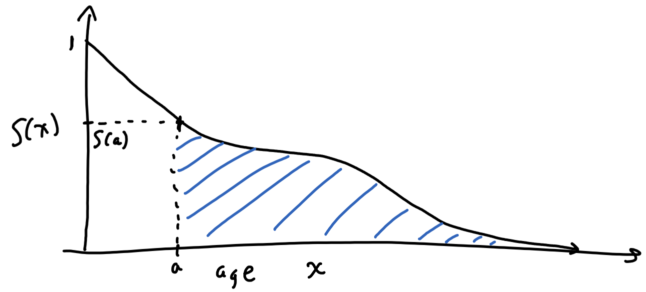
Discussion question 1 continued

Life expectancy from age 0

$$e_0 = \int_0^\infty S(x)\,dx .$$

What is $\int_a^\infty S(x)\,dx$ for $a \geq 0$?



Given that an individual has survived to age a, how many more years is it expected to live?

$$e_a = \frac{\int_a^\infty S(x)\,dx}{S(a)}$$ = expected # of years to live post age a, so $e_a \in [0, \infty)$

Tuesday, August 15, 2017 10:17 AM

$n = n(a)$

$\cancel{\frac{\partial n}{\partial t}} + \frac{\partial n}{\partial a} = \frac{\partial n}{\partial a}$

$\frac{dn}{da} = -\mu(a)\, n(a)$

$n = n(0) \exp\left(-\int_0^a \mu(u)\, du\right)$

$n(0) = \int_0^\infty \beta(a)\, n(a)\, da$

$1 = \int_0^\infty \beta(a) \exp\left(-\int_0^a \mu(u)\, du\right) da$

$1 = \beta \int_{15}^{45} \exp\left(-\int_0^a \mu(u)\, du\right) da$

$\beta = \frac{1}{\int \ldots da}$

end of Discussion question 1

Tuesday, August 15, 2017 10:20 AM

Back to the main lectures

The characteristic equation is

$$F'(N_0)\left(e^{-\lambda\tau} - e^{-\lambda(\tau+X)}\right) = \lambda.$$

Since $F'(N_0) < 0$, there are no real roots.

(Note: the root $\lambda = 0$ is spurious. It corresponds to the steady state solution $N(t) = N_0(1+\varepsilon)$, which is not a steady state unless $\varepsilon = 0$.)

So all roots are complex, so the system will oscillate.

The only way to transition from unstable to stable is for a complex root to change the sign of its real part (Hopf bifurcation).

If this occurs, it happens when $\lambda = i\omega$ (purely imaginary).

Substitute $\lambda = i\omega$ into the characteristic equation, so

$$F'(N_0)\left(e^{-i\omega\tau} - e^{-i\omega(\tau+X)}\right) = i\omega.$$

Separate real and imaginary parts:

(A) $F'(N_0)\left(\cos(\omega\tau) - \cos(\omega(\tau+X))\right) = 0,$

(B) $F'(N_0)\left(\sin(\omega\tau) - \sin(\omega(\tau+X))\right) = -\omega.$

Since $F'(N_0) < 0$,

$$\cos(\omega\tau) - \cos(\omega(\tau+X)) = 0$$

$$\Rightarrow \left.\begin{array}{l} \omega\tau + \omega X = \omega\tau + 2n\pi \\ \text{or } \omega\tau + \omega X = -\omega\tau + 2n\pi \end{array}\right\}$$

for $n = 1, 2, 3, \ldots$

$$\Rightarrow \left.\begin{array}{l} \omega X = 2n\pi \\ \text{or } (2\tau + X)\omega = 2n\pi \end{array}\right\}$$

But $\omega X = 2n\pi$ won't work in (B).

$\Rightarrow$ (B) becomes

$$2F'(N_0)\sin(\omega\tau) = -\omega$$

$$\Rightarrow 2\tau F'(N_0) = -\frac{\omega\tau}{\sin(\omega\tau)}$$

$$\Rightarrow 2\tau F'(N_0) = -\frac{2n\pi}{2+\frac{X}{\tau}} \cdot \frac{1}{\sin\left(\frac{2n\pi}{2+\frac{X}{\tau}}\right)}$$

$$\Rightarrow X F'(N_0) = -\frac{1}{2}\frac{X}{\tau} \cdot \frac{2n\pi}{2+\frac{X}{\tau}} \cdot \frac{1}{\sin\left(\frac{2n\pi}{2+\frac{X}{\tau}}\right)}$$

Tuesday, August 15, 2017 10:47 AM

Use $N_0 = F(N_0)X \Rightarrow X = \dfrac{N_0}{F(N_0)}$

$$\Rightarrow \frac{N_0 F'(N_0)}{F(N_0)} = -\frac{1}{2}\left(\frac{X}{\tau}\right)\frac{2n\pi}{2+\left(\frac{X}{\tau}\right)} \cdot \frac{1}{\sin\left(\frac{2n\pi}{2+\left(\frac{X}{\tau}\right)}\right)} \quad \text{C}$$

C defines a relationship between the steady state N_0 and the ratio

$$\frac{X}{\tau} = \text{nondimensional parameter}$$
$$= \frac{\text{time in circulation}}{\text{time in bone marrow}}$$

Suppose

$$F(x) = \frac{A}{1+x^7}$$

Then,

$$F'(N_0) = -\frac{A}{(1+N_0^7)^2} \cdot 7N_0^6,$$

so C + some algebra

$$\Rightarrow N_0 = \sqrt[7]{\left[\left(\frac{-7}{f\left(\frac{X}{\tau}\right)}\right) - 1\right]^{-1}} \quad \text{D}$$

where $f(y) = -\frac{1}{2}\, y\, \frac{2n\pi}{2+y} \cdot \dfrac{1}{\sin\left(\frac{2n\pi}{2+y}\right)}$,

which expresses N_0 as a function of $\frac{X}{\tau}$.

Since $F(N_0) = \dfrac{N_0}{X}$, $\dfrac{A}{1+N_0^7} = \dfrac{N_0}{X}$

$$\Rightarrow \tau \underbrace{A}_{F(0)} = N_0(1+N_0^7)\left(\frac{X}{\tau}\right)^{-1}$$

$$\Rightarrow \tau F(0) = N_0(1+N_0^7)\left(\frac{X}{\tau}\right)^{-1}$$

and use D for N_0 to get an implicit expression of $\frac{X}{\tau}$.