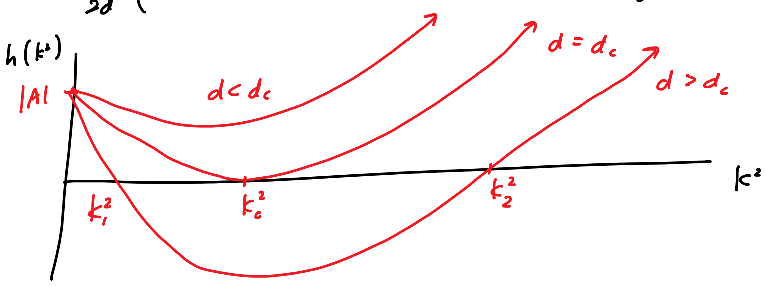
When $d > d_c$, the range of unstable wave numbers k is constrained by

$$k_1^2 < k^2 < k_2^2 ,$$

where k_1^2 and k_2^2 are the zeros of $h(k^2) = 0$, i.e.,

$$k_1^2 = \frac{\gamma}{2d}\left(d f_u + g_v - \sqrt{(d f_u + g_v)^2 - 4d|A|} \right),$$

$$k_2^2 = \frac{\gamma}{2d}\left(\quad '' \quad + \quad '' \quad \right).$$



Then, the solution

$$\vec{w} = \sum_k c_k e^{\lambda(k^2) t} \vec{W}_k(\vec{r})$$

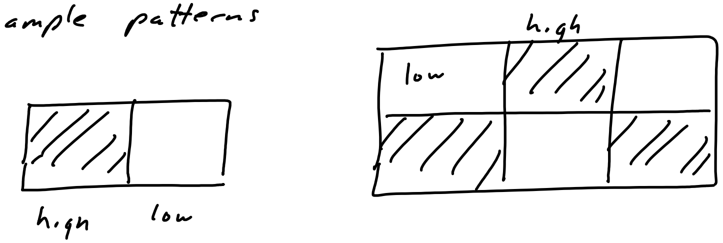
approaches

$$\sum_{k_1}^{k_2} c_k e^{\lambda(k^2) t} \vec{W}_k(\vec{r})$$

for large t.

Note that the system is nonlinear, so a key assumption is that these linearly unstable modes will eventually be bounded by nonlinear terms, resulting in a spatially inhomogeneous steady state.

Example patterns



In summary, the conditions for spatial pattern formation for

$$u_t = \gamma f(u,v) + \nabla^2 u,$$
$$v_t = \gamma g(u,v) + d\nabla^2 v$$

with no-flux boundary conditions are

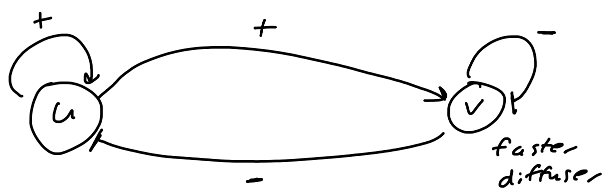
1. $f_u + g_v < 0$
2. $f_u g_v - f_v g_u > 0$
3. $d f_u + g_v > 0$
4. $(d f_u + g_v)^2 - 4d(f_u g_v - f_v g_u) > 0$

From (1) and (3), f_u and g_v must have opposite signs. Suppose $f_u > 0$ and $g_v < 0$, then (3) $\Rightarrow d > 1$.

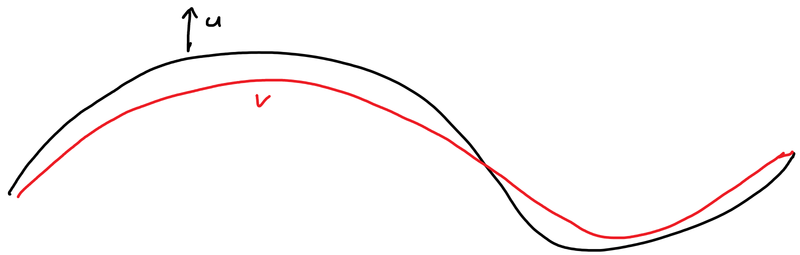
Since $f_u g_v < 0$, (2) $f_v g_u < f_u g_v < 0$, so f_v and g_u also have opposite signs.

We have 2 cases:

i) $\begin{bmatrix} f_u & f_v \\ g_u & g_v \end{bmatrix} = \begin{bmatrix} + & - \\ + & - \end{bmatrix}$

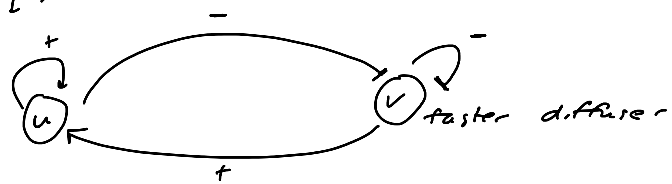


u is the activator of v (and itself), and v is the inhibitor of u (and itself).



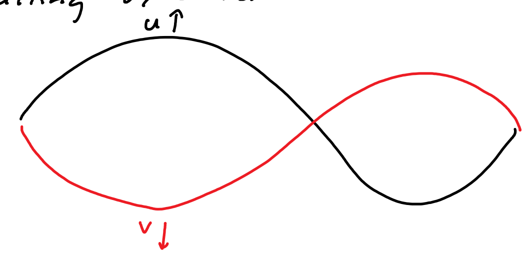
They are in phase.

ii) $\begin{bmatrix} f_u & f_v \\ g_u & g_v \end{bmatrix} = \begin{bmatrix} + & + \\ - & - \end{bmatrix}$



u is the inhibitor of v, and v is the activator of u.

Resulting dynamics look like



They are out of phase

Specific example

Consider the 1-D system

$$u_t = \gamma f(u,v) + u_{xx},$$

$$v_t = \gamma g(u,v) + d v_{xx},$$

where $f(u,v) = a - u + u^2 v,$

$g(u,v) = b - u^2 v,$

for some positive constants γ, a, b, d.

The positive steady state is

$$u_0 = a+b,$$

$$v_0 = \frac{b}{(a+b)^2}.$$

At steady state,

$$f_u = \frac{b-a}{a+b}$$

$$f_v = (a+b)^2 > 0$$

$$g_u = -\frac{2b}{a+b} < 0$$

$$g_v = -(a+b)^2 < 0.$$

Since f_u and g_v must have opposite signs, we must have $f_u > 0 \Rightarrow b > a$.

From the 4 "Turing conditions", we have

$$f_u + g_v < 0 \Rightarrow b - a < (a+b)^3$$

and

$$f_u g_v - f_v g_u > 0 \Rightarrow (a+b)^2 > 0 \quad \text{automatically satisfied}$$

and

$$d f_u + g_v > 0 \Rightarrow d(b-a) > (a+b)^3$$

and

$$(d f_u + g_v)^2 - 4d(f_u g_v - f_v g_u) > 0$$

$$\Rightarrow \left(d(b-a) - (a+b)^3\right)^2 > 4d(a+b)^4$$

These conditions define a domain in (a,b,d)-parameter space, called the Turing space, in which patterns from diffusion-driven instability can occur.

Consider the domain $x \in (0,L)$ and consider the eigenvalue problem

$$W_{xx} + k^2 W = 0$$

with BCs $\dfrac{\partial W(0,t)}{\partial x} = \dfrac{\partial W(L,t)}{\partial x} = 0.$

The solutions are

$$W_n(x) = A_n \cos\left(\frac{n\pi x}{L}\right)$$

where A_n is constant.

So, the wave numbers are $k = \frac{n\pi}{L}$.

Following the same reasoning as for the general case, we get

$$k_1^2 < k^2 < k_2^2$$

where

$$k_1^2 = \gamma\left(\frac{d(b-a) - (a+b)^3 - \sqrt{(d(a-b) - (a+b)^3)^2 - 4d(a+b)^4}}{2d(a+b)}\right)$$

$$k_2^2 = \gamma\left(\frac{\text{''} + \text{''}}{\text{''}}\right)$$

In terms of wavelength,

$$\omega = \frac{2\pi}{k}$$

and $\omega_2^2 < \omega^2 < \omega_1^2$ where $\omega_i = \frac{2\pi}{k_i}$

Note that both sides of the inequalities for k^2 are proportional to γ. Also, the allowed wave numbers $k = \frac{n\pi}{L}$ are discrete.

So, for small γ, there are no unstable modes.

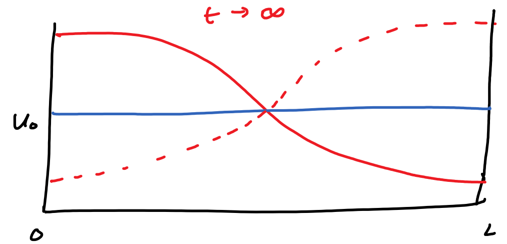
Suppose γ is such that the only unstable wave number corresponds to $n=1$.

Then,

$$\vec{w}(x,t) \sim C_1 \exp\left(\lambda\left(\frac{\pi^2}{L^2}\right)t\right)\cos\frac{\pi x}{L}$$

as $t \to \infty$.

So, one would expect a spatial pattern like



Recall that unstable modes satisfy

$$k_1^2 < k^2 < k_2^2$$

where $k_1^2 = \gamma(\text{''} - \text{''}) = \gamma P$
$k_2^2 = \gamma(\text{''} + \text{''}) = \gamma Q$

and $k = \frac{n\pi}{L}$.

So,

$$\gamma P\left(\frac{L}{\pi}\right)^2 < n^2 < \gamma Q\left(\frac{L}{\pi}\right)^2$$

and suppose $n=1$ is the only $n \in \mathbb{N}$ that satisfies this inequality.

Suppose we double the spatial domain size.
Then γ will change to 4γ.
Then $n=2$ satisfies

$$(4\gamma)\left(\frac{L}{\pi}\right)^2 P < n^2 < (4\gamma)\left(\frac{L}{\pi}\right)^2 Q,$$

so the $n=2$ mode can arise