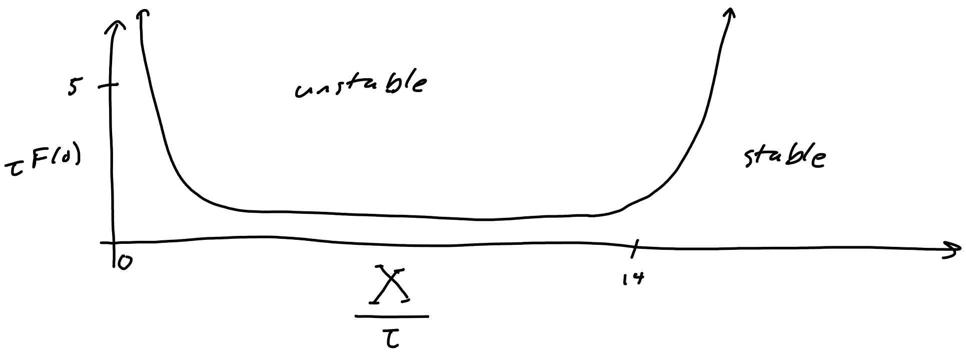
Continuing with the blood model...

We plot the results numerically.



For human red blood cells, $\tau = 5$ days and $X = 120$ days. So, under normal circumstances, the system is stable, since $\frac{X}{\tau} = 24$.

Basic diffusion

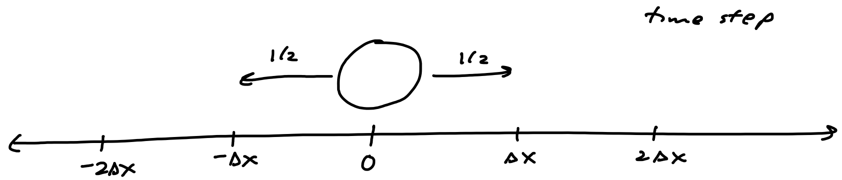
Diffusion is used to describe spread of particles through random motion.

In biology / social sciences / medicine, it has been used to model

- spread of a species
- spread of a favoured genetic trait
- propagation of ideas

Diffusion is connected to random walks.

Consider a 1-D space.



An object has ½ chance of moving a step Δx to the left or right in one time step Δt.

What is the probability $p(m, n)$ that an individual arrives at point $m\Delta x$ at time $n\Delta t$?

Method 1: Track individuals (Lagrangian approach)

A particle gets to point m by moving a steps right and b steps left.

Assume $m \pm n$ is even for convenience.

Then,
$$\begin{cases} a - b = m, \\ a + b = n \end{cases} \Rightarrow \begin{cases} a = \frac{n+m}{2}, \\ b = \frac{n-m}{2}. \end{cases}$$

The probability of having such a path is

$$p(m,n) = \binom{n}{a}\left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^n \frac{n!}{a!\,b!}.$$

Using Stirling's formula,
$$n! \sim (2\pi)^{1/2} n^{n+1/2} e^{-n}, \qquad n \gg 1,$$

and many algebraic manipulations
$$p(m,n) \approx \left(\frac{2}{\pi n}\right)^{1/2} e^{-m^2/(2n)}, \qquad m \gg 1, \quad n \gg 1.$$

Let $x = m\Delta x$, $t = n\Delta t$ and

$u(x,t)$ = prob. density of being at point x at time t

Note that $p(m,n)$ is the prob. of finding the individual in the interval $[x-\Delta x, \Delta + \Delta x)$, since $m \pm n$ has to be even.

Hence,
$$u(x,t) \sim \frac{p(m,n)}{2\Delta x}$$
$$\sim \left(\frac{\Delta t}{2\pi t (\Delta x)^2}\right)^{1/2} \exp\left\{-\frac{x^2}{2t} \cdot \frac{\Delta t}{(\Delta x)^2}\right\}$$

(we substituted $m = \frac{x}{\Delta x}$, $n = \frac{t}{\Delta t}$).

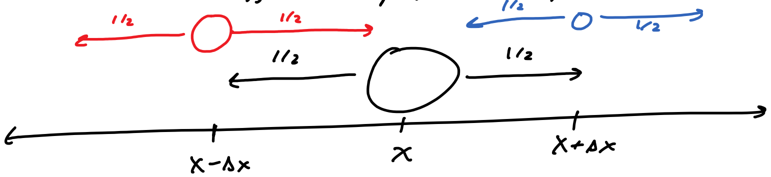
If we assume
$$D = \lim_{\substack{\Delta x \to 0 \\ \Delta t \to 0}} \frac{(\Delta x)^2}{2\Delta t} \neq 0,$$

then $u(x,t) = \lim_{\substack{\Delta x \to 0 \\ \Delta t \to 0}} \frac{p(m,n)}{2\Delta x} = \frac{1}{\sqrt{4\pi D t}} e^{-x^2/(4Dt)}.$

We call D the diffusion coefficient.

Method 2: Track probability distribution of the population.

Let $u(x,t)$ = prob. density that an individual is at position x at time t.



Thus, we have the recursion

$$u(x,t) = \underbrace{\tfrac{1}{2}u(x-\Delta x,\ t-\Delta t)} + \underbrace{\tfrac{1}{2}u(x+\Delta x,\ t-\Delta t)}.$$

Taylor expanding, we get

$$u(x,t) = u(x,t) - \frac{\partial u}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(\Delta x)^2 + o(\Delta t^2) + \cdots$$

So, $$\frac{\partial u}{\partial t} = \left(\frac{\Delta x^2}{2\Delta t}\right)\frac{\partial^2 u}{\partial x^2} + \text{(higher order terms)}$$

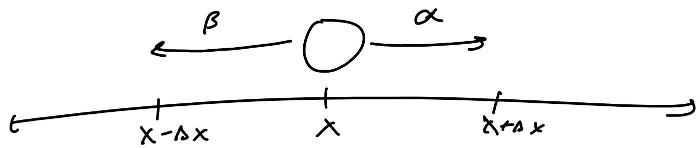
Defining $D = \lim\limits_{\substack{\Delta x \to 0 \\ \Delta t \to 0}} \dfrac{(\Delta x)^2}{2\Delta t} \neq 0$ as before

and letting $\Delta t, \Delta x \to 0$, we get the diffusion/heat equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}.$$

We can check that $u(x,t) = \dfrac{1}{\sqrt{4\pi D t}} e^{-\frac{x^2}{4Dt}}$ solves this equation.

Biased random walk



$\alpha + \beta \le 1$

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + V\frac{\partial u}{\partial x}$$