = (\geq\leq)$.

Let Σ be the set of order types, and for each $\sigma \in \Sigma$ set

$$\Lambda_{\mathcal{B}}(\sigma) = \{(\mu, \lambda) \in \Lambda_{\mathcal{B}} : (\mu, \lambda) \text{ is of order type } \sigma\}.$$

It's easy to check that $\Lambda_{\mathcal{B}}(\sigma)$ is a sub-semigroup of $\Lambda_{\mathcal{B}}$. Therefore

$$\mathcal{B}_\sigma = \bigoplus_{(\mu,\lambda) \in \Lambda_{\mathcal{B}}(\sigma)} W^{\lambda/\mu}$$

is a subalgebra of $\mathcal{B}$. Moreover, $\mathcal{B}_\sigma$ is SL_2-invariant. We now have all the ingredients to state our second theorem.

Theorem 3.5. *There is a unique representation* $(\Phi, \mathcal{B})$ *of* L *satisfying the following two properties:*

(1) *for all* $(\mu,\lambda)\in\Lambda_{\mathcal{B}}$, $W^{\lambda/\mu}$ *is an irreducible* L*-invariant subspace of* $\mathcal{B}$ *isomorphic to* $A^{\lambda/\mu}$*, and*
(2) *for all* $\sigma\in\Sigma$, L *acts as algebra automorphisms on* $\mathcal{B}_\sigma$.

Moreover, $\mathrm{Res}^{\mathrm{L}}_{\mathrm{SL}_2}(\Phi)$ *recovers the natural action of* SL_2 *on* $\mathcal{B}$.

We will now give an overview of the proof of this theorem. From the statement of Theorem 3.5 it's clear that the subalgebras $\mathcal{B}_\sigma$ are intrinsic to the definition of the representation $(\Phi,\mathcal{B})$ of L. Therefore it is not surprising that the proof requires a thorough understanding of these subalgebras.

To investigate the subalgebras $\mathcal{B}_\sigma$ we define the L-module $(\theta_\sigma,\mathcal{A}_\sigma)$, where

$$\mathcal{A}_\sigma=\bigoplus_{(\mu,\lambda)\in\Lambda_{\mathcal{B}}(\sigma)}A^{\lambda/\mu}. \tag{8}$$

The crucial observation is that since we are restricting to a fixed order type there is a natural product on $\mathcal{A}_\sigma$. The multiplication

$$A^{\lambda/\mu}\otimes A^{\lambda'/\mu'}\to A^{\lambda+\lambda'/\mu+\mu'}$$

is given by Cartan product of irreducible L-modules, which induces an algebra structure on $\mathcal{A}_\sigma$ (cf. Lemma 6.1). The algebra $\mathcal{A}_\sigma$ is a naturally occurring L-algebra. Indeed, in Lemma 6.2 we show that $\mathcal{A}_\sigma$ is isomorphic as a graded L-algebra to the ring of polynomials $\mathcal{O}(V)$ on a certain L-module V.

A priori the algebras $\mathcal{B}_\sigma$ and $\mathcal{A}_\sigma$ seem to be quite different. Indeed, the product maps $A^{\lambda/\mu}\otimes A^{\lambda'/\mu'}\to A^{\lambda+\lambda'/\mu+\mu'}$ in $\mathcal{A}_\sigma$ are surjective, while the Cartan product of multiplicity spaces need not be surjective. For example, consider the case $\lambda=\lambda'=(2,1,0)$, $\mu=(2,0)$, and $\mu'=(0,0)$. By Corollary 3.3, $\dim W^{\lambda/\mu}=\dim W^{\lambda'/\mu'}=2$, and $\dim W^{\lambda+\lambda'/\mu+\mu'}=9$. Therefore the product map cannot be surjective in this case.

Notice that in the above example (μ,λ) and (μ',λ') do not satisfy a common order type. An important result for us is that if the multiplicity spaces do satisfy a common order type, then their product is surjective. This will be our most crucial application of Theorem 3.1.

Proposition 3.6. *Let* $\sigma\in\Sigma$ *and let* $(\mu,\lambda),(\mu',\lambda')\in\Lambda_{\mathcal{B}}(\sigma)$. *Then the map*

$$W^{\lambda/\mu}\otimes W^{\lambda'/\mu'}\overset{\pi_{\lambda,\lambda'}}{\to}W^{\lambda+\lambda'/\mu+\mu'}$$

is surjective.

By Proposition 3.3, $\mathcal{B}_\sigma$ and $\mathcal{A}_\sigma$ are isomorphic as SL_2 modules. The above proposition shows, moreover, that their products behave similarly. In fact, we have the following theorem:

Proposition 3.7. *Let* $\sigma\in\Sigma$. *Then,*

(1) *There is an isomorphism of graded* SL_2*-algebras,* $\phi_\sigma:\mathcal{B}_\sigma\to\mathcal{A}_\sigma$*, which is unique up to scalars.*
(2) *By part (1) we can transfer the action of* L *on* $\mathcal{A}_\sigma$ *to* $\mathcal{B}_\sigma$*, and the resulting representation,* $(\Phi_\sigma,\mathcal{B}_\sigma)$*, of* L *is canonical, i.e. independent of the choice of scalars.*

We now have a family of L-algebras $\{(\Phi_\sigma, \mathcal{B}_\sigma)\}_{\sigma\in\Sigma}$. By showing that the action of L is well-defined on the intersection of these subalgebras, we obtain the representation $(\Phi, \mathcal{B})$ of L and prove Theorem 3.5.

We conclude this section by describing how to use our results to resolve the multiplicities that occur in branching of symplectic groups. Recall that generically the branching of the symplectic groups is not multiplicity free. It's a fundamental problem in classical invariant theory to resolve these multiplicities. Theorem 3.5 provides a solution to this problem that is rooted in classical invariant theoretic techniques. Another solution to this problem using the theory of quantum groups, in particular Yangians, appears in [Mol99].

Now, it is well known that irreducible L-modules have one dimensional weight spaces. Therefore, by Theorem 3.5 we obtain a canonical decomposition of the multiplicity spaces $W^{\lambda/\mu}$ into one-dimensional spaces. A priori, it seems that this decomposition depends on a choice of torus of L. We will show that in fact this choice is induced by the torus of Sp_{2n} that is fixed throughout. More precisely, we have:

Corollary 3.8. *Let $(\mu,\lambda) \in \Lambda$ There is a canonical decomposition of $W^{\lambda/\mu}$ into one dimensional spaces*

$$W^{\lambda/\mu} = \bigoplus_{\substack{\gamma\in\Lambda_n \\ \mu<\gamma<\lambda^+}} W^{\lambda/\gamma/\mu}.$$

In particular, $\mathcal{B}$ has a basis which is unique up to scalar.

Properties of the basis of $\mathcal{B}$ appearing in Corollary 3.8 are studied in [KY]; in particular we show that it is a standard monomial basis, i.e. it satisfies a straightening law. We then use that to describe an explicit toric deformation of $\mathrm{Spec}(\mathcal{B})$.

4. Proof of Theorem 3.1

4.1. **Some results of Zhelobenko.** Let G be a connected classical group. We use freely the notation from Section 2.1. Let $\lambda \in \Lambda_G$. Then $F_G^\lambda \subset \mathcal{R}_G$ embeds linearly in $\mathcal{O}(U_G)$ via $res : f \mapsto f|_{U_G}$. Set $Z_\lambda(U_G) = res_{U_G}(F_G^\lambda)$. If there is no cause for confusion, we write simply $Z_\lambda = Z_\lambda(U_G)$.

We define a representation of G on Z_λ as follows. Let $e^\lambda : T_G \to \mathbb{C}$ be the character of T_G given by $t \mapsto t^\lambda$. We extend this character to $\overline{U}_G T_G U_G$ by defining $e^\lambda(\overline{u}tu) = t^\lambda$. Then by continuity e^λ is defined on all of G. Now let $u \in U_G$, $g \in u^{-1}\overline{U}_G T_G U_G$, and $f \in Z_\lambda$. Write $ug = \overline{u}_1 t_1 u_1 \in \overline{U}_G T_G U_G$. Then define

$$g.f(u) = e^\lambda(t_1) f(u_1). \tag{9}$$

Since $u^{-1}\overline{U}_G T_G U_G$ is dense, we extend this action to all of G. Note that the constant function $z_\lambda : u \mapsto 1$ is a canonical highest weight vector in Z_λ of weight λ, and $res : F_G^\lambda \to Z_\lambda$ is an isomorphism of G-modules.

Let $\{\alpha_1, ..., \alpha_n\}$ be a set of simple roots relative to the positive roots Φ^+. For each α_i choose a nonzero root vector $X_i \in \mathfrak{g}_{\alpha_i}$. Let D_i be the differential operator on $\mathcal{O}(U_G)$ corresponding to the infinitesimal action of X_i acting on $\mathcal{O}(U_G)$ by left translation.

Finally, let $\{\varpi_1, ..., \varpi_n\}$ be the fundamental weights and suppose $\lambda = m_1\varpi_1 + \cdots + m_n\varpi_n \in \Lambda_G$.

Proposition 4.1 (Theorem 1, §65, Chapter X, [Zh73]). *The space $Z_\lambda \subset \mathcal{O}(U_G)$ is the solutions to the system of differential equations $\{D_i^{m_i+1} = 0 : i = 1, ..., n\}$. In other words,*

$$Z_\lambda = \{f \in \mathcal{O}(U_G) : D_i^{m_i+1} f = 0 \textit{ for } i = 1, ..., n\}.$$

In [Zh73] the system of differential equations $\{D_i^{m_i+1} = 0 : i = 1, ..., n\}$ is termed the "indicator system". Notice that by the Leibniz rule $Z_\lambda Z_{\lambda'} = Z_{\lambda+\lambda'}$.

Consider the ring $\mathcal{O}(U_G \times \mathbb{C}^n)$. Let $t_1, ..., t_n$ be the standard coordinates on $\mathbb{C}^n$. Then $\mathcal{O}(U_G \times \mathbb{C}^n) = \bigoplus_{\overline{m} \in \mathbb{N}^n} \mathcal{O}(U_G) \otimes t_1^{m_1} \cdots t_n^{m_n}$. Set $\overline{m}_\lambda = (m_1, ..., m_n)$ where $\lambda = m_1\varpi_1 + \cdots + m_n\varpi_n$. We form the subring

$$\mathcal{Z}_G = \bigoplus_{\lambda \in \Lambda_G} Z_\lambda \otimes t^{\overline{m}_\lambda} \tag{10}$$

of $\mathcal{O}(U_G \times \mathbb{C}^n)$. This is a G-ring, with G acting on the left factor. Define a map

$$\mathrm{res} : \mathcal{R}_G \to \mathcal{Z}_G \tag{11}$$

by

$$f \in F_G^\lambda \mapsto f|_{U_G} \otimes t^{\overline{m}_\lambda} \in Z_\lambda \otimes t^{\overline{m}_\lambda}.$$

It's easy to see that:

Proposition 4.2. *The map $\mathrm{res} : \mathcal{R}_G \to \mathcal{Z}_G$ is an isomorphism of G-rings.*

4.2. **Preparatory lemmas.** We now specialize the results of the previous section to our setting.

For $\lambda \in \Lambda_n$ we consider Zhelobenko's realization of V^{λ^+}, which is denoted $Z_{\lambda^+}(U_{n+1})$. So $Z_{\lambda^+}(U_{n+1})$ is an irreducible GL_{n+1}-module. Let x_{ij} be the standard coordinates on U_{n+1}.

Zhelobenko's realization of the irreducible Sp_{2n}-module of highest weight λ is denoted $Z_\lambda(U_{C_n})$. For the affine space U_{C_n}, the following can be taken as coordinates:

$$\begin{matrix} 1 & u_{12} & & \cdots & & u_{1n-1} & u_{1n} \\ & 1 & u_{23} & \cdots & & u_{2n-1} & \\ & & \ddots & \ddots & & \iddots & \\ & & & 1 & u_{n,n+1} & & \end{matrix} \tag{12}$$

(The one's are retained here in order to preserve the symmetry of the entries.) The other entries of U_{C_n} are polynomials in these coordinates.

The group U_{n-1} acts on $\mathcal{O}(U_{n+1})$ by right translation, and a straight-forward calculation shows that $f \in \mathcal{O}(U_{n+1})^{U_{n-1}}$ if, and only if, it's a polynomial in the variables

$$\{x_{i,j} : i = 1, ..., n \ j = n, n+1 \text{ and } i < j\}.$$

Similarly, $\mathcal{O}(U_{C_n})^{U_{C_{n-1}}}$ is the polynomial ring in the variables

$$\{u_{i,j} : i = 1, ..., n \ j = n, n+1 \text{ and } i < j\}.$$

Hence both $\mathcal{O}(U_{n+1})^{U_{n-1}}$ and $\mathcal{O}(U_{C_n})^{U_{C_{n-1}}}$ are polynomial rings in $2n-1$ variables.

Let $\psi_{a,b} : M_{m,n} \to M_{a,b}$ be the map assigning a matrix its principal $a \times b$ submatrix, and set $\psi_a = \psi_{a,a}$. Let $\psi_Z = \psi_{n+1}|_{U_{C_n}} : U_{C_n} \to U_{n+1}$. The induced map on functions $\psi_Z^* : \mathcal{O}(U_{n+1}) \to \mathcal{O}(U_{C_n})$ satisfies $\psi_Z^*(x_{ij}) = u_{ij}$. By our descriptions of the rings $\mathcal{O}(U_{C_n})^{U_{C_{n-1}}}$ and $\mathcal{O}(U_{n+1})^{U_{n-1}}$, $\psi_Z^* : \mathcal{O}(U_{n+1})^{U_{n-1}} \to \mathcal{O}(U_{C_n})^{U_{C_{n-1}}}$ is a ring isomorphism.

Lemma 4.3. *Let $\lambda \in \Lambda_n$. The map ψ_Z^* restricts to a linear isomorphism:*

$$\psi_Z^* : Z_{\lambda^+}(U_{n+1})^{U_{n-1}} \xrightarrow{\cong} Z_\lambda(U_{C_n})^{U_{C_{n-1}}}$$

Proof. Let $\overline{m}_\lambda = (m_1, ..., m_n)$. In the proof of Theorem 4, §114 in [Zh73], Zhelobenko shows that $Z_\lambda(U_{C_n})^{U_{C_{n-1}}}$ equals

$$\left\{ f \in \mathcal{O}(U_{C_n})^{U_{C_{n-1}}} : (u_{i+1,n}\frac{\partial}{\partial u_{i,n}} + u_{i+1,n+1}\frac{\partial}{\partial u_{i,n+1}})^{m_i+1}(f) = 0 \text{ for } i = 1, ..., n \right\}$$

while $Z_{\lambda^+}(U_{n+1})^{U_{n-1}}$ equals

$$\left\{ f \in \mathcal{O}(U_{n+1})^{U_{n-1}} : (x_{i+1,n}\frac{\partial}{\partial x_{i,n}} + x_{i+1,n+1}\frac{\partial}{\partial x_{i,n+1}})^{m_i+1}(f) = 0 \text{ for } i = 1, ..., n \right\}.$$

With these descriptions in hand, and the fact that $\psi_Z^*(x_{ij}) = u_{ij}$, it follows that

$$\psi_Z^*(Z_{\lambda^+}(U_{n+1})^{U_{n-1}}) = Z_\lambda(U_{C_n})^{U_{C_{n-1}}}.$$

□

Now, $T_{C_{n-1}}$ acts on $Z_\lambda(U_{C_n})^{U_{C_{n-1}}}$, while T_{n-1} acts on $Z_{\lambda^+}(U_{n+1})^{U_{n-1}}$. In the next lemma, these tori are both identified with $(\mathbb{C}^\times)^{n-1}$.

Lemma 4.4. *Let $\lambda \in \Lambda_n$. Then $\psi_Z^* : Z_{\lambda^+}(U_{n+1})^{U_{n-1}} \to Z_\lambda(U_{C_n})^{U_{C_{n-1}}}$ is a $(\mathbb{C}^\times)^{n-1}$-isomorphism.*

Proof. By Lemma 4.3, it remains to show only that ψ_Z^* intertwines the $(\mathbb{C}^\times)^{n-1}$-action. Let $f \in Z_{\lambda^+}(U_{n+1})^{U_{n-1}}$, $t \in (\mathbb{C}^\times)^{n-1}$, and $u \in U_{C_n}$. By definition of the action of the tori (cf. (9)),

$$\psi_Z^*(t.f)(u) = e^\lambda(t) f(t^{-1}\psi_Z(u)t),$$

while

$$t.\psi_Z^*(f)(u) = e^\lambda(t) f(\psi_Z(t^{-1}ut)).$$

It's easy to check now that $\psi_Z^*(t.f)(u) = t.\psi_Z^*(f)(u)$. □

For $\mu \in \Lambda_{n-1}$ let $Z_{\lambda/\mu}(U_{C_n})$ (resp. $Z_{\lambda^+/\mu}(U_{n+1})$) denote the μ weight spaces of $Z_\lambda(U_{C_n})^{U_{C_{n-1}}}$ (resp. $Z_{\lambda^+}(U_{n+1})^{U_{n-1}}$). Lemma 4.4 implies that

$$\psi_Z^* : Z_{\lambda^+/\mu}(U_{n+1}) \xrightarrow{\cong} Z_{\lambda/\mu}(U_{C_n})$$

is a linear isomorphism. These spaces are isomorphic to the multiplicity spaces $V^{\lambda^+/\mu}$ and $W^{\lambda/\mu}$. Recall that SL_2 acts on these multiplicity spaces.

Lemma 4.5. *Let $\lambda \in \Lambda_n$ and $\mu \in \Lambda_{n-1}$. Then*

$$\psi_Z^* : Z_{\lambda^+/\mu}(U_{n+1}) \to Z_{\lambda/\mu}(U_{C_n})$$

is an SL_2-isomorphism.

Proof. As mentioned above, the map $\psi_Z^* : Z_{\lambda^+/\mu}(U_{n+1}) \to Z_{\lambda/\mu}(U_{C_n})$ is a linear isomorphism, so we just need to show it intertwines the SL_2 action. Now, the action of of SL_2 on the multiplicity spaces is defined via its embeddings in GL_{n+1} and Sp_{2n} as explained in Section 2.2. For the sake of clarity, let us denote these embeddings by $\alpha : SL_2 \hookrightarrow GL_{n+1}$ and $\beta : SL_2 \hookrightarrow Sp_{2n}$. Note that for $x \in SL_2$, we have $\psi_{n+1}(\beta(x)) = \alpha(x)$.

Let $f \in Z_{\lambda^+}(U_{n+1})^{U_{n-1}}$, $x \in SL_2$, and $u \in U_{C_n}$. We want to show that

$$\psi_{n+1}^*(\alpha(x).f)(u) = (\beta(x).(\psi_{n+1}^*(f)))(u). \tag{13}$$

By definition of the action of Sp_{2n} on $Z_\lambda(U_{C_n})$ (see (9)), we have

$$(\beta(x).(\psi_{n+1}^*(f)))(u) = e^\lambda(t_1)f(\psi_{n+1}(u_1)),$$

where

$$u\beta(x) = \overline{u}_1 t_1 u_1 \in \overline{U}_{C_n} T_{C_n} U_{C_n}.$$

To describe the left hand side of (13), we need to first decompose $\psi_{n+1}(u)\alpha(x)$ into a product compatible with $\overline{U}_{n+1}T_{n+1}U_{n+1}$. To wit,

$$\begin{aligned}
\psi_{n+1}(u)\alpha(x) &= \psi_{n+1}(u)\psi_{n+1}(\beta(x)) \\
&= \psi_{n+1}(u\beta(x)) \\
&= \psi_{n+1}(\overline{u}_1 t_1 u_1) \\
&= \psi_{n+1}(\overline{u}_1)\psi_{n+1}(t_1)\psi_{n+1}(u_1).
\end{aligned}$$

Therefore

$$\psi_{n+1}^*(\alpha(x).f)(u) = e^{\lambda^+}(\psi_{n+1}(t_1))f(\psi_{n+1}(u_1)).$$

Since clearly, $e^\lambda(t_1) = e^{\lambda^+}(\psi_{n+1}(t_1))$, (13) holds. □

4.3. **Proof of Theorem 3.1.** Recall the homomorphism $\psi^* : \mathcal{O}(M_{n,n+1}) \to \mathcal{O}(Sp_{2n})$ (see (7)), which is induced from the map ψ taking an element of Sp_{2n} to its principal $n \times (n+1)$ submatrix.

Lemma 4.6. *We have $\psi^*(\mathcal{B}') \subset \mathcal{B}$. Moreover, $\psi^* : \mathcal{B}' \to \mathcal{B}$ is an $\Lambda_{n-1,n}$-graded map of SL_2 algebras.*

Proof. Let $f \in \mathcal{B}', \overline{u} \in \overline{U}_{C_n}, g \in Sp_{2n}$, and $u \in U_{C_{n-1}}$. We must show that $f(\psi(\overline{u}gu)) = f(\psi(g))$. Indeed, a straight-forward computation using block matrices shows that

$$\psi(\overline{u}gu) = \psi_n(\overline{u})\psi(g)\psi_{n+1}(u).$$

Since clearly $\psi_n(\overline{u}) \in \overline{U}_n$ and $\psi_{n+1}(u) \in U_{n-1}$, the first statement follows.

Now suppose $f \in V^{\lambda^+/\mu}$. Let $t \in T_{C_n}$, $s \in T_{C_{n-1}}$, and $g \in Sp_{2n}$. Then

$$\begin{aligned}(t,s).\psi^*(f)(g) &= \psi^*(f)(t^{-1}gs)\\ &= f(\psi(t^{-1}gs))\\ &= f(\psi_n(t^{-1})\psi(g)\psi_{n-1}(s))\\ &= \psi_n(t)^{-\lambda}\psi_{n-1}(s)^{\mu}f(\psi(g))\\ &= t^{-\lambda}s^{\mu}f(\psi(g))\\ &= t^{-\lambda}s^{\mu}.\psi^*(f)(g).\end{aligned}$$

Therefore $\psi^*(f) \in W^{\lambda/\mu}$, and hence ψ^* is graded.

Finally, we must show ψ^* intertwines the SL_2-action. Indeed, let $f \in \mathcal{B}'$, $x \in SL_2$, and $g \in Sp_{2n}$. Then another computation with block matrices shows that $f(\psi(g)x) = f(\psi(gx))$. (Here one has to be careful to use the correct embeddings of SL_2 in GL_{n+1} and Sp_{2n} that define the corresponding actions.) Therefore, $x.\psi^*(f) = \psi^*(x.f)$. □

By the above lemma,$\psi^* : \mathcal{B}' \to \mathcal{B}$ is a morphism of $\Lambda_{n-1,n}$-graded algebras. To complete the proof of Theoreme 3.1 we must it is an isomorphism.

Let $U_{n,n+1} = \psi_{n,n+1}(U_{n+1})$. We identify the affine spaces U_{n+1} and $U_{n,n+1}$ in the obvious way. For $\lambda \in \Lambda_n$, consider the embedding $Z_{\lambda^+}(U_{n+1}) \hookrightarrow \mathcal{O}(U_{n,n+1})$. Denote the image of this embedding by $Z_{\lambda^+}(U_{n,n+1})$. (Similarly, denote by $Z_{\lambda^+/\mu}(U_{n,n+1})$ the image of $Z_{\lambda^+/\mu}(U_{n+1})$ under the embedding.) By transfer of structure, $Z_{\lambda^+}(U_{n,n+1})$ is an irreducible GL_{n+1}-module of highest weight λ^+, i.e. we decree that

$$\psi^*_{n,n+1} : Z_{\lambda^+}(U_{n,n+1}) \to Z_{\lambda^+}(U_{n+1})$$

is an isomorphism of GL_{n+1}-modules. Of course, the inverse of this isomorphism is the map induced by the embedding $\phi : U_{n,n+1} \to U_{n+1}$. Combining this isomorphism with Lemma 4.5 we obtain that

$$\psi^*_{n,n+1} : Z_{\lambda^+/\mu}(U_{n,n+1}) \to Z_{\lambda/\mu}(U_{C_n}) \tag{14}$$

is an isomorphism of SL_2-modules.

Consider now

$$\mathcal{B}'_Z = \bigoplus_{(\mu,\lambda)\in\Lambda_{n-1,n}} Z_{\lambda^+/\mu}(U_{n,n+1}) \otimes t^{\overline{m}_{\lambda^+}}$$

and

$$\mathcal{B}_Z = \bigoplus_{(\mu,\lambda)\in\Lambda_{n-1,n}} Z_{\lambda/\mu}(U_{C_n}) \otimes t^{\overline{m}_{\lambda}}.$$

These are subalgebras of $\mathcal{Z}_{GL_{n+1}}$ and $\mathcal{Z}_{Sp_{2n}}$, respectively (see (10)).

We define a map $\mathcal{B}'_Z \to \mathcal{B}_Z$ which on graded components is simply given by

$$\psi^*_{n,n+1} \otimes 1 : Z_{\lambda^+/\mu}(U_{n,n+1}) \otimes t^{\overline{m}_{\lambda^+}} \to Z_{\lambda/\mu}(U_{C_n}) \otimes t^{\overline{m}_{\lambda}}.$$

Let us denote the total map by $\psi^*_{n,n+1} \otimes 1 : \mathcal{B}'_Z \to \mathcal{B}_Z$ also.

By (14) $\psi^*_{n,n+1} \otimes 1 : \mathcal{B}'_Z \to \mathcal{B}_Z$ is a isomorphism of SL_2-modules. But clearly the map is a morphism of graded algebras, so in fact it is an isomorphism of $\Lambda_{n-1,n}$-graded SL_2-algebras. Now, by Proposition 4.2, the restriction of $res : \mathcal{R}_{GL_{n+1}} \to \mathcal{Z}_{GL_{n+1}}$ to $\mathcal{B}'$

gives an isomorphism $res : \mathcal{B}' \to \mathcal{B}'_Z$ of $\Lambda_{n-1,n}$-graded SL_2-algebras. Similarly, we have the isomorphism $res : \mathcal{B} \to \mathcal{B}_Z$ of $\Lambda_{n-1,n}$-graded SL_2-algebras.

We now have a diagram:

$$\begin{array}{ccc} \mathcal{B}' & \longrightarrow & \mathcal{B} \\ \downarrow & & \downarrow \\ \mathcal{B}'_Z & \longrightarrow & \mathcal{B}_Z \end{array}$$

where the vertical arrows are given by res, the bottom arrow is $\psi^*_{n,n+1} \otimes 1$, and the top arrow is ψ^*. Clearly, this diagram commutes. Indeed, this follows from the simple fact that for $f \in \mathcal{O}(M_{n,n+1})$,

$$(f \circ \psi)|_{U_{C_n}} = f|_{U_{n,n+1}} \circ \psi_{n,n+1}$$

as elements of $\mathcal{O}(U_{C_n})$. Moreover, by the previous paragraph the bottom three maps are isomorphisms of $\Lambda_{n-1,n}$-graded SL_2-algebras by. We conclude that $\psi^* : \mathcal{B}' \to \mathcal{B}$ is an isomorphism of $\Lambda_{n-1,n}$-graded SL_2-algebras, thus completing the proof of Theorem 3.1.

5. Proof of Proposition 3.6

In this section we prove various structural results about the multiplicity spaces $V_{\lambda/\mu}$ where $(\mu, \lambda) \in \Lambda_{n-1,n+1}$. Notice that these are multiplicity spaces that occur in branching from GL_{n+1} to GL_{n-1}. By virtue of Theorem 3.1, these results have analogous in the setting of branching of symplectic groups, and it is for this reason that these results are important for us.

We will work in slightly greater generality than strictly necessary, and consider the semigroup

$$\begin{aligned} \Omega &= \{(\mu, \lambda) \in \Lambda_{n-1,n+1} : V_{\lambda/\mu} \neq \{0\}\} \\ &= \{(\mu, \lambda) \in \Lambda_{n-1,n+1} : \mu \ll \lambda\}. \end{aligned}$$

The second equality follows from (3).

5.1. The rearrangement function.

We begin by introducing the **rearrangement function** on Ω. Define $f : \Omega \to \Lambda_{2n}$ by:

$$(\mu, \lambda) \overset{f}{\mapsto} (x_1, y_1, ..., x_n, y_n)$$

where $\{x_1 \geq y_1 \geq \cdots \geq x_n \geq y_n\}$ is the non-increasing rearrangement of $(\mu_1, ..., \mu_{n-1}, \lambda_1, ..., \lambda_{n+1})$. Notice that $f(\mu, \lambda)$ equals

$$(\lambda_1 \geq \max(\mu_1, \lambda_2) \geq \min(\mu_1, \lambda_2) \geq \cdots \geq \max(\mu_{n-1}, \lambda_n) \geq \min(\mu_{n-1}, \lambda_n) \geq \lambda_{n+1}).$$

This easily implies:

Lemma 5.1. *Let $(\mu, \lambda) \in \Omega$. Suppose $f(\mu, \lambda) = (x_1, y_1, ..., x_n, y_n)$ and $\gamma \in \Lambda_n$. Then $\mu < \gamma < \lambda$ if, and only if, $y_i \leq \gamma_i \leq x_i$ for $i = 1, ..., n$, where $\gamma = (\gamma_1, ..., \gamma_n)$.*

For $\sigma \in \Sigma$ let $\Omega(\sigma)$ be the sub-semigroup of Ω consisting of the pairs of order type σ. Let f_σ denote the restriction of f to $\Omega(\sigma)$.

Lemma 5.2. *Let $\sigma \in \Sigma$. Then $f_\sigma : \Omega(\sigma) \to \Lambda_{2n}$ is a semigroup isomorphism.*

Proof. For $(\mu,\lambda) \in \Omega(\sigma)$ let $f_\sigma(\mu,\lambda) = (f_\sigma(\mu,\lambda)_1, ..., f_\sigma(\mu,\lambda)_{2n})$. Define functions

$$a : \{1, ..., n-1\} \to \{1, ..., 2n\}$$
$$b : \{1, ..., n+1\} \to \{1, ..., 2n\}$$

by $b(1) = 1$ and $b(n+1) = 2n$, and for $i = 1, ..., n-1$

$$(\sigma_i \text{ is } \geq\ ) \Longrightarrow a(i) = 2i \text{ and } b(i+1) = 2i+1$$
$$(\sigma_i \text{ is } \leq\ ) \Longrightarrow a(i) = 2i+1 \text{ and } b(i+1) = 2i.$$

Then for all $(\mu,\lambda) \in \Omega(\sigma)$

$$\mu_i = f_\sigma(\mu,\lambda)_{a(i)} \text{ for } i = 1, ..., n-1$$
$$\lambda_j = f_\sigma(\mu,\lambda)_{b(j)} \text{ for } j = 1, ..., n+1.$$

This implies that f_σ is an injective semigroup homomorphism.

Now suppose $(z_1, ..., z_{2n}) \in \Lambda_{2n}$ is given. Define μ and λ by the formulas

$$\mu_i = z_{a(i)} \text{ for } i = 1, ..., n-1$$
$$\lambda_j = z_{b(j)} \text{ for } j = 1, ..., n+1.$$

Since $a(1) < a(2) < \cdots < a(n-1)$, it follows that $\mu \in \Lambda_{n-1}$. Similarly, $\lambda \in \Lambda^+_{n+1}$. Since $b(i) < a(i) < b(i+2)$, we get that $\mu \ll \lambda$. Finally, suppose σ_i is $\geq$. Then $a(i) < b(i+1)$, and so $\mu_i \geq \lambda_{i+1}$. Similarly, if σ_i is $\leq$ then $\mu_i \leq \lambda_{i+1}$. Therefore $(\mu,\lambda) \in \Omega(\sigma)$. Since $f_\sigma(\mu,\lambda) = (z_1, ..., z_{2n})$ we conclude that f_σ is surjective. □

Lemma 5.3. *Let $\sigma \in \Sigma$ and let $(\mu,\lambda), (\mu',\lambda') \in \Omega(\sigma)$. Suppose that $\gamma \in \Lambda_n$ satisfies*

$$\mu + \mu' < \gamma < \lambda + \lambda'.$$

Then there exist $\nu,\nu' \in \Lambda_n$ such that $\gamma = \nu + \nu'$, $\mu < \nu < \lambda$, and $\mu' < \nu' < \lambda'$.

Proof. Set $f_\sigma(\mu,\lambda) = (x_1, y_1, ..., x_n, y_n)$ and $f_\sigma(\mu',\lambda') = (x'_1, y'_1, ..., x'_n, y'_n)$. By Lemma 5.2,

$$f_\sigma(\mu+\mu', \lambda+\lambda') = (x_1 + x'_1, y_1 + y'_1, ..., x_n + x'_n, y_n + y'_n).$$

Therefore by Lemma 5.1, $y_i + y'_i \leq \gamma_i \leq x_i + x'_i$. Now choose ν_i, ν'_i such that $\gamma_i = \nu_i + \nu'_i$, $y_i \leq \nu_i \leq x_i$, and $y'_i \leq \nu'_i \leq x'_i$. Set $\nu = (\nu_1, ..., \nu_n)$ and $\nu' = (\nu'_1, ..., \nu'_n)$. Clearly $\nu, \nu' \in \Lambda_n$ and $\gamma = \nu + \nu'$. Moreover, by Lemma 5.1, $\mu < \nu < \lambda$, and $\mu' < \nu' < \lambda'$. □

5.2. **Proof of Proposition 3.2.** For $(\mu,\lambda) \in \Omega$ recall that $r_i(\mu,\lambda) = x_i - y_i$, where $f(\mu,\lambda) = (x_1, y_1, ..., x_n, y_n)$. Let $(\mu,\lambda) \in \Lambda_{n-1,n+1}$. Then as a $GL_1 \times GL_1$-module,

$$V^{\lambda/\mu} \cong \bigoplus_{\substack{\gamma \in \Lambda_n \\ \mu < \gamma < \lambda}} V^{\gamma/\mu} \otimes V^{\lambda/\gamma}.$$

Now, $d(q) = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$ acts on $V^{\gamma/\mu} \otimes V^{\lambda/\gamma}$ by the scalar $q^{2|\gamma|-|\lambda|-|\mu|}$. Moreover, by (2) and (3), $\dim V^{\gamma/\mu} \otimes V^{\lambda/\gamma} = 1$ for γ such that $\mu < \gamma < \lambda$. Therefore the character

equals
$$\mathrm{ch}(V^{\lambda/\mu}) = \sum_{\substack{\gamma\in\Lambda_n\\ \mu<\gamma<\lambda}} q^{2|\gamma|-|\lambda|-|\mu|}.$$
Set $f(\mu,\lambda) = (x_1, y_1, \ldots, x_n, y_n)$ and $r_i = r_i(\mu,\lambda)$. By Lemma 5.1,
$$\begin{aligned}
\sum_{\substack{\gamma\in\Lambda_n\\ \mu<\gamma<\lambda}} q^{2|\gamma|-|\lambda|-|\mu|} &= \sum_{0\le j_i\le r_i} q^{2(y_1+\cdots+y_n+j_1+\cdots+j_n)-(x_1+\cdots+x_n+y_1+\cdots+y_n)}\\
&= \sum_{0\le j_i\le r_i} q^{(-r_1+2j_1)+\cdots+(-r_n+2j_n)}\\
&= \prod_{i=1}^{n}\sum_{j=0}^{r_i} q^{-r_i+2j}\\
&= \prod_{i=1}^{n}\mathrm{ch}(F^{r_i})
\end{aligned}$$
Therefore $\mathrm{ch}(V^{\lambda/\mu}) = \mathrm{ch}\left(\bigotimes_{i=1}^{n} F^{r_i(\mu,\lambda)}\right)$. This proves Proposition 3.2.

5.3. **A technical lemma.** Suppose $(\gamma,\lambda) \in \Lambda_{n,n+1}$. We may view V^λ as a GL_n-module by restriction, and, as such, define $V^\lambda[\gamma]$ to be the γ-isotypic component of V^λ. Let $p^\lambda_\gamma : V^\lambda \to V^\lambda[\gamma]$ be the corresponding projection.

Before stating and proving the lemma we first make a simple observation. Suppose Θ is a semigroup and $\mathcal{V},\mathcal{W}$ are Θ-graded vector spaces:
$$\mathcal{V} = \bigoplus_{i\in\Theta} V_i,\ \mathcal{W} = \bigoplus_{i\in\Theta} W_i.$$
Suppose there are linear maps $\pi_{i,j} : V_i \otimes V_j \to V_{i+j}$ and $\tau_{i,j} : W_i \otimes W_j \to W_{i+j}$ for every $i, j \in \Theta$. We refer to these maps as "products" on the vector spaces. Finally, suppose also there is an Θ-graded isomorphism $T : \mathcal{V} \to \mathcal{W}$ that preserves the products on $\mathcal{V}$ and $\mathcal{W}$ in the following sense: for all $i, j \in \Theta$ the following diagram commutes:
$$\begin{array}{ccc}
V_i\otimes V_j & \xrightarrow{\pi_{i,j}} & V_{i+j}\\
\downarrow T & & \downarrow T\\
W_i\otimes W_j & \xrightarrow{\tau_{i,j}} & W_{i+j}
\end{array}$$
Then if $x \in V_i$ and $y \in V_j$ and $\tau_{i,j}(T(x) \otimes T(y)) \neq 0$, then $\pi_{i,j}(x \otimes y) \neq 0$.

For the purposes of the following lemma we will use the branching semigroup
$$\Theta = \{(\gamma,\lambda) \in \Lambda_{n,n+1} : \gamma < \lambda\}.$$
We introduce three Θ-graded vector spaces, each of which is equipped with product maps.

The first space is $\mathcal{V}_1 = \bigoplus_{(\gamma,\lambda)\in\Theta} V^\lambda[\gamma]$. The product is defined as follows: for $x \in V^\lambda[\gamma]$ and $x' \in V^{\lambda'}[\gamma']$, define
$$xx' = p^{\lambda+\lambda'}_{\gamma+\gamma'}(\pi_{\lambda,\lambda'}(x\otimes x')).$$

The second space is $\mathcal{V}_2 = \bigoplus_{(\gamma,\lambda)\in\Theta} V^\gamma \otimes \mathrm{Hom}_{GL_n}(V^\gamma, V^\lambda)$. The product is defined as follows: for $v \otimes f \in V^\gamma \otimes \mathrm{Hom}_{GL_n}(V^\gamma, V^\lambda)$ and $v' \otimes f' \in V^{\gamma'} \otimes \mathrm{Hom}_{GL_n}(V^{\gamma'}, V^{\lambda'})$, define

$$(v \otimes f)(v' \otimes f') = \pi_{\gamma,\gamma'}(v \otimes v') \otimes (\pi_{\lambda,\lambda'} \circ (f \otimes f') \circ j_{\gamma,\gamma'}).$$

The third space is $\mathcal{V}_3 = \bigoplus_{(\gamma,\lambda)\in\Theta} V^\gamma \otimes (V^\lambda)^{U_n}(\gamma)$. The product is defined as follows: for $v \otimes w \in V^\gamma \otimes (V^\lambda)^{U_n}(\gamma)$ and $v' \otimes w' \in V^{\gamma'} \otimes (V^{\lambda'})^{U_n}(\gamma')$, define

$$(v \otimes w)(v' \otimes w') = \pi_{\gamma,\gamma'}(v \otimes v') \otimes \pi_{\lambda,\lambda'}(w \otimes w').$$

Finally, we can state and prove the lemma. We note that it can also be obtained as a consequence of Theorem 1 in [Vin95]. We include an elementary proof for the sake of completeness.

Lemma 5.4. *Let $(\nu,\lambda), (\nu',\lambda') \in \Lambda_{n,n+1}$. Suppose that $0 \neq x \in V^\lambda[\nu]$ and $0 \neq x' \in V^{\lambda'}[\nu']$. Then $p^{\lambda+\lambda'}_{\nu+\nu'}(\pi_{\lambda,\lambda'}(x \otimes x')) \neq 0$.*

Proof. Define $T : \mathcal{V}_2 \to \mathcal{V}_1$ by $T(v \otimes f) = f(v)$. This is clearly a linear isomorphism. Let $v \otimes f, v' \otimes f'$ be chosen as in the definition of $\mathcal{V}_2$. Then

$$T((v \otimes f)(v' \otimes f')) = (\pi_{\lambda,\lambda'} \circ (f \otimes f') \circ j_{\gamma,\gamma'})(\pi_{\gamma,\gamma'}(v \otimes v'))$$

while

$$T(v \otimes f)T(v' \otimes f') = p^{\lambda+\lambda'}_{\gamma+\gamma'}(\pi_{\lambda,\lambda'}(f(v) \otimes f(v'))).$$

Let $z = v \otimes v'$. Since $z \in V^\gamma \otimes V^{\gamma'}$ we can write $z = z_0 + z_1$, where $z_0 \in (V^\gamma \otimes V^{\gamma'})[\gamma+\gamma']$ and $z_1 \in \sum_{\tau\neq\gamma+\gamma'} (V^\gamma \otimes V^{\gamma'})[\tau]$. By the definition of $\pi_{\gamma,\gamma'}$ and $j_{\gamma,\gamma'}$, the composition $j_{\gamma,\gamma'} \circ \pi_{\gamma,\gamma'}$ is the projection of $V^\gamma \otimes V^{\gamma'}$ onto its isotypic component $(V^\gamma \otimes V^{\gamma'})[\gamma+\gamma']$. Therefore,

$$(\pi_{\lambda,\lambda'} \circ (f \otimes f') \circ j_{\gamma,\gamma'})(\pi_{\gamma,\gamma'}(z)) = (\pi_{\lambda,\lambda'} \circ (f \otimes f'))(z_0).$$

On the other hand,

$$\begin{aligned} p^{\lambda+\lambda'}_{\gamma+\gamma'}(\pi_{\lambda,\lambda'}(f(v) \otimes f(v'))) &= p^{\lambda+\lambda'}_{\gamma+\gamma'}(\pi_{\lambda,\lambda'}((f \otimes f')(z_0))) + p^{\lambda+\lambda'}_{\gamma+\gamma'}(\pi_{\lambda,\lambda'}((f \otimes f')(z_1))) \\ &= \pi_{\lambda,\lambda'}((f \otimes f')(z_0)). \end{aligned}$$

Therefore T preserves the products on $\mathcal{V}_1$ and $\mathcal{V}_2$.

Next define $S : \mathcal{V}_2 \to \mathcal{V}_3$ by $S(v \otimes f) = v \otimes f(v_\gamma)$ (recall that v_γ is the canonical highest weight vector in V^γ). This is clearly a linear isomorphism. We show S preserves the product maps. Let $v \otimes f, v' \otimes f'$ be chosen as in the definition of $\mathcal{V}_2$. Then

$$\begin{aligned} S((v \otimes f)(v' \otimes f')) &= S(\pi_{\gamma,\gamma'}(v \otimes v') \otimes (\pi_{\lambda,\lambda'} \circ (f \otimes f') \circ j_{\gamma,\gamma'})) \\ &= \pi_{\gamma,\gamma'}(v \otimes v') \otimes \pi_{\lambda,\lambda'}(f(v_\gamma) \otimes f(v_{\gamma'})) \\ &= S(v \otimes f)S(v' \otimes f'). \end{aligned}$$

Therefore S preserves the products on $\mathcal{V}_2$ and $\mathcal{V}_3$.

Now $S \circ T^{-1}$ is a graded isomorphism of $\mathcal{V}_1$ and $\mathcal{V}_3$ that respects products. Consider $0 \neq x \in V^\lambda[\nu]$. Under the isomorphism $S \circ T^{-1}$, x is mapped to a simple tensor $v \otimes w$. Indeed, by (2) $\dim(V^\lambda)^{U_n}(\nu) = 1$, and x is mapped to the summand $V^\nu \otimes (V^\lambda)^{U_n}(\nu)$.

Similarly, $0 \neq x' \in V^{\lambda'}[\nu']$ is mapped to a simple tensor $v' \otimes w'$. By the definition of multiplication in $\mathcal{V}_3$, we have that $(v \otimes w)(v' \otimes w') = \pi_{\nu,\nu'}(v \otimes v') \otimes \pi_{\lambda,\lambda'}(w \otimes w')$. Now, $\pi_{\nu,\nu'}(v \otimes v')$ (resp. $\pi_{\lambda,\lambda'}(w \otimes w')$) is simply the product of v and v' (resp. w and w') in $\mathcal{R}_{\mathrm{GL}_n}$ (resp. $\mathcal{R}_{\mathrm{GL}_{n+1}}$). Since $\mathcal{R}_{\mathrm{GL}_n}$ (resp. $\mathcal{R}_{\mathrm{GL}_{n+1}}$) has no zero divisors, it follows that $(v \otimes w)(v' \otimes w') \neq 0$. By the observation above we conclude that $xx' \neq 0$ in $\mathcal{V}_1$, i.e. $p^{\lambda+\lambda'}_{\gamma+\gamma'}(\pi_{\lambda,\lambda'}(x \otimes x')) \neq 0$. □

5.4. **Proof of Proposition 3.6.** Let $(\mathfrak{t}_n^*)_{\mathbb{R}}$ be the real form of $\mathfrak{t}_n^*$ spanned by $\{\varepsilon_i : i = 1, \ldots, n\}$, where ε_i is the functional mapping a diagonal matrix to its i^{th} entry. Let $(\cdot,\cdot)$ be the inner product on $(\mathfrak{t}_n^*)_{\mathbb{R}}$ defined by $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$, and let $\|\gamma\|^2 = (\gamma,\gamma)$ define the associated norm. Denote by $\preceq$ the positive root ordering on $\mathfrak{t}_n^*$, defined relative to the set of positive roots: $\{\varepsilon_i - \varepsilon_j : i < j\}$. In other words, $\alpha \preceq \beta$ means $\beta - \alpha$ is a nonnegative integer combination of positive roots. Recall that for $\nu, \nu', \gamma \in \Lambda_n$, $\mathrm{Hom}_{\mathrm{GL}_n}(V^{\gamma}, V^{\nu} \otimes V^{\nu'}) \neq \{0\}$ implies $\gamma \preceq \nu + \nu'$.

By Theorem 3.1, Proposition 3.6 is an immediate corollary of the following result:

Proposition 5.5. *Let* $\sigma \in \Sigma$ *and let* $(\mu,\lambda), (\mu',\lambda') \in \Omega(\sigma)$. *Then the map*

$$V^{\lambda/\mu} \otimes V^{\lambda'/\mu'} \xrightarrow{\pi_{\lambda,\lambda'}} V^{\lambda+\lambda'/\mu+\mu'}$$

is surjective.

Proof. To ease notation let $X = V^{\lambda/\mu}$, $X' = V^{\lambda'/\mu'}$, $Y = V^{\lambda+\lambda'/\mu+\mu'}$, and $\pi = \pi_{\lambda,\lambda'}$. For $\gamma \in \Lambda_n$ set $Y[\gamma] = p^{\lambda+\lambda'}_{\gamma}(Y)$, $X[\gamma] = p^{\lambda}_{\gamma}(X)$, and $X'[\gamma] = p^{\lambda'}_{\gamma}(X')$.

Note that $Y = \bigoplus_{\gamma\in\Lambda_n} Y[\gamma]$, and $\dim Y[\gamma]$ is zero or one. Moreover, $Y[\gamma] \neq \{0\}$ if, and only if, $\mu + \mu' < \gamma < \lambda + \lambda'$. We will prove by induction on $\|\gamma\|$ that $Y[\gamma]$ is in the image of π.

Let $\gamma \in \Lambda_n$ be of minimal norm such that $Y[\gamma] \neq \{0\}$. Our base case is to show that $Y[\gamma]$ is in the image of π. Since $(\mu,\lambda), (\mu',\lambda') \in \Omega(\sigma)$ we can apply Lemma 5.3 to obtain $\nu,\nu' \in \Lambda_n$ such that $\gamma = \nu + \nu'$, $\mu < \nu < \lambda$, and $\mu' < \nu' < \lambda'$. Choose $0 \neq x \in X[\nu]$ and $0 \neq x' \in X'[\nu']$, and let $z = x \otimes x'$.

Now, $\pi(z) = \sum_{\tau\in\Lambda_n} p^{\lambda+\lambda'}_{\tau}(\pi(z))$ is a decomposition of $\pi(z)$ in $Y = \bigoplus_{\gamma\in\Lambda_n} Y[\gamma]$. Since $p^{\lambda+\lambda'}_{\tau}(\pi(z)) = 0$ for $\tau \succ \gamma$, $\pi(z) = \sum_{\tau\preceq\gamma} p^{\lambda+\lambda'}_{\tau}(\pi(z))$. Now $\tau \prec \gamma$ implies $\|\tau\| < \|\gamma\|$, and by hypothesis γ is of minimal norm such that $Y[\gamma] \neq \{0\}$. Therefore $p^{\lambda+\lambda'}_{\tau}(\pi(z)) = 0$ for $\tau \prec \gamma$, and hence $\pi(z) = p^{\lambda+\lambda'}_{\gamma}(\pi(z)) \in Y[\gamma]$. By definition, $\pi(z)$ is the product of x and x' in $\mathcal{R}_{\mathrm{GL}_{n+1}}$. Therefore, since $\mathcal{R}_{\mathrm{GL}_{n+1}}$ has no zero divisors, $\pi(z) \neq 0$. Since $\dim Y[\gamma] = 1$, we conclude that $Y[\gamma]$ is in the image of π. This completes the base case.

Now fix $\gamma \in \Lambda_n$ such that $Y[\gamma] \neq \{0\}$, and suppose $Y[\tau]$ is in the image of π for all τ such that $\|\tau\| < \|\gamma\|$. Using Lemma 5.3 again, we choose $\nu,\nu' \in \Lambda_{n-1}$ such that $\gamma = \nu + \nu'$, $\mu < \nu < \lambda$, and $\mu' < \nu' < \lambda'$. Also choose $0 \neq y \in Y[\gamma]$, $0 \neq x \in X[\nu]$, and $0 \neq x' \in X'[\nu']$. By Lemma 5.4, $p^{\lambda+\lambda'}_{\gamma}(\pi(x \otimes x')) \neq 0$. Therefore

$$\pi(x \otimes x') \in \mathbb{C}^{\times} y + \sum_{\tau\prec\gamma} Y[\tau].$$

Since $\tau \prec \gamma$ implies $\|\tau\| < \|\gamma\|$, by the inductive hypothesis we obtain an element $\xi \in X \otimes X'$ such that $\pi(\xi) = y$. Since $\dim Y[\gamma] = 1$, this shows that $Y[\gamma]$ is in the image of π. This completes the induction. $\square$

6. Proof of Theorem 3.5

In this section it will be convenient for us to introduce the following convention. Elements of the branching semigroup $\Lambda_{\mathcal{B}}$ will be thought of as "skew shapes", and so instead of writing $(\mu, \lambda) \in \Lambda_{\mathcal{B}}$, we will write $\lambda/\mu \in \Lambda_{\mathcal{B}}$. In this way, for $p = \lambda/\mu \in \Lambda_{\mathcal{B}}$ we associate the spaces W^p, A^p, etc...

6.1. **A filtration on the branching semigroup.** Let $h : \Lambda_{\mathcal{B}} \to \Lambda_{2n}$ be given by $h(\lambda/\mu) = f(\mu, \lambda^+)$, where $f(\mu, \lambda^+)$ is the rearrangement function defined in Section 5.1. The image of h is thus all sequences in Λ_{2n} ending in zero. As before, we define the functions $r_i : \Lambda_{\mathcal{B}} \to \mathbb{Z}$ by $r_i(p) = x_i - y_i$, where $h(p) = (x_1, y_1, ..., x_n, y_n)$. Moreover, for $\sigma \in \Sigma$, let h_σ denote the restriction of h to $\Lambda_{\mathcal{B}}(\sigma)$.

The same argument as in Lemma 5.2 shows:

Lemma 6.1. *Let $\sigma \in \Sigma$. Then $h_\sigma : \Lambda_{\mathcal{B}}(\sigma) \to \Lambda_{2n}$ is a semigroup embedding, with image the sequences in Λ_{2n} ending in zero. In particular, h_σ^{-1} is defined on the set of such sequences.*

In this section we will only deal with sequences ending in zero, so h_σ^{-1} will always be well-defined. By the above lemma we endow the L-module

$$\mathcal{A}_\sigma = \bigoplus_{p \in \Lambda_{\mathcal{B}}(\sigma)} A^p$$

with a product given by Cartan product of irreducible L-modules: $A^p \otimes A^{p'} \to A^{p+p}$.

We now show that $\mathcal{A}_\sigma$ is a very naturally occurring L-algebra. Consider the L-module $V = U \times W$, where $U = \mathbb{C}^2 \times \cdots \times \mathbb{C}^2$ (n copies) and $W = \mathbb{C} \times \cdots \times \mathbb{C}$ ($n-1$ copies). Here, L acts on U diagonally, on W trivially, and on the ring of functions $\mathcal{O}(V)$ by right translation. Let $t_1, ..., t_{n-1}$ be the standard coordinate functions on $\mathbb{C}^{n-1}$. Decompose $\mathcal{O}(V)$ into graded components:

$$\mathcal{O}(V) \cong \bigoplus_{\substack{r_j \geq 0 \\ j=1,...,n}} \bigoplus_{\substack{s_k \geq 0 \\ k=1,...,n-1}} F^{r_1} \otimes \cdots \otimes F^{r_n} \otimes t_1^{s_1} \cdots t_{n-1}^{s_{n-1}}. \tag{15}$$

This is also a decomposition of $\mathcal{O}(V)$ into irreducible L-modules.

For $\sigma \in \Sigma$, we can consider $\mathcal{O}(V)$ as an $\Lambda_{\mathcal{B}}(\sigma)$-graded algebra as follows. Set $s_i : \Lambda_{\mathcal{B}}(\sigma) \to \mathbb{Z}$ by $s_i(p) = y_i - x_{i+1}$, where, as usual, $h(p) = (x_1, y_1, ..., x_n, y_n)$. Then define the p-component of $\mathcal{O}(V)$ by:

$$\mathcal{O}(V)^p = F^{r_1(p)} \otimes \cdots \otimes F^{r_n(p)} \otimes t_1^{s_1(p)} \cdots t_{n-1}^{s_{n-1}(p)}.$$

Clearly, we have

$$\mathcal{O}(V) = \bigoplus_{p \in \Lambda_{\mathcal{B}}(\sigma)} \mathcal{O}(V)^p.$$

One easily proves the following lemma.

Lemma 6.2. *Let* $\sigma \in \Sigma$ *and regard* $\mathcal{O}(V)$ *as an* $\Lambda_{\mathcal{B}}(\sigma)$*-graded* L*-algebra. Then* $\mathcal{A}_\sigma$ *and* $\mathcal{O}(V)$ *are isomorphic as* $\Lambda_{\mathcal{B}}(\sigma)$*-graded* L*-algebras, and the isomorphism is unique up to scalars.*

The main step in proving Theorem 3.5 is showing that $\mathcal{A}_\sigma$ and $\mathcal{B}_\sigma$ are isomorphic as SL_2-algebras, and the isomorphism is unique up to scalars. We will prove this by induction on a certain filtration of $\Lambda_{\mathcal{B}}(\sigma)$, which we now describe.

For $p \in \Lambda_{\mathcal{B}}$ let $p_{max} = \lambda_1$ where $p = \lambda/\mu$ and $\lambda = (\lambda_1, ..., \lambda_n)$. For every $\sigma \in \Sigma$ we define the set

$$\Lambda_{\mathcal{B}}(\sigma, m) = \{p \in \Lambda_{\mathcal{B}}(\sigma) : p_{max} \leq m\}.$$

Clearly $\Lambda_{\mathcal{B}}(\sigma, m)$ is finite, $\Lambda_{\mathcal{B}}(\sigma, m-1) \subset \Lambda_{\mathcal{B}}(\sigma, m)$, and $\bigcup_{m \geq 0} \Lambda_{\mathcal{B}}(\sigma, m) = \Lambda_{\mathcal{B}}(\sigma)$.

Lemma 6.3. *Let* $m > 1$*,* $\sigma \in \Sigma$*, and suppose* $p \in \Lambda_{\mathcal{B}}(\sigma, m)$ *satisfies* $p_{max} = m$*.*

(1) *There exist* $p', p'' \in \Lambda_{\mathcal{B}}(\sigma, m-1)$ *such that* $p = p' + p''$*.*
(2) *Suppose moreover that* $\tau \in \Sigma$ *and* $p \in \Lambda_{\mathcal{B}}(\tau, m)$*. Then there exist* $p', p'' \in \Lambda_{\mathcal{B}}(\sigma, m-1) \cap \Lambda_{\mathcal{B}}(\tau, m-1)$ *such that* $p = p' + p''$*.*

Proof. Let $h_\sigma(p) = (z_1, ..., z_{2n})$. Define

$$z_i' = \begin{cases} 1 \text{ if } z_i \geq 1 \\ 0 \text{ if } z_i = 0 \end{cases}$$

and $z_i'' = z_i - z_i'$. It's trivial to check that $\xi' = (z_1', ..., z_{2n}'), \xi'' = (z_1'', ..., z_{2n}'') \in \Lambda_{2n}$. Let $p' = f_\sigma^{-1}(\xi')$ and $p'' = f_\sigma^{-1}(\xi'')$. This is well-defined by Lemma 6.1. Lemma 6.1 also shows that $p = p' + p''$. Since $m > 1$, $p', p'' \in \Lambda_{\mathcal{B}}(\sigma, m-1)$. This proves (1).

Let $p', p'' \in \Lambda_{\mathcal{B}}(\sigma, m-1)$ be constructed as in the previous paragraph. We must show that $p', p'' \in \Lambda_{\mathcal{B}}(\sigma) \cap \Lambda_{\mathcal{B}}(\tau)$. If $\sigma = (\sigma_1 \cdots \sigma_{n-1})$ and $\tau = (\tau_1 \cdots \tau_{n-1})$, then every i such that $\sigma_i \neq \tau_i$ forces the equality $\mu_i = \lambda_{i+1}$ among the entries of p. Therefore, if $h_\sigma(p) = (z_1, ..., z_{2n})$ and $\sigma_i \neq \tau_i$, then $z_{2i+1} = z_{2i+2}$. Now note that in the definition of ξ', ξ'', if $z_{2i+1} = z_{2i+2}$ then $z_{2i+1}' = z_{2i+2}'$ and $z_{2i+1}'' = z_{2i+2}''$. Hence the entries of ξ', ξ'' satisfy the same equalities that $h_\sigma(p)$ satisfies, which implies that $p', p'' \in \Lambda_{\mathcal{B}}(\sigma) \cap \Lambda_{\mathcal{B}}(\tau)$. This proves (2). □

Lemma 6.4. *Let* $m > 1$*,* $\sigma \in \Sigma$*, and suppose* $p \in \Lambda_{\mathcal{B}}(\sigma, m)$ *satisfies* $p_{max} = m$*. Then there exist* $q_1, ..., q_n \in \Lambda_{\mathcal{B}}(\sigma, m-1)$ *such that*

(1) $p = q_1 + \cdots + q_n$
(2) A^{q_i} *is an irreducible* SL_2*-module.*
(3) *Either* $A^{q_1} \otimes \cdots \otimes A^{q_n} \cong A^p$ *as* SL_2*-modules, or* A^p *is irreducible as an* SL_2*-module, and the multiplication map* $A^{q_1} \otimes \cdots \otimes A^{q_n} \to A^p$ *is a projection onto the Cartan component* A^p *of* $A^{q_1} \otimes \cdots \otimes A^{q_n}$*.*

Proof. Let $h_\sigma(p) = (x_1, y_1, ..., x_n, y_n)$. Define

$$\xi_i = (\underbrace{x_i - x_{i+1}, ..., x_i - x_{i+1}}_{2i-1}, y_i - x_{i+1}, 0, ..., 0)$$

for $i = 1, ..., n-1$, and set $\xi_n = (x_n, ..., x_n, 0)$. The argument breaks into cases.

Case 1: Suppose $h_\sigma^{-1}(\xi_i) \notin \Lambda_{\mathcal{B}}(\sigma, m-1)$ for some $i \le n$. Then $x_i - x_{i+1} = m$ and therefore

$$h_\sigma(p) = (m, ..., m, b, 0, ..., 0)$$

for some $b \le m$ in the $(2i)^{\text{th}}$ entry. Therefore A^p is irreducible as an SL_2-module.

Now choose ξ', ξ'' as in the proof of Lemma 6.3 and consider the associated p', p''. By the lemma $p', p'' \in \Lambda_{\mathcal{B}}(\sigma, m-1)$. Moreover, by our construction of ξ', ξ'' from ξ, $\mathcal{A}^{p'}, \mathcal{A}^{p''}$ are irreducible SL_2-modules. Therefore the map $A^{p'} \otimes A^{p''} \to A^p$ is a projection onto the Cartan component of $A^{p'} \otimes A^{p''}$, and the lemma is satisfied with $q_1 = p', q_2 = p''$, and $q_i = 0$ for $i > 2$.

Case 2: Suppose that $h_\sigma^{-1}(\xi_i) \in \Lambda_{\mathcal{B}}(\sigma, m-1)$ for $i = 1, ..., n$. Then set $q_i = h_\sigma^{-1}(\xi_i)$. Since $\xi = \xi_1 + \cdots + \xi_n$, by Lemma 5.2, $p = q_1 + \cdots + q_n$. By the definition of ξ_i we also have that

$$A^{q_i} = F^0 \otimes \cdots \otimes \underbrace{F^{x_i - y_i}}_{i^{th}} \otimes \cdots \otimes F^0.$$

Therefore A^{q_i} is an irreducible SL_2-module, and $A^{q_1} \otimes \cdots \otimes A^{q_n} \cong A^p$. □

Remark 6.5. *In the proof of Lemma 6.4 all we used was the* SL_2*-module structure of* A^p*. Therefore, by Corollary 3.3, the statement holds with* A^p *replaced by* W^p *and* A^{q_i} *replaced by* W^{q_i}*.*

Lemma 6.6. *Let* $m > 1$*,* $\sigma \in \Sigma$*, and suppose* $p \in \Lambda_{\mathcal{B}}(\sigma, m)$ *satisfies* $p_{\max} = m$*. Let* $q_1, ..., q_n \in \Lambda_{\mathcal{B}}(\sigma, m-1)$ *be given as in Lemma 6.4. Suppose also we are given* SL_2*-isomorphisms* $\phi_i : W^{q_i} \to A^{q_i}$ *for* $i = 1, ..., n$*. Let*

$$\begin{aligned} K &= \ker(W^{q_1} \otimes \cdots \otimes W^{q_n} \xrightarrow{\tau} W^p) \\ J &= \ker(A^{q_1} \otimes \cdots \otimes A^{q_n} \xrightarrow{\kappa} A^p) \end{aligned}$$

be the kernels of the multiplication maps coming from the rings $\mathcal{B}_\sigma$ *and* $\mathcal{A}_\sigma$*, which we denote here by* τ *and* κ*. Set* $\phi = \phi_1 \otimes \cdots \otimes \phi_n$*. Then* $\phi(K) = J$*. Consequently, there is an* SL_2*-isomorphism* $\psi : W^p \to A^p$ *making the following diagram commute:*

$$\begin{array}{ccc} W^{q_1} \otimes \cdots \otimes W^{q_n} & \longrightarrow & W^p \\ \downarrow \phi & & \downarrow \psi \\ A^{q_1} \otimes \cdots \otimes A^{q_n} & \longrightarrow & A^p \end{array}$$

Proof. Clearly κ is surjective. By Proposition 3.6, τ is surjective. Therefore we have the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & W^{q_1} \otimes \cdots \otimes W^{q_n} & \longrightarrow & W^p & \longrightarrow & 0 \\ & & & & \downarrow \phi & & & & \\ 0 & \longrightarrow & J & \longrightarrow & A^{q_1} \otimes \cdots \otimes A^{q_n} & \longrightarrow & A^p & \longrightarrow & 0 \end{array}$$

According to Lemma 6.4 there are two possibilities. Either $A^{q_1} \otimes \cdots \otimes A^{q_n} \cong A^p$ as SL_2-modules, or A^p is irreducible as a SL_2-module, and the multiplication map $A^{q_1} \otimes \cdots \otimes A^{q_n} \to A^p$ is a projection onto the Cartan component A^p of $A^{q_1} \otimes \cdots \otimes A^{q_n}$. If $A^{q_1} \otimes \cdots \otimes A^{q_n} \cong A^p$ then $J = \{0\}$, and by the above remark $W^{q_1} \otimes \cdots \otimes W^{q_n} \cong W^p$. Therefore $K = \{0\}$, and so clearly $\phi(K) = J$.

In the other case, $A^p \cong W^p$ are irreducible SL_2-modules. Choose k so that $A^p \cong W^p \cong F^k$. Since the maps κ, τ are both projections onto the Cartan component F^k, their kernels are given as sums of SL_2-isotypic components

$$\begin{aligned} K &= \sum_{n<k} (W^{q_1} \otimes \cdots \otimes W^{q_n})[j] \\ J &= \sum_{j<k} (A^{q_1} \otimes \cdots \otimes A^{q_n})[j]. \end{aligned}$$

Since ϕ intertwines the SL_2-action, $\phi(K) \subset J$. Moreover, κ and τ are both Cartan multiplications of the same SL_2-modules, and so $\dim K = \dim J$. Therefore $\phi(K) = J$. $\square$

Lemma 6.7. *Let* $m > 1$, $\sigma \in \Sigma$, *and suppose we are given* $p', p'', q', q'' \in \Lambda_{\mathcal{B}}(\sigma, m)$ *such that* $p' + p'' = q' + q''$. *Then there exist* $t', t'', r', r'' \in \Lambda_{\mathcal{B}}(\sigma, m)$ *such that*

$$\begin{gathered} t' + r' = p', t'' + r'' = p'' \\ t' + t'' = q', r' + r'' = q'' \end{gathered}$$

Proof. For some nonnegative integers n_i', n_i'', m_i', m_i'' we have

$$\begin{aligned} h_\sigma(p') &= \textstyle\sum_{i=1}^{2n} n_i' \varpi_i,\ h_\sigma(p'') = \sum_{i=1}^{2n} n_i'' \varpi_i \\ h_\sigma(q') &= \textstyle\sum_{i=1}^{2n} m_i' \varpi_i,\ h_\sigma(q'') = \sum_{i=1}^{2n} m_i'' \varpi_i \end{aligned}$$

Define:

$$\begin{aligned} \tau' &= \sum_{i=1}^{2n} (m_i'' - \min(n_i'', m_i''))\varpi_i \\ \rho' &= \sum_{i=1}^{2n} (n_i' - m_i'' + \min(n_i'', m_i''))\varpi_i \\ \tau'' &= \sum_{i=1}^{2n} \min(n_i'', m_i'')\varpi_i \\ \rho'' &= \sum_{i=1}^{2n} (n_i'' - \min(n_i'', m_i''))\varpi_i. \end{aligned}$$

Clearly $\tau', \tau'', \rho'' \in \Lambda_{2n}$. Since $p' + p'' = q' + q''$, $\rho' \in \Lambda_{2n}$ as well. Now, note that

$$\begin{gathered} h_\sigma(p') = \tau' + \rho', h_\sigma(p'') = \tau'' + \rho'' \\ h_\sigma(q') = \rho' + \rho'', h_\sigma(q'') = \tau' + \tau''. \end{gathered}$$

Set

$$\begin{aligned} t' &= h_\sigma^{-1}(\tau'),\ \ t'' = h_\sigma^{-1}(\tau'') \\ r' &= h_\sigma^{-1}(\rho'),\ \ r'' = h_\sigma^{-1}(\rho'') \end{aligned}$$

Since h_σ is a semigroup isomorphism and $p', p'', q', q'' \in \Lambda_{\mathcal{B}}(\sigma)$, it follows that $t', t'', r', r'' \in \Lambda_{\mathcal{B}}(\sigma)$ and they satisfy the desired equations. $\square$

6.2. **Proof of Proposition 3.7.**

Definition 6.8. *Let $\mathcal{F} = \{\phi_p : W^p \to A^p\}_{p\in\Lambda_{\mathcal{B}}}$ be a family of SL_2-isomorphisms indexed by $\Lambda_{\mathcal{B}}$. Then $\mathcal{F}$ is a* ***compatible family*** *if for any $\sigma \in \Sigma$ and $p', p'' \in \Lambda_{\mathcal{B}}(\sigma)$ the following diagram commutes:*

$$\begin{array}{ccc} W^{p'} \otimes W^{p''} & \longrightarrow & W^{p'+p''} \\ \downarrow & & \downarrow \\ A^{p'} \otimes A^{p''} & \longrightarrow & A^{p'+p''} \end{array} \tag{16}$$

Here the vertical maps are given by $\phi_{p'} \otimes \phi_{p''}$ and $\phi_{p'+p''}$, and the horizontal maps are the product maps in the rings $\mathcal{B}_\sigma$ and $\mathcal{A}_\sigma$.

Proposition 6.9. *There exists a compatible family $\mathcal{F} = \{\phi_p : W^p \to A^p\}_{p\in\Lambda_{\mathcal{B}}}$ of SL_2-isomorphisms. Moreover, each map $\phi_p \in \mathcal{F}$ is unique up to scalar.*

Proof. For a nonnegative integer m set $\Lambda_{\mathcal{B}}(m) = \{p \in \Lambda_{\mathcal{B}} : p_{max} \le m\}$. We first prove by induction on m that there is a family of SL_2 isomorphisms

$$\mathcal{F}_m = \{\phi_p : W^p \to A^p\}_{p\in\Lambda_{\mathcal{B}}(m)}$$

such that for any $p \in \Lambda_{\mathcal{B}}(m)$, $\sigma \in \Sigma$, and $p', p'' \in \Lambda_{\mathcal{B}}(\sigma)$ such that $p = p' + p''$, diagram (16) commutes.

For the base case we construct $\mathcal{F}_1$. If $p_{max} = 0$ then $p = p_0 = 0/0$. We define $\phi_{p_0} : W^{p_0} \to A^{p_0}$ by $1 \in W^{p_0} \mapsto 1 \otimes \cdots \otimes 1 \in A^{p_0}$. Of course, if $p' + p'' = p_0$ then $p' = p'' = p_0$ and (16) trivially commutes. Suppose now that $p_{max} = 1$. Then $p = \lambda/\mu$ with λ a fundamental weight, and μ either zero or a fundamental weight. In any case, A^p is an irreducible SL_2-module, and by Corollary 3.3, $W^p \cong A^p$ as SL_2-modules. We choose an SL_2-isomorphism $\phi_p : W^p \to A^p$. By Schur's Lemma, ϕ_p is unique up to scalar. Now suppose $\sigma \in \Sigma$, $p', p'' \in \Lambda_{\mathcal{B}}(\sigma)$, and $p' + p'' = p$. Then either $p'_{max} = 0$ or $p''_{max} = 0$. Assume, without loss of generality, that $p'_{max} = 0$. Then $p' = p_0$, and by our construction of ϕ_{p_0}, diagram (16) commutes. Set $\mathcal{F}_1 = \{\phi_p : W^p \to A^p\}_{p\in\Lambda_{\mathcal{B}}(1)}$; this completes the base case.

Let $m > 1$ and suppose that $\mathcal{F}_{m-1}$ exists and satisfies the desired properties. We must construct $\mathcal{F}_m$. For $p \in \Lambda_{\mathcal{B}}(m)$ such that $p_{max} < m$, there exists $\phi_p \in \mathcal{F}_{m-1}$ by hypothesis. We include these ϕ_p in $\mathcal{F}_m$. For such p we have the following: if $\sigma \in \Sigma$, $p', p'' \in \Lambda_{\mathcal{B}}(\sigma)$, and $p = p' + p''$, then diagram (16) commutes. Indeed, $p = p' + p''$ implies that $p', p'' \in \Lambda_{\mathcal{B}}(m-1)$. Therefore $\phi_{p'}$ and $\phi_{p''}$ are also obtained from $\mathcal{F}_{m-1}$, and diagram (16) commutes by hypothesis.

Suppose $p \in \Lambda_{\mathcal{B}}(m)$ satisfies $p_{max} = m$. Choose an order type $\sigma \in \Sigma$ such that $p \in \Lambda_B(\sigma)$. Note that σ may not be unique. Choose $q_1, ..., q_n \in \Lambda_{\mathcal{B}}(\sigma, m-1)$ by Lemma 6.4. Now apply Lemma 6.6 to obtain an SL_2-isomorphism $\psi : W^p \to A^p$ such that the following diagram commutes:

$$\begin{array}{ccc} W^{q_1} \otimes \cdots \otimes W^{q_n} & \longrightarrow & W^p \\ \downarrow \phi & & \downarrow \psi \\ A^{q_1} \otimes \cdots \otimes A^{q_n} & \longrightarrow & A^p \end{array} \tag{17}$$

where $\phi = \phi_{q_1} \otimes \cdots \otimes \phi_{q_n}$. We now show that (i) if $p', p'' \in \Lambda_{\mathcal{B}}(\sigma)$ satisfy $p = p' + p''$ then

$$\begin{array}{ccc} W^{p'} \otimes W^{p''} & \longrightarrow & W^{p} \\ \downarrow & & \downarrow \psi \\ A^{p'} \otimes A^{p''} & \longrightarrow & A^{p} \end{array} \tag{18}$$

commutes, (ii) ψ is independent of the choice of $q_1, \ldots, q_n$, and (iii) ψ is independent of the choice of σ.

First note that (i) implies (ii). Indeed, suppose $q_1', \ldots, q_n' \in \Lambda_{\mathcal{B}}(\sigma, m-1)$ is another collection of shapes satisfying the conditions of Lemma 6.4, and $\psi' : W^p \to A^p$ is the associated SL_2-isomorphism obtained by Lemma 6.6. By (i) both ψ and ψ' would make (18) commute. But since all the maps in the diagram are surjective, there is a unique map making (18) commute. Therefore $\psi = \psi'$.

Now we prove (i). If $p'_{\max} = m$ (resp. $p''_{\max} = m$) then $p'' = p_0$ (resp. $p' = p_0$), and (18) commutes by our choice of ϕ_{p_0}. Therefore we may assume that $p'_{\max}, p''_{\max} < m$. By renumbering the q_j if necessary, we may assume that $(q_1)_{\max} \neq 0$. Let $q' = q_1$ and $q'' = q_2 + \cdots + q_n$. Then $q', q'' \in \Lambda_{\mathcal{B}}(\sigma, m-1)$ and $q' + q'' = p$. By inductive hypothesis the following diagram commutes:

$$\begin{array}{ccc} W^{q_1} \otimes \cdots \otimes W^{q_n} & \longrightarrow & W^{q'} \otimes W^{q''} \\ \downarrow & & \downarrow \\ A^{q_1} \otimes \cdots \otimes A^{q_n} & \longrightarrow & A^{q'} \otimes A^{q''} \end{array} \tag{19}$$

where the vertical map on the left is $\phi = \phi_{q_1} \otimes \cdots \otimes \phi_{q_n}$, and the one on the right is $\phi_{q'} \otimes \phi_{q''}$. Combining (17) and (19) and the fact that all the maps are surjective (Proposition 3.6), we conclude that

$$\begin{array}{ccc} W^{q'} \otimes W^{q''} & \longrightarrow & W^{p} \\ \downarrow & & \downarrow \psi \\ A^{q'} \otimes A^{q''} & \longrightarrow & A^{p} \end{array} \tag{20}$$

commutes.

Since $p' + p'' = q' + q''$, by Lemma 6.7 there exist $t', t'', r', r'' \in \Lambda_{\mathcal{B}}(\sigma, m)$ such that

$$\begin{gathered} t' + r' = p', t'' + r'' = p'' \\ t' + t'' = q', r' + r'' = q''. \end{gathered}$$

Consider the following diagram:

$$\begin{tikzcd}
& W^{r'}\otimes W^{t'}\otimes W^{r''}\otimes W^{t''} \arrow[rr] \arrow[dl] \arrow[dd] & & W^{q'}\otimes W^{q''} \arrow[dl] \arrow[dd] \\
W^{p'}\otimes W^{p''} \arrow[rr, crossing over] \arrow[dd] & & W^{p} & \\
& A^{r'}\otimes A^{t'}\otimes A^{r''}\otimes A^{t''} \arrow[rr] \arrow[dl] & & A^{q'}\otimes A^{q''} \arrow[dl] \\
A^{p'}\otimes A^{p''} \arrow[rr] & & A^{p} \arrow[from=uu, crossing over] &
\end{tikzcd}$$

The top square commutes by associativity of the product in $\mathcal{B}_\sigma$. The left and back squares commute by inductive hypothesis. The right square commutes since it is the diagram (20). The bottom square commutes by associativity of the product in $\mathcal{A}_\sigma$. By chasing this diagram and repeatedly using Proposition 3.6, it follows that the front square commutes. This proves (ii).

We now prove (iii), namely that ψ is independent of σ. Indeed, suppose $\tau \in \Sigma$ is another order type such that $p \in \Lambda_{\mathcal{B}}(\tau)$. By the above argument we obtain an SL_2 isomorphism $\zeta : W^p \to A^p$ such that (18) commutes for all $p', p'' \in \Lambda_{\mathcal{B}}(\tau)$. By Lemma 6.3(2) there exist $p', p'' \in \Lambda_{\mathcal{B}}(\sigma) \cap \Lambda_{\mathcal{B}}(\tau)$ such that $p = p' + p''$. Therefore both ψ and ζ make the following diagram commute:

$$\begin{array}{ccc}
W^{p'} \otimes W^{p''} & \longrightarrow & W^{p} \\
\downarrow & & \downarrow \psi, \zeta \\
A^{p'} \otimes A^{p''} & \longrightarrow & A^{p}
\end{array}$$

Hence $\psi = \zeta$.

At this point we've shown for any $p \in \Lambda_{\mathcal{B}}(m)$ there is a canonical SL_2 isomorphism $\psi : W^p \to A^p$ satisfying the property: for any $\sigma \in \Sigma$ and $p', p'' \in \Lambda_{\mathcal{B}}(\sigma)$ such that $p = p' + p''$, diagram (18) commutes. Set $\phi_p = \psi$ and define $\mathcal{F}_m = \{\phi_p : W^p \to A^p\}_{p \in \Lambda_{\mathcal{B}}(m)}$. This completes the induction.

Let $\mathcal{F} = \bigcup_{m=1}^{\infty} \mathcal{F}_m$. By construction, $\mathcal{F}$ is a compatible family of SL_2-isomorphisms. This completes the proof of the first statement of the proposition.

Now suppose $\widetilde{\mathcal{F}} = \{\widetilde{\phi}_p : W^p \to A^p\}_{p \in \Lambda_{\mathcal{B}}}$ is another compatible family of SL_2-isomorphisms. We will show by induction on $p_{\max}$ that there exist a set of nonzero scalars $\{c_p \in \mathbb{C}^\times : p \in \Lambda_{\mathcal{B}}\}$, such that $\phi_p = c_p \widetilde{\phi}_p$ for all $p \in \Lambda_{\mathcal{B}}$.

We already noted that by Schur's Lemma each isomorphism ϕ_p with $p_{\max} = 1$ is unique up to scalar. Therefore there exist $c_p \in \mathbb{C}^\times$ such that

$$\phi_p = c_p \widetilde{\phi}_p \tag{21}$$

for all p with $p_{\max} = 1$. Let $m > 1$. Suppose now that there exist scalars so that (21) holds for all $p \in \Lambda_{\mathcal{B}}$ such that $p_{\max} < m$. Let $p \in \Lambda_{\mathcal{B}}$ with $p_{\max} = m$. Choose some $\sigma \in \Sigma$ such that $p \in \Lambda_{\mathcal{B}}(\sigma)$. By Lemma 6.3, there exist $p', p'' \in \Lambda_{\mathcal{B}}(\sigma, m-1)$ such that

$p = p' + p''$. Then by the compatibility of $\mathcal{F}$ the following diagram commutes:

$$\begin{array}{ccc} W^{p'} \otimes W^{p''} & \longrightarrow & W^{p} \\ \downarrow & & \downarrow \\ A^{p'} \otimes A^{p''} & \longrightarrow & A^{p} \end{array} \tag{22}$$

where the vertical maps are $\phi_{p'} \otimes \phi_{p''}$ and ϕ_p. By hypothesis,

$$\phi_{p'} \otimes \phi_{p''} = c_{p'} c_{p''} \widetilde{\phi}_{p'} \otimes \widetilde{\phi}_{p''}.$$

Therefore (22) commutes with the vertical maps replaced by $\widetilde{\phi}_{p'} \otimes \widetilde{\phi}_{p''}$ and $\frac{1}{c_{p'} c_{p''}} \phi_p$. Hence $\frac{1}{c_{p'} c_{p''}} \phi_p = \widetilde{\phi}_p$, or, in other words, $c_p = c_{p'} c_{p''}$ and $\phi_p = c_p \widetilde{\phi}_p$. This completes the induction, and shows that $\phi_p \in \mathcal{F}$ is unique up to scalar. $\square$

We can now prove Proposition 3.7. Indeed, let $\mathcal{F} = \{\phi_p : W^p \to A^p\}_{p \in \Lambda_{\mathcal{B}}}$ be a compatible family of SL_2-isomorphisms guaranteed by the above proposition. Define a map

$$\phi_\sigma : \mathcal{B}_\sigma \to \mathcal{A}_\sigma \tag{23}$$

by

$$\phi_\sigma|_{W^p} = \phi_p$$

for all $p \in \Lambda_{\mathcal{B}}(\sigma)$, and extend linearly. Since $\mathcal{F}$ is a compatible family, ϕ_σ is an isomorphism of SL_2-algebras. Indeed, the commutativity of diagram (16) means precisely that ϕ_σ is an algebra homomorphism.

Now suppose $\widetilde{\phi}_\sigma : \mathcal{B}_\sigma \to \mathcal{A}_\sigma$ is some other isomorphism of SL_2-algebras. Set $\widetilde{\phi}_p = \widetilde{\phi}_\sigma|_{W^p}$. Then

$$\{\widetilde{\phi}_p : p \in \Lambda_{\mathcal{B}}(\sigma)\}$$

is a compatible family of SL_2 isomorphisms. Therefore, by the above proposition, there scalars c_p such that $\phi_p = c_p \widetilde{\phi}_p$. Therefore the graded components of $\widetilde{\phi}_\sigma$ are scalar multiples of the graded components of ϕ_σ, i.e. ϕ_σ is unique up to scalars. This proves part (1) of Proposition 3.7.

To prove part (2) we define a representation of L on $\mathcal{B}_\sigma$, denoted Φ_σ, by the formula

$$\Phi_\sigma(g) = \phi_\sigma^{-1} \circ \theta_\sigma(g) \circ \phi_\sigma.$$

Here $g \in L$, ϕ_σ is the algebra isomorphism coming from Proposition 6.9 as in (23), and θ_σ is the action of L on $\mathcal{A}_\sigma$ defined in (8). We must show that Φ_σ is independent of the choice of ϕ_σ.

Indeed, suppose we are given another isomorphism $\widetilde{\phi}_\sigma$ and use it to define the corresponding representation of L on $\mathcal{B}_\sigma$, which we denote $\widetilde{\Phi}_\sigma$. Now, choose scalars c_p as above. Then for any $g \in L$ and $p \in \Lambda_{\mathcal{B}}(\sigma)$,

$$\begin{aligned} \Phi_\sigma(g)|_{W^p} &= \phi_p^{-1} \circ \theta_p(g) \circ \phi_p \\ &= (c_p \widetilde{\phi}_p)^{-1} \circ \theta_p(g) \circ (c_p \widetilde{\phi}_p) \\ &= (\widetilde{\phi}_p)^{-1} \circ \theta_p(g) \circ \widetilde{\phi}_p \\ &= \widetilde{\Phi}_\sigma(g)|_{W^p}. \end{aligned}$$

Therefore $\Phi_\sigma = \widetilde{\Phi}_\sigma$. This proves part (2) of Proposition 3.7.

6.3. **Proof of Theorem 3.5. Existence**: Consider the representations $(\Phi_\sigma, \mathcal{B}_\sigma)$ of L from Proposition 3.7, and let mathcalF be the compatible family guaranteed by Proposition 6.9. These representations satisfy four desirable properties, all of which are almost tautologies.

(i) For any $p \in \Lambda_{\mathcal{B}}(\sigma)$, W^p is an irreducible L-submodule isomorphic to $\bigotimes_{i=1}^n F^{r_i(p)}$. Indeed, by definition of Φ_σ, $\phi_p : W^p \to A^p$ is an isomorphism of L-modules.

(ii) L acts as algebra automorphisms on $\mathcal{B}_\sigma$. In other words, we claim that for $p, p' \in \Lambda_{\mathcal{B}}(\sigma)$, the product map, $W^p \otimes W^{p'} \to W^{p+p'}$, is a homomorphism of L-modules. Indeed, by the compatibility of $\mathcal{F}$, the product map factors as follows:

$$\begin{array}{ccc} W^p \otimes W^{p'} & \longrightarrow & W^{p+p'} \\ \downarrow & & \uparrow \phi_{p+p'}^{-1} \\ A^p \otimes A^{p'} & \longrightarrow & \mathcal{A}_{p+p'} \end{array}$$

Since the three lower maps are L-module morphisms, it follows that the top map is too.

(iii) $\mathrm{Res}^L_{SL_2}(\Phi_\sigma)$ is the natural action of SL_2 on $\mathcal{B}_\sigma$. In other words, for $x \in SL_2$

$$x|_{\mathcal{B}_\sigma} = \Phi_\sigma(\delta(x))$$

where $x|_{\mathcal{B}_\sigma}$ denotes the natural action of x on $\mathcal{B}_\sigma$ and δ is the diagonal embedding of SL_2 into L. Indeed, ϕ_σ intertwines the natural action of SL_2 on $\mathcal{B}_\sigma$ with the diagonal SL_2-action on $\mathcal{A}_\sigma$. This means that

$$\phi_\sigma \circ (x|_{\mathcal{B}_\sigma}) = \theta_\sigma(\delta(x)) \circ \phi_\sigma.$$

Therefore

$$\begin{aligned} \Phi_\sigma(\delta(x)) &= \phi_\sigma^{-1} \circ \theta_\sigma(\delta(x)) \circ \phi_\sigma \\ &= \phi_\sigma^{-1} \circ \phi_\sigma \circ (x|_{\mathcal{B}_\sigma}) \\ &= x|_{\mathcal{B}_\sigma}. \end{aligned}$$

(iv) For any $\sigma_1, \sigma_2 \in \Sigma$ and $g \in L$

$$\Phi_{\sigma_1}(g)|_{\mathcal{B}_{\sigma_1} \cap \mathcal{B}_{\sigma_2}} = \Phi_{\sigma_2}(g)|_{\mathcal{B}_{\sigma_1} \cap \mathcal{B}_{\sigma_2}}. \tag{24}$$

Indeed, suppose $p \in \Lambda_{\mathcal{B}}(\sigma_1) \cap \Lambda_{\mathcal{B}}(\sigma_2)$. Then $\phi_{\sigma_1}|_{W^p} = \phi_p = \phi_{\sigma_2}|_{W^p}$ from which (24) immediately follows.

We now construct from $\mathcal{F}$ the representation $(\Phi_{\mathcal{F}}, \mathcal{B})$ of L satisfying the conditions of Theorem 3.5. Let $g \in L$. Define $\Phi_{\mathcal{F}}(g)$ on $\mathcal{B}$ by

$$\Phi_{\mathcal{F}}(g)|_{\mathcal{B}_\sigma} = \Phi_\sigma(g).$$

By (24) this is well-defined, and since $\sum_\sigma \mathcal{B}_\sigma = \mathcal{B}$, this gives an action of L on all of $\mathcal{B}$. Moreover, by properties (i) and (ii), $(\Phi_{\mathcal{F}}, \mathcal{B})$ satisfies the conditions of Theorem 3.5. By property (iii) $(\Phi_{\mathcal{F}}, \mathcal{B})$ extends the natural action of SL_2 on $\mathcal{B}$.

Uniqueness: Suppose $(\Phi, \mathcal{B})$ is some representation of L satisfying the conditions of Theorem 3.5. By Proposition 3.7, to show uniqueness it suffices to show that there exists a compatible family $\widetilde{\mathcal{F}} = \{\widetilde{\phi}_p : W^p \to A^p\}_{p \in \Lambda_{\mathcal{B}}}$ such that $\Phi = \Phi_{\widetilde{\mathcal{F}}}$.

By condition (1) of the theorem, W^p is isomorphic to A^p as L-modules for every $p \in \Lambda_{\mathcal{B}}$. In particular, we can choose a set of L-isomorphisms $\{\widetilde{\phi}_p : W^p \to A^p\}_{p_{\max}=1}$. From this set we construct a compatible family $\widetilde{\mathcal{F}} = \{\widetilde{\phi}_p : W^p \to A^p\}_{p \in \Lambda_{\mathcal{B}}}$ as in the proof of Proposition 6.9.

To show that $\Phi = \Phi_{\widetilde{\mathcal{F}}}$, we need to show that for all $g \in L$ and $p \in \Lambda_{\mathcal{B}}$, the following diagram commutes:

$$\begin{array}{ccc} W^p & \xrightarrow{\Phi(g)} & W^p \\ \downarrow & & \downarrow \\ A^p & \xrightarrow{\theta_p(g)} & A^p \end{array} \tag{25}$$

where the vertical maps are both $\widetilde{\phi}_p$. We prove this by induction on $p_{\max}$.

Let $p \in \Lambda_{\mathcal{B}}$. If $p_{\max} = 1$, then (25) commutes by our choice of $\{\widetilde{\phi}_p : W^p \to A^p\}_{p_{\max}=1}$ above. Let $m > 1$ and assume (25) commutes for all p such that $p_{\max} < m$. Suppose then that $p_{\max} = m$. Choose some $\sigma \in \Sigma$ such that $p \in \Lambda_{\mathcal{B}}(\sigma)$. By Lemma 6.3, there exist $p', p'' \in \Lambda_{\mathcal{B}}(\sigma, m-1)$ such that $p = p' + p''$. Consider the following cube:

$$\begin{tikzcd}
& W^{p'} \otimes W^{p''} \arrow[rr, "\Phi(g)\otimes\Phi(g)"] \arrow[dl] \arrow[dd] & & W^{p'} \otimes W^{p''} \arrow[dl] \arrow[dd] \\
W^p \arrow[rr, "\Phi(g)" near end] \arrow[dd] & & W^p \arrow[dd] & \\
& A^{p'} \otimes A^{p''} \arrow[rr, "\theta_{p'}(g)\otimes\theta_{p''}(g)" near start] \arrow[dl] & & A^{p'} \otimes A^{p''} \arrow[dl] \\
A^p \arrow[rr, "\theta_p(g)"] & & A^p &
\end{tikzcd}$$

The top square commutes since Φ satisfies condition (2) of Theorem 3.5. The left and right squares commute by the compatibility of $\mathcal{F}$. The bottom square commutes since the product on $\mathcal{A}_\sigma$ intertwines the L-action. Finally, the back square commutes by inductive hypothesis. Since all the maps are surjective, we conclude that the front square commutes. This completes the induction, and proves that $\Phi = \Phi_{\widetilde{\mathcal{F}}}$.

This completes the proof of Theorem 3.5.

7. Proof of Corollary 3.8

Let T_{SL_2} be the torus of SL_2 consisting of diagonal matrices. Let $T_L = T_{SL_2} \times \cdots \times T_{SL_2}$ be the diagonal torus of L. From elementary representation theory of SL_2 we know that irreducible L-modules decompose canonically into one dimensional T_L weight spaces. Now let $(\Phi, \mathcal{B})$ be the representation of L afforded by Theorem 3.5. Then by part (1) of Theorem 3.5, the multiplicity spaces W^p, $(p \in \Lambda_{\mathcal{B}})$, decompose into one dimensional spaces.

The decomposition is canonical. Indeed, T_L is the unique torus of L containing T_{SL_2}. Moreover, the choice of the torus T_{SL_2} is induced by our choice of torus of Sp_n, i.e.

$$T_{SL_2} = SL_2 \cap T_{C_n}.$$

Therefore the decomposition of $W^{\lambda/\mu}$ into one dimensional spaces depends only the choice of torus of Sp_n. We now make this decomposition more precise.

Lemma 7.1. *Let* $\lambda/\mu \in \Lambda_{\mathcal{B}}$. *The weight spaces of* T_L *on* $W^{\lambda/\mu}$ *are indexed by the set* $\{\gamma \in \Lambda_n : \mu < \gamma < \lambda^+\}$. *An element* γ *corresponds to the weight*

$$(t_1, ..., t_n) \mapsto \prod_{i=1}^{n} t_i^{2\gamma_i - x_i - y_i}.$$

where $h(\lambda/\mu) = (x_1, y_1, ..., x_n, y_n)$.

Proof. Set $p = \lambda/\mu$. Since W^p is isomorphic as an L-module to $\bigotimes_{i=1}^n F^{r_i(p)}$, the weight spaces of T_L on W^p are indexed by

$$\{(j_1, ..., j_n) : 0 \leq j_i \leq r_i(p) \text{ for } i = 1, ..., n\}.$$

Indeed, such a sequence $(j_1, ..., j_n)$ corresponds to the weight

$$(t_1, ..., t_n) \mapsto \prod_{i=1}^{n} t_i^{(-r_i(p)+2j_i)}.$$

Now there is a one-to-one correspondence between

$$\{(j_1, ..., j_n) : 0 \leq j_i \leq r_i(p) \text{ for } i = 1, ..., n\}$$

and

$$\{(\gamma_1, ..., \gamma_n) : y_i \leq \gamma_i \leq x_i \text{ for } i = 1, ..., n\}$$

given by $(j_1, ..., j_n) \mapsto (j_1 + y_1, ..., j_n + y_n)$. Therefore, by Lemma 5.1, the weight spaces of T_L on W^p are indexed by $\{\gamma \in \Lambda_n : \mu < \gamma < \lambda^+\}$. Unwinding these identifications, we see that a pattern γ corresponds to the weight

$$(t_1, ..., t_n) \mapsto \prod_{i=1}^{n} t_i^{-(x_i+y_i)+2\gamma_i}.$$

□

For $\lambda/\mu \in \Lambda_{\mathcal{B}}$ and $\gamma \in \{\gamma \in \Lambda_n : \mu < \gamma < \lambda^+\}$, let $W^{\lambda/\gamma/\mu}$ be the T_L weight space of $W^{\lambda/\mu}$ corresponding to the weight γ by the above lemma. Then we obtain a decomposition of $W^{\lambda/\mu}$ into one dimensional weight spaces:

$$W^{\lambda/\mu} = \bigoplus_{\substack{\gamma \in \Lambda_n \\ \mu < \gamma < \lambda^+}} W^{\lambda/\gamma/\mu}.$$

This decomposition is canonical in the sense that it only depends on the choice of torus $T_{C_n} \subset Sp_{2n}$. This completes the proof of Corollary 3.8.

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