Tuesday, August 08, 2017 9:58 AM

Last time, we wrote

$$\underbrace{n(0,t)}_{\text{birth rate}} = F(N(t-\tau))$$

time delay

$F(N)$ is a monotone decreasing function of N.

F is related to the secretion rate of growth inducer (e.g. erythropoietin for red blood cells) in response to population.

Suppose we have some initial condition

$$n(x,0) = n_0(x).$$

So, the system is

$$\begin{cases} \dfrac{\partial n}{\partial t} + \dfrac{\partial n}{\partial x} = -\mu n \\ n(0,t) = F(N(t-\tau)) \quad \text{where} \quad N(t) = \displaystyle\int_0^X n(x,t)\,dx \\ n(x,0) = n_0(x) \end{cases}$$

Steady state calculation

Set $\dfrac{\partial n}{\partial t} = 0$

$$\Rightarrow \frac{dn}{dx} = -\mu n \Rightarrow \begin{cases} n(0)e^{-\mu x}, & x < X \\ 0, & x \geq X \end{cases}$$

Let $N_0 = \displaystyle\int_0^X n(x)\,dx$ = total population at steady state

$$= \frac{n(0)}{\mu}\left(1 - e^{-\mu X}\right).$$

At steady state, we also have

$$F(N_0) = n(0)$$

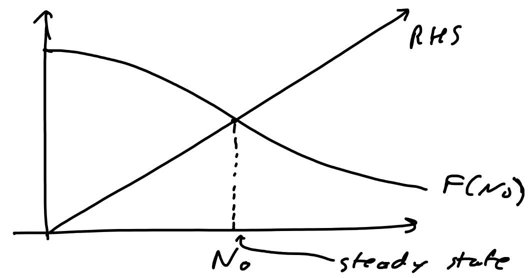
So,

$$F(N_0) = \frac{\mu N_0}{1 - e^{-\mu X}}$$

decreasing

linear and increasing with respect to N_0

Since $F(N_0)$ is decreasing and the RHS is increasing, the equation has a unique solution.



Q: Is the steady state stable or unstable

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Integrate the original PDE:

$$\int_0^X \left(\underline{\frac{\partial n}{\partial t} + \frac{\partial n}{\partial x}}\right) dx = \int_0^X \underline{-\mu n}\, dx$$

$$\Rightarrow \frac{\partial}{\partial t} \underbrace{\int_0^X n\, dx}_{N} + n(X,t) - n(0,t) = -\mu N$$

$$\Rightarrow \frac{dN}{dt} + \underline{n(X,t)} - \underline{n(0,t)} = -\mu N$$

$$\Rightarrow \frac{dN}{dt} + \underline{e^{-\mu X} F(N(t-\tau-X))} - \underline{F(N(t-\tau))} = -\mu N$$

Note: We have assumed that the initial condition has "washed out", i.e., $t > X$.

The equation above is a <u>delay differential equation</u> (DDE).

To analyse stability, linearise around the steady state by looking for solutions of the form

$$N(t) = N_0(1 + \varepsilon e^{\lambda t}) \quad \text{for} \quad \varepsilon \ll 1.$$

(Assume $N_0 > 0$ to be biologically relevant and interesting.)

Substituting the expression into the DDE and ignoring terms of $O(\varepsilon^2)$, we get

$$\lambda + \underbrace{F'(N_0) e^{-\lambda(\tau+X)}} e^{-\mu X} - F'(N_0) e^{-\lambda\tau} = -\mu$$

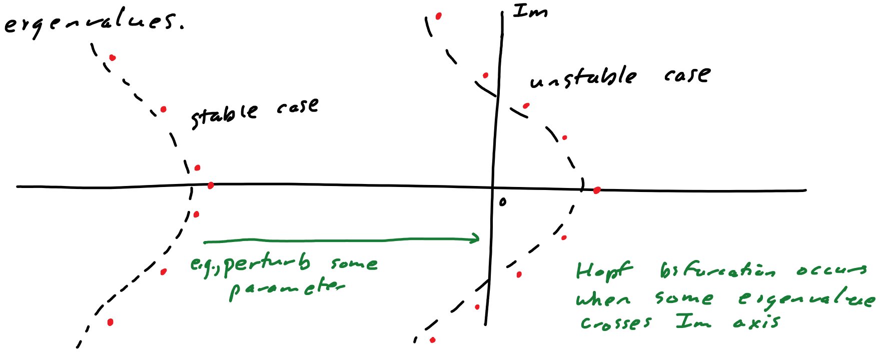
comes from Taylor expansion of $F(N_0(1+\varepsilon e^{\lambda(t-\tau-X)}))$

$$\Rightarrow \boxed{F'(N_0) e^{-\lambda\tau} \cdot \frac{1 - e^{-(\lambda+\mu)X}}{\lambda+\mu} = 1}$$

characteristic equation

Roots of the characteristic equation determine stability of the linearised system.

In general, DDEs have infinitely many eigenvalues.



It is hard (nearly impossible) to solve for eigenvalues explicitly, so we do it numerically.

For simplicity, consider the case $\mu = 0$ (i.e., no death until age $x = X$).