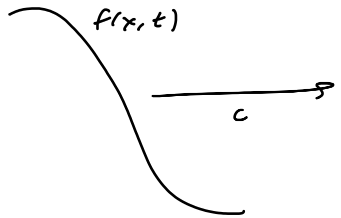
Travelling Wave Solutions

Let $f(x,t)$ be a function that represents a wave moving to the right at a constant rate c while maintaining a fixed shape.



An observer moving at the same speed and direction of motion would see an unchanging picture given by a function $F(z)$.

The relation between F and f is

$F(z) = f(x,t)$ where $z = x - ct$.

For simplicity, we will assume $c > 0$ and use $z = x + ct$ for motion to the left.

By the chain rule,

$$\frac{\partial f}{\partial x} = \frac{dF}{dz}\frac{\partial z}{\partial x} = \frac{dF}{dz},$$

$$\frac{\partial f}{\partial t} = \frac{dF}{dz}\cdot\frac{\partial z}{\partial t} = -c\frac{dF}{dz}.$$

Fisher's equation

The spread of genes in a population.

Consider a population of individuals carrying an advantageous allele "A" and migrating randomly into a region where only allele "a" is initially present.

If p is the frequency of A, and $q = 1-p$ is the frequency of a, under Hardy-Weinberg genetics, the rate of change in p is governed by

$$\frac{\partial p}{\partial t} = D\frac{\partial^2 p}{\partial x^2} + \underbrace{\alpha p(1-p)}_{\text{logistic growth}}$$

where D = diffusion coefficient,
α = intensity of selection.

Let $P(z) = p(x,t)$ where $z = x - ct$.

Then,

$$-c\frac{dP}{dz} = D\frac{d^2P}{dz^2} + \alpha P(1-P).$$

Let $-S = \frac{dP}{dz}$.

Then,

$$\frac{dP}{dz} = -S$$
$$\frac{dS}{dz} = \frac{\alpha}{D}P(1-P) - \frac{c}{D}S \qquad (*)$$

P nullcline: $S = 0$

S nullcline: $S = \frac{\alpha}{c}P(1-P)$

Steady states: $(\bar{P}, \bar{S}) = (0,0)$ and $(1,0)$.

Jacobian of $(*)$

$$J = \begin{pmatrix} 0 & -1 \\ \frac{\alpha}{D}(1-2P) & -\frac{c}{D} \end{pmatrix}$$

At steady state $(0,0)$, $J = \begin{pmatrix} 0 & -1 \\ \frac{\alpha}{D} & -\frac{c}{D} \end{pmatrix}$

$\Rightarrow$ eigenvalues are $\lambda = \dfrac{-\frac{c}{D} \pm \sqrt{\frac{c^2}{D^2} - \frac{4\alpha}{D}}}{2}$

So, for $\frac{c^2}{D^2} - \frac{4\alpha}{D} \geq 0 \iff c \geq 2\sqrt{\alpha D}$, the point $(0,0)$ is a stable node.

For $c < 2\sqrt{\alpha D}$, the point $(0,0)$ is a stable spiral.

($c=0$ gives a center.)

$\boxed{c_{min} = 2\sqrt{\alpha D} \text{ is called the minimum wave speed for Fisher's equation.}}$

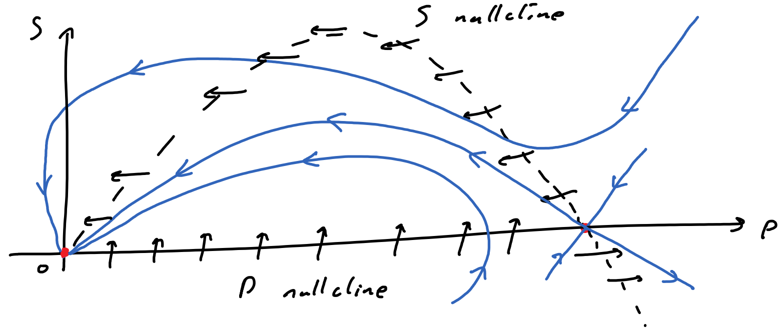
At steady state $(1,0)$,

$$J = \begin{pmatrix} 0 & -1 \\ -\frac{\alpha}{D} & -\frac{c}{D} \end{pmatrix}.$$

So, eigenvalues $\lambda = \dfrac{-\frac{c}{D} \pm \sqrt{\frac{c^2}{D^2} + \frac{4\alpha}{D}}}{2}$,

which means $(1,0)$ is a saddle.

Consider the case where $c \geq c_{min}$, so that $(0,0)$ is a stable node.

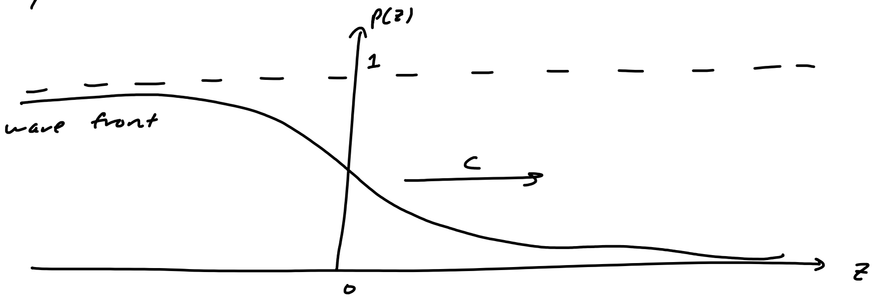


The heteroclinic trajectory from $(1,0)$ to $(0,0)$ has the following properties:

$P(z) \longrightarrow 1$ as $z \longrightarrow -\infty$

$P(z) \to 0$ as $z \longrightarrow \infty$

Also $0 < P < 1$ for $-\infty < z < \infty$



What about the other trajectories in the phase plane and what about the case when $(0,0)$ is a spiral or a center?

These cases lead to values of P that are >1 or <0, which are biologically unrealistic.

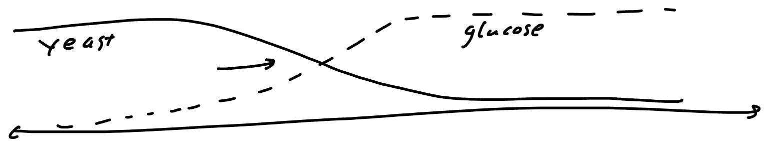
Regardless of the biology, if you start with initial conditions between 0 and 1, you will stay there.

Hence, any travelling wave solutions must move at speeds

$$c \geq c_{min} = 2\sqrt{\alpha D}.$$

Spreading colonies of microorganisms

We consider a model of yeast spreading through a nutrient agar of glucose. Consider the 1-D case.



$g(x,t)$ = concentration of glucose in the medium at x at time t

$n(x,t)$ = density of yeast cells at x at time t

Assume yeast cells undergo only very slight random motion and only multiply when enough glucose is available.

A simple model is

$$\frac{\partial n}{\partial t} = D_0 \frac{\partial^2 n}{\partial x^2} + kn(g - g_1)$$

$$\frac{\partial g}{\partial t} = D \frac{\partial^2 g}{\partial x^2} - ckn(g - g_1)$$

g_1 represents the min amount of glucose necessary for proliferation

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k = proportional reproduction rate

c = consumption rate

D_0, D_1 = diffusion coefficients of yeast and glucose.