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# Age-structured models

Sometimes the age-structure of a population matters, and we want to know the age distribution of a population over time.

Let $n(a,t)$ = population density of individuals of age $a$ at time $t$.

The number of individuals between ages $a$ and $a+da$ at time $t$ is

$$\int_a^{a+da} n(u,t)\,du.$$

Let $\mu(a)$ = age-specific death rate
$\beta(a)$ = age-specific birth rate

The equation for death is

$$\frac{dn}{dt} = -\mu(a)n(a,t)$$

By the chain rule,

$\frac{da}{dt}=1$ because an individual ages at the same rate as time $t$.

$$\frac{dn}{dt} = \frac{\partial n}{\partial t} + \frac{\partial n}{\partial a}\cdot\frac{da}{dt} = \frac{\partial n}{\partial t} + \frac{\partial n}{\partial a}.$$

The total number of births, i.e., the number of individuals of age 0, at time $t$ is

$$\underline{n(0,t)} = \int_0^\infty \beta(a)n(a,t)\,da.$$

Suppose the age distribution at time 0 is given by $n(a,0)=f(a)$ for some function $f(a)\geq 0$. Then the system is given by

$$\begin{cases} \dfrac{\partial n}{\partial t} + \dfrac{\partial n}{\partial a} = -\mu(a)n(a,t) & \text{PDE} \\ n(0,t) = \displaystyle\int_0^\infty \beta(a)n(a,t)\,da & \text{BC} \\ n(a,0) = f(a) & \text{IC} \end{cases}$$

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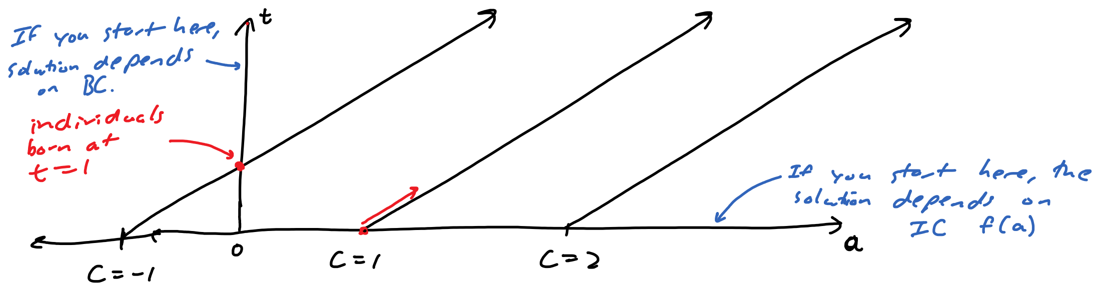
This PDE can be solved by the method of characteristics.

The characteristic curve $a(t) = t + c$ for some constant $c$.




Let $n_c(t) = n(t+c, t)$ for $t \geq t_c$

where $t_c := \max(0, -c)$.

The function $n_c(t)$ is called a cohort function.

(Note: $c \in \mathbb{R}$, so $c$ can be $+$, $-$, or $0$.)

The variable $n_c(t)$ tracks the population of individuals who were at age $c$ at time $t=0$ or who were born (with age $0$) at time $t=-c$.

Substituting $n_c(t)$ into the PDE, we obtain

$$\frac{dn_c}{dt} = -\mu(a)\, n_c(t)$$

for $t \geq t_c$ and $a = t + c$.

This yields the solution

$$n_c(t) = n_c(t_c) \exp\left(-\int_{t_c}^{t} \mu(u+c)\, du\right).$$