Recall that $a(t) = t + c$ on the characteristic lines.

If $a \geq t$, then $c \geq 0$, so $t_c = \max(0, -c) = 0$.

Then,

$$n_c(t) = n_c(0) \exp\left(-\int_0^t \mu(u+c)\,du\right)$$

for $t \geq 0$. So,

$$n(a,t) = n(a-t, 0) \exp\left(-\int_0^t \mu(u+a-t)\,du\right)$$

$$\boxed{= f(a-t) \exp\left(-\int_{a-t}^a \mu(v)\,dv\right)}$$

If $a < t$, then $c < 0$, so $t_c = -c$.

Then,

$$n_c(t) = n_c(-c) \exp\left(-\int_{-c}^t \mu(u+c)\,du\right)$$

for $t \geq -c$.

So,

$$n(a,t) = n(0, t-a) \exp\left(-\int_{t-a}^t \mu(u+a-t)\,du\right)$$

$$= \underbrace{n(0, t-a)}_{\text{get this from BC}} \exp\left(-\int_0^a \mu(v)\,dv\right)$$

where $n(0, t-a) = \int_0^\infty \beta(a) n(a, t-a)\,da$.

The complete solution is

$$n(a,t) = \begin{cases} n(0,t-a) \exp\left(-\int_0^a \mu(v)\,dv\right) & \text{for } a < t \\ f(a-t) \exp\left(-\int_{a-t}^a \mu(v)\,dv\right) & \text{for } a \geq t \end{cases}$$

These age-structured PDEs are called McKendrick-von Foerster Equations.

When dealing with age-structured models, we are often interested in the stable age distribution, $A(a)$.

For these solutions, the total population can change over time, but the proportion in any age bracket remains constant, i.e., the age distribution

$$\frac{n(a,t)}{\int_0^\infty n(a,t)\,da} = A(a)$$

is independent of time.

These solutions have the form

$$n(a,t) = A(a) T(t).$$

Thursday, August 03, 2017 10:34 AM

Substituting this expression into the PDE, we get

$$A(a)\frac{dT(t)}{dt} + T(t)\frac{dA(a)}{da} = -\mu(a)A(a)T(t).$$

Separating variables, we get

$$\frac{T'}{T} = -\frac{A' + \mu(a)A}{A} = \lambda,$$

where λ is constant.

Solving for T, we get

$$T(t) = T(0)e^{\lambda t}$$

and solving for A, we get

$$A(a) = A(0)\exp\left(-\lambda a - \int_0^a \mu(u)\,du\right).$$

Hence,

$$n(a,t) = A(0)T(0)\exp\left(\lambda(t-a) - \int_0^a \mu(u)\,du\right).$$

Applying the boundary condition

$$n(0,t) = \int_0^\infty \beta(a)\,n(a,t)\,da,$$

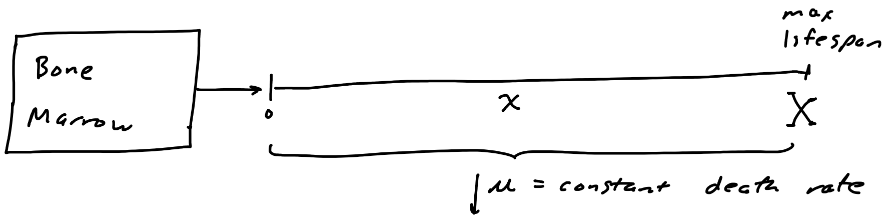
we get

$$\boxed{1 = \int_0^\infty \beta(a)\exp\left(-\lambda a - \int_0^a \mu(u)\,du\right)da}$$

This is called the characteristic equation. It relates λ to the known/given functions $\mu(a)$ and $\beta(a)$.

Model of Blood Cell Growth

$n(x,t)$ = density of blood cells at time t that are x time units old.



The PDE is

$$\frac{\partial n}{\partial t} + \frac{\partial n}{\partial x} = -\mu n$$

Total number of circulating blood cells is

$$N(t) = \int_0^X n(x,t)\,dx$$

Suppose blood cell production is controlled by $N(t)$ (negative feedback), and once a cohort of cells is formed in the bone marrow, it emerges into the blood some τ time units later.

Thus,

$$\underbrace{n(0,t)}_{\text{birth rate}} = F(N(t-\tau)) \quad \text{time delay}$$

where $F(N)$ is a monotone decreasing function of N.