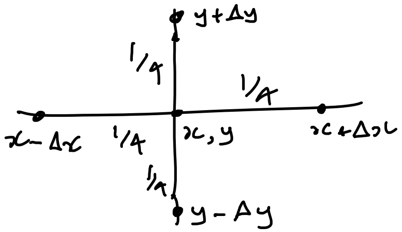
Discussion Q 3



$$u(x,y,t) = \frac{1}{4} u(x-\Delta x, y, t-\Delta t)$$
$$+ \frac{1}{4} u(x+\Delta x, y, t-\Delta t) + \frac{1}{4} \ldots$$

$$u(x, y, t) = \frac{1}{4}\Big[u(x,y,t) + u_x \Delta x + \frac{1}{2} u_{xx} (\Delta x)^2 + u_t(-\Delta t) + h.o.t.\Big]$$
$$+ \frac{1}{4}\Big[u(x,y,t) + u_x(-\Delta x) + \frac{1}{2} u_{xx} (\Delta x)^2 + u_t(-\Delta t) + h.o.t.\Big]$$
$$+ \frac{1}{4}\Big[u(x,y,t) + u_y(\Delta y) + \frac{1}{2} u_{yy} (\Delta y)^2 + u_t(-\Delta t)\Big] + h.o.t$$
$$+ \frac{1}{4}\Big[u(x,y,t) + u_y(-\Delta y) + \frac{1}{2} u_{yy} (\Delta y)^2 + u_t(-\Delta t)\Big]$$
$$+ h.o.t$$

$$0 = \frac{1}{4} u_{xx} (\Delta x)^2 + \frac{1}{4} u_{yy} (\Delta y)^2 - u_t \Delta t$$

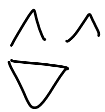
$$u_t = \frac{1}{4}\frac{(\Delta x)^2}{\Delta t} u_{xx} + \frac{1}{4}\frac{(\Delta y)^2}{\Delta t} u_{yy}$$

$$D_1 = \lim_{\substack{\Delta x \to 0 \\ \Delta t \to 0}} \frac{(\Delta x)^2}{4\Delta t}, \quad D_2 = \lim_{\substack{\Delta y \to 0 \\ \Delta t \to 0}} \frac{(\Delta y)^2}{4\Delta t}$$

Assume $D_1 = D_2 = D$,

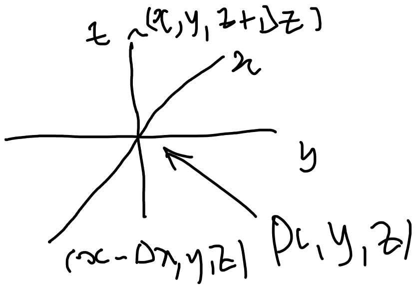
then taking our PDE and letting $\Delta t, \Delta x, \Delta y \to 0$ we have!

$$u_t = D(u_{xx} + u_{yy})$$



We can assume $\Delta X = \Delta y$. Otherwise diffusion would be directionally dependent.

Tuesday, August 29, 2017 10:26 AM



$u(x,y,z,t)=$

$$\frac{1}{6}\left[6\,u(x,y,z,t)+\frac{\partial^2 u}{\partial x^2}(\Delta x)^2 + \frac{\partial^2 u}{\partial y^2}(\Delta y)^2 + \frac{\partial^2 u}{\partial z^2}(\Delta z)^2 - 6\,\frac{\partial u}{\partial t}\Delta t\right]$$

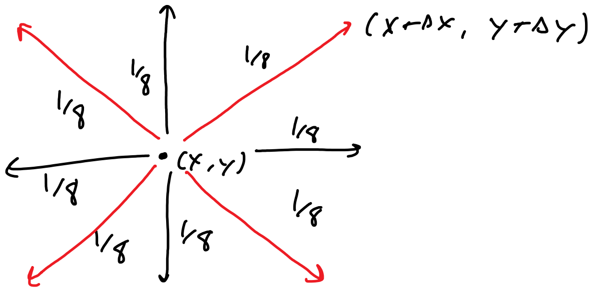
$$\frac{\partial u}{\partial t} = \frac{(\Delta x)^2}{6\Delta t}\frac{\partial^2 u}{\partial x^2} + \frac{(\Delta y)^2}{6\Delta t}\frac{\partial^2 u}{\partial y^2} + \frac{(\Delta z)^2}{6\Delta t}\frac{\partial^2 u}{\partial z^2}$$

$$D = \lim_{\substack{\Delta x\to 0\\ \Delta t\to 0}} \frac{(\Delta x)^2}{6\Delta t} = \dots$$

$$\frac{\partial u}{\partial t} = D\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

We can assume $\Delta x = \Delta y = \Delta z$.

What if the rule is (for 2D)



Moving along the diagonals, makes the particle move further, because the diagonals have length $\sqrt{2}\,\Delta x = \sqrt{2}\,\Delta y$.

In the N, S, E, W direction, the diffusion is $\frac{1}{2}D$ where $D = \lim_{\substack{\Delta t \to 0 \\ \Delta x \to 0}} \frac{(\Delta x)^2}{4\Delta t}$.

In the 4 diagonal directions, the diffusion is D, because it is $\lim_{\substack{\Delta t \to 0 \\ \Delta x \to 0}} \frac{1}{2} \cdot \frac{(\sqrt{2}\,\Delta x)^2}{4\Delta t}$

$$= \lim_{\substack{\Delta t \to 0 \\ \Delta x \to 0}} \frac{(\Delta x)^2}{4\Delta t}.$$

So, the random walk system with diagonals converges to the diffusion equation

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$$\frac{\partial u}{\partial t} = \left(\frac{1}{2}D + D\right)\nabla^2 u = \frac{3}{2}D\nabla^2 u$$

where $D = \lim_{\substack{\Delta t \to 0 \\ \Delta x \to 0}} \frac{(\Delta x)^2}{4\Delta t}$ is the diffusion from the first part of the question.

Reaction-Diffusion Turing Mechanisms

Turing's idea:

Suppose one has 2 chemical species A and B that obey certain (nonlinear) reaction-diffusion equations

$$\frac{\partial A}{\partial t} = F(A, B) + D_A \nabla^2 A$$

$$\frac{\partial B}{\partial t} = G(A, B) + D_B \nabla^2 B$$

If in the absence of diffusion (i.e., $D_A = D_B = 0$), the system is linearly stable, then under certain conditions, two different diffusion rates ($D_A \neq D_B$) can give rise to spatially inhomogeneous patterns, which are now called Turing patterns or Turing instabilities.