Fick's laws (for diffusion)

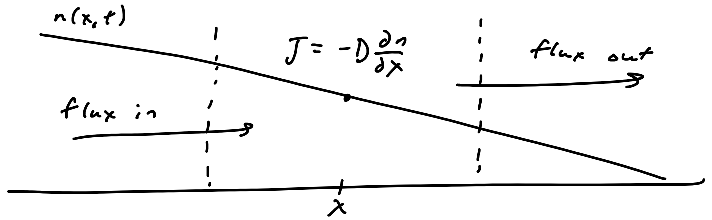
Fick's 1st law:

Diffusive flux goes from regions of high to low concentration with a magnitude proportional to the concentration gradient.

In 1-D,

flux $J_{Diffusion} = -D \frac{\partial n}{\partial x}$

where D is the diffusive coefficient.



Mass conservation:

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x} J = 0 \quad \left(\text{or} \quad \frac{\partial n}{\partial t} + \nabla \cdot J = 0 \text{ in higher dimensions} \right)$$

So,

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x} J_{Diffusion} = 0$$

$$\Rightarrow \frac{\partial n}{\partial t} = \underbrace{\frac{\partial}{\partial x}\left(D \frac{\partial n}{\partial x} \right)}_{\text{flux part}} \quad \text{in 1-D.}$$

Density-dependent diffusion

Some animals diffuse differently in response to population pressure. One extension of the classical diffusion model that is relevant to insect dispersal is when there is an increase in diffusion due to population pressure.

For example, the flux

$$J = -D(n)\nabla n$$

where $\frac{dD}{dn} > 0$.

A typical form is $D(n) = D_0 \left(\frac{n}{n_0}\right)^m$

where $m > 0$, $D_0 > 0$, $n_0 > 0$.

The dispersal equation is

$$\frac{\partial n}{\partial t} = D_0 \nabla \cdot \left(\left(\frac{n}{n_0}\right)^m \nabla n \right)$$

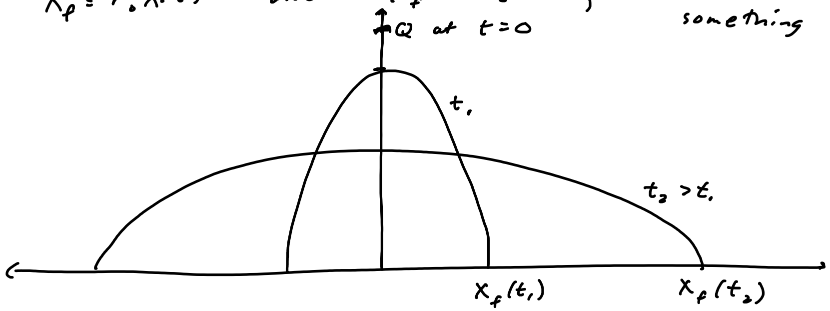
or in 1-D,

$$\frac{\partial n}{\partial t} = D_0 \frac{\partial}{\partial x} \left(\underbrace{\left(\frac{n}{n_0}\right)^m}_{\text{not constant with respect to } x} \frac{\partial n}{\partial x} \right)$$

with initial condition $n(x,0) = Q\delta_0(x)$, point mass of size Q at $x = 0$ at time 0.

The solution is fundamentally different from that of the classical case when $m=0$, because $D(0)=0$. The solution has a wave front at

$x_f = r_0 \lambda(t)$ and $-x_f = -r_0 \lambda(t)$, where $\lambda(t)$ is something messy.



Chemotaxis

Many insects, animals, cells, etc are attracted up or down a chemical gradient of pheremones, cytokines, etc.

Motion up a chemical concentration is called chemotaxis.

Suppose the presence of a gradient in an attractant $a(x,t)$ gives rise to the movement of individuals up the gradient.

Also, assume the flux of individuals will increase with the number of individuals $n(x,t)$.

So, we may reasonably take the chemotactic flux to be

$$J = n \, \chi(a) \, \underbrace{\nabla a}$$

population ← gradient of chemoattractant

where $\chi(a)$ is a function of a called the chemotactic coefficient.

Tuesday, September 12, 2017 10:33 AM

The "pure" chemotaxis equation is

$$\frac{\partial n}{\partial t} + \nabla \cdot J = 0$$

$$\Rightarrow \frac{\partial n}{\partial t} + \nabla \cdot \left(n \chi(a) \nabla a \right) = 0$$

Suppose we also have diffusion and a growth term $f(n)$, then the basic reaction-diffusion-chemotaxis equation is

$$\frac{\partial n}{\partial t} = f(n) - \nabla \cdot \left(n \chi(a) \nabla a \right) + \nabla \cdot (D \nabla n) .$$

Assume attractant $a(x,t)$ is governed by

$$\frac{\partial a}{\partial t} = \underbrace{g(a,n)}_{\substack{\text{production} \\ \text{+ decay}}} + \underbrace{\nabla \cdot (D_a \nabla a)}_{\text{diffusion}}$$

The classical Keller-Segel model (1971) for a slime mould is

$$\frac{\partial n}{\partial t} = D \frac{\partial^2 n}{\partial x^2} - \chi_0 \frac{\partial}{\partial x}\left(n \frac{\partial a}{\partial x} \right)$$

$$\frac{\partial a}{\partial t} = \underbrace{hn}_{\text{production}} - \underbrace{ka}_{\text{decay}} + D_a \frac{\partial^2 a}{\partial x^2}$$

A problem is that this model can give rise to solutions that blow up in finite time.

Chemotaxis modelling and analysis is a big are in mathematical bio.

People try to "fix" the blow up problem by using

- volume-filling method: there is a max density of cells/individuals $n(x,t)$ at any point.
- one could saturate the effect of the gradient, so that extremely steep gradients don't attract better than very steep gradients.