Thursday, August 24, 2017 9:57 AM

Discussion question 2

$$\frac{\delta n}{\delta t} + \frac{\delta n}{\delta a} = -\mu\, n(a,t)$$

stage 1; $-\mu_1 n_1$

stage 2; $-\mu_2 n_2$

n_1 is n for $0<a<\tau_1$

n_2 is n for $\tau_1<a<\tau_2$

$$\int_{\tau_2}^{\tau_3} \beta n_3\, da = n(0,t) \qquad n(a,0) = f(a)$$

Steady state: $\frac{\partial n}{\partial t} = 0$

Stage 1: $\frac{\partial n}{\partial t} + \frac{\partial n}{\partial a} = -\mu_1 n, \quad 0<a<\tau_1$

$$\Rightarrow \frac{\partial n}{\partial a} = -\mu_1 n$$

$$\Rightarrow n(a) = \begin{cases} n(0)e^{-\mu_1 a} & \text{for } 0<a<\tau_1 \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow N = \int_0^{\tau_1} n(a)\,da = \frac{n(0)}{\mu}\left(1-e^{-\mu_1\tau_1}\right)$$

Stage 1. $\frac{\partial n}{\partial t} + \frac{\partial n}{\partial a} = -\mu_1 n \qquad 0 \le a < \tau_1$

Integrating gives:

$$\int_0^{\tau_1}\left(\frac{\partial n}{\partial t} + \frac{\partial n}{\partial a}\right)da = \int_0^{\tau_1} -\mu_1 n\, da$$

$$\Rightarrow \frac{\partial}{\partial t}\underbrace{\int_0^{\tau_1} n\, da}_{= N_1} + n(\tau_1,t) - n(0,t) = -\mu_1 \underbrace{N_1}_{\text{total population of stage 1 at } t}$$

$$\Rightarrow \frac{dN_1}{dt} + n(\tau_1,t) - n(0,t) = -\mu_1 N_1$$

$$n(\tau_1,t) = n(0, t-\tau_1)\cdot e^{-\mu_1\tau_1}$$

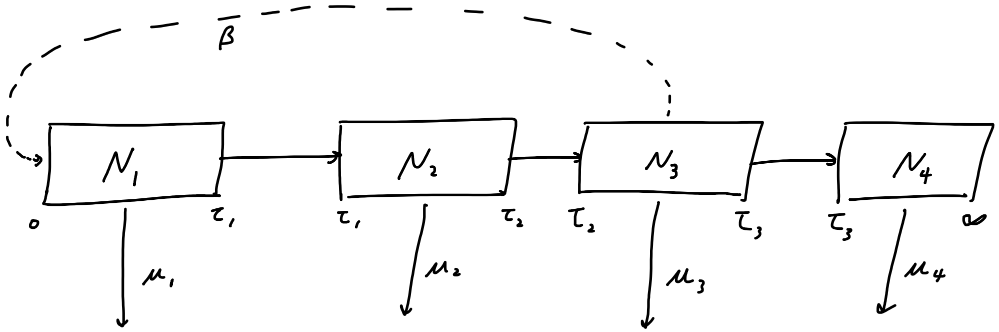
By B.C. : $n(0, t-\tau_1) = \int_{\tau_2}^{\tau_3} \beta n(a, t-\tau)\,da$

$$= \beta N_3(t-\tau_1)$$

$$n(0,t) = \beta N_3(t)$$

DDE. $\frac{dN_1}{dt} + \beta N_3(t-\tau_1)e^{-\mu_1\tau_1} - \beta N_3(t)$

$$= -\mu_1 N_1$$



Stage 1

$$\frac{\partial n}{\partial t} + \frac{\partial n}{\partial a} = -\mu_1 n, \qquad a \in (0, \tau_1]$$

$$n(0,t) = \beta \int_{\tau_2}^{\tau_3} n(a,t)\,da.$$

Let $N_i = \int_{\tau_{i-1}}^{\tau_i} n(a,t)\,da$

where $\tau_0 = 0$ and $\tau_4 = \infty$.

$$\int_0^{\tau_1} \left(\frac{\partial n}{\partial t} + \frac{\partial n}{\partial a} \right) da = \mu_1 \int_0^{\tau_1} n(a,t)\,da$$

$$\frac{dN_1}{dt} + \underbrace{n(\tau_1, t)}_{\beta e^{-\mu_1 \tau_1} N_3(t-\tau_1)} - \underbrace{n(0,t)}_{\beta N_3} = -\mu_1 N_1$$

(Assume $t > \tau_1$)

So,

$$\frac{dN_1}{dt} = -\mu_1 N_1 + \beta N_3 - \beta e^{-\mu_1 \tau_1} N(t-\tau_1)$$

which is a DDE.

Stage 2

$$\frac{dN_2}{dt} + \underbrace{n(\tau_2, t)}_{\beta e^{-\mu_1 \tau_1} e^{-\mu_2(\tau_2-\tau_1)} N_3(t-\tau_2)} - \underbrace{n(\tau_1, t)}_{\beta e^{-\mu_1 \tau_1} N_3(t-\tau_1)} = -\mu_2 N_2$$

Stage 3

$$\frac{dN_3}{dt} = -\mu_3 N_3 + \beta D_2 N_3(t-\tau_2) - \beta D_3 N_3(t-\tau_3)$$

where

$$D_2 = e^{-\mu_1 \tau_1 - \mu_2(\tau_2 - \tau_1)},$$

$$D_3 = e^{-\mu_1 \tau_1 - \mu_2(\tau_2 - \tau_1) - \mu_3(\tau_3 - \tau_2)}.$$

This gives a DDE for N_3.

Basically, a self-contained DDE for N_3 (or N_1, N_2, N_4) will govern the dynamics of the whole system.

For the characteristic equation, substitute

$$N_3(t) = 0 + \varepsilon e^{\lambda t}$$

into the DDE for N_3 to get

$$\underbrace{\lambda = -\mu_3 + \beta D_2 e^{-\lambda \tau_2} - \beta D_3 e^{-\lambda \tau_3}}_{\text{quasipolynomial (it involves } \lambda \text{ and } e^{-\lambda \cdot \text{constant}})}$$

There are 2 distinct delays