General conditions for Turing instability

Nondimensionalise the reaction-diffusion equations to the form

$$u_t = \gamma f(u,v) + \nabla^2 u,$$
$$v_t = \gamma g(u,v) + d\nabla^2 v,$$

where $d = \frac{D_B}{D_A}$ is the diffusion ratio.

Also, γ is a useful scaling constant that has to do with the size of the spatial domain.

Also, assume no-flux boundary conditions

$$(\vec{n}\cdot\nabla)\begin{pmatrix} u \\ v \end{pmatrix} = 0 \qquad \text{on} \qquad \partial B$$

where ∂B is the closed boundary of the domain B and $\vec{n}$ is unit outward normal. And, there is some initial condition $u(\vec{r},0)$ and $v(\vec{r},0)$.

Note: We are interested in positive solutions, because we want to keep the system biologically relevant.

① First, assume no spatial variation, so that u and v are only functions of t. Then,

$$u_t = \gamma f(u,v),$$
$$v_t = \gamma g(u,v).$$

Set $\vec{w} = \begin{pmatrix} u-u_0 \\ v-v_0 \end{pmatrix}$,

where (u_0, v_0) is a steady state.

If we linearise for small $|\vec{w}|$, we get $\vec{w}_t = \gamma A \vec{w}$

where $A = \begin{bmatrix} f_u & f_v \\ g_u & g_v \end{bmatrix}\Bigg|_{u_0,v_0}$.

Looking for solutions of the form $\vec{w} \propto e^{\lambda t}$, we get the characteristic equation

$$0 = \det(\gamma A - \lambda I)$$
$$0 = \lambda^2 - \gamma(f_u + g_v)\lambda + \gamma^2(f_u g_v - f_v g_u).$$

Note that we mean $f_u(u_0,v_0), \ldots$

Linear stability occurs when

$\operatorname{tr} A = f_u + g_v < 0$ and

$|A| = f_u g_v - f_v g_u > 0.$

(2) Now, consider the system

$$u_t = \gamma f(u,v) + \nabla^2 u,$$
$$v_t = \gamma g(u,v) + d\nabla^2 v.$$

Set $\vec{w} = \begin{pmatrix} u - u_0 \\ v - v_0 \end{pmatrix}$

and linearise to get

$$\vec{w}_t = \gamma A \vec{w} + D\nabla^2 \vec{w} \qquad (A)$$

where $D = \begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix}$.

Let $\vec{W}(\vec{r})$ be the time-independent solution of the spatial eigenvalue problem

$$\nabla^2 \vec{W} + k^2 \vec{W} = 0 \qquad (B)$$

with no-flux boundary conditions.

The eigenvalue k is called the wave number and $1/k$ is proportional to the wavelength.

Since the equation is linear, we look for solutions of the form

$$\vec{w}(\vec{r},t) = \sum_k c_k e^{\lambda_k t} \vec{W}_k(\vec{r})$$

Note that λ depends on k.

Constants c_k are determined by Fourier expansion of initial conditions.

Substituting $e^{\lambda t}\vec{W}_k(\vec{r})$ into (A) and (B), we get

$$\lambda e^{\lambda t}\vec{W}_k = \gamma A e^{\lambda t}\vec{W}_k + D\nabla^2(e^{\lambda t}\vec{W}_k)$$

$$\Rightarrow \lambda \vec{W}_k = \gamma A \vec{W}_k - Dk^2 \vec{W}_k.$$

Nontrivial solutions satisfy the characteristic equation

$$0 = |\lambda I - \gamma A + Dk^2|$$
$$= \lambda^2 + \lambda\left(k^2(1+d) - \gamma(f_u + g_v)\right) + h(k^2)$$

where $h(k^2) = dk^4 - \gamma(df_u + g_v)k^2 + \gamma^2|A|.$

If $k=0$, the characteristic equation reduces to

$$0 = \lambda^2 - \gamma(f_u + g_v)\lambda + \gamma^2 |A|$$

which is the same as in case 1 without spatial effects. We assumed this system is stable.

To have the steady state be unstable due to spatial disturbances, we have to find values of $k \neq 0$ for which

$$Re(\lambda(k)) > 0.$$

For this, we need

$$k^2(1+d) - \gamma(f_u + g_v) < 0 \quad \text{or} \quad h(k^2) < 0.$$

We have $tr\, A = f_u + g_v < 0$ from 1, and we know $k^2(1+d) > 0$ for all $k \neq 0$, so

$$k^2(1+d) - \gamma(f_u + g_v) > 0,$$

so we need to look for k such that $h(k^2) < 0$.

Recall from 1, we required that

$$|A| > 0 \quad \text{and} \quad tr\, A = f_u + g_v < 0.$$

Then,

$$h(k^2) = dk^4 - \gamma(df_u + g_v)k^2 + \gamma^2 |A| < 0$$

$$\Rightarrow \quad df_u + g_v > 0.$$

Since $f_u + g_v < 0$, it follows that $d \neq 1$ and f_u and g_v have opposite signs.

For convenience, let's say $d > 1$ and $f_u > 0$ and $g_v < 0$. (The situation is symmetric otherwise.)

To have $h(k^2) < 0$ for some k, the minimum h_{min} must be negative, so we find h_{min}:

$$\frac{dh}{d(k^2)} = 2dk_m^2 - \gamma(df_u + g_v) = 0$$

$$\Rightarrow k_m^2 = \gamma \frac{df_u + g_v}{2d} \quad (*).$$

So,

$$h_{min} = dk_m^4 - \gamma(df_u + g_v)k_m^2 + \gamma^2 |A|$$

$$= \gamma^2 \left(|A| - \frac{(df_u + g_v)^2}{4d} \right).$$

Thus, $h(k^2) < 0$ for some $k^2 \neq 0$, when

$$\frac{(df_u + g_v)^2}{4d} > |A|.$$

At the critical point, we have $h_{min} = 0$, so

$$|A| = \frac{(df_u + g_v)^2}{4d} \quad (*)$$

You can solve this last (quadratic) equation to find the critical diffusion ratio d_c in terms of f_u, f_v, g_u, g_v.

From the two ⊛ equations, the critical wave number k_c is given by

$$k_c^2 = \gamma \frac{d_c f_u + g_v}{2 d_c} = \gamma \sqrt{\frac{|A|}{d_c}}.$$