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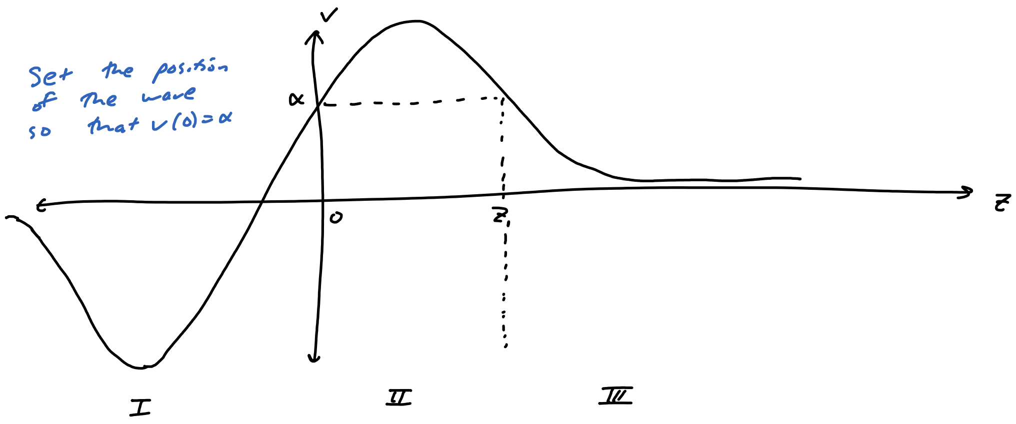
We look for solutions of the form

Set the position of the wave so that $v(0)=\alpha$




We assume that $v$ and $w$ are continuous and that $v$ has a continuous derivative at $z=0$ and $z=z_1$.

In regions I and III,

$$\varepsilon^2 v'' + c\varepsilon v' - v - w = 0$$
$$c w' + v = 0$$

Looking for solutions $v = Ae^{\lambda z}$ and $w = Be^{\lambda z}$, we find

$$\begin{bmatrix} \varepsilon^2\lambda^2 + \varepsilon c\lambda - 1 & -1 \\ 1 & c\lambda \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ characteristic equation

$$\det \begin{vmatrix} \varepsilon^2\lambda^2 + \varepsilon c\lambda - 1 & -1 \\ 1 & c\lambda \end{vmatrix} = 0$$

$$\Rightarrow \varepsilon^2 p(\lambda) = \varepsilon^2\lambda^3 + \varepsilon c\lambda^2 - \lambda + \frac{1}{c} = 0$$

This polynomial has exactly
1 negative real root $\lambda$ and
2 other roots $\lambda_2$ and $\lambda_3$ with positive real parts.

(You can try different kinds of roots to see that this is true.)

In region II, we have

$$\varepsilon^2 v'' + c\varepsilon v' + 1 - v - w = 0$$
$$c w' + v = 0$$

The inhomogeneous solution is $w=1$, $v=0$, and the homogeneous solution is a sum of $e^{\lambda_i z}$, $i=1,2,3$.

Since we want the solution to approach $0$ as $z \to \pm\infty$, the travelling pulse solution has the form

$$w(z) = \begin{cases} A e^{\lambda_1 z} & \text{for } z \geq z_1 \\ 1 + \sum_{i=1}^{3} B_i e^{\lambda_i z} & \text{for } 0 \leq z \leq z_1 \\ \sum_{i=2}^{3} C_i e^{\lambda_i z} & \text{for } z \leq 0 \end{cases}$$

and $v = -c w_z$.

We also require $w, v$ and $v_z$ to be continuous at $z=0, z_1$ and that $v(0) = v(z_1) = \alpha$.

So, we have 6 continuity conditions and 2 constraints to help us find 8 unknown coefficients $A, B_i, C_i$ and parameters $c$ and $z_1$.

After much calculation, we get

$$e^{\lambda_1 z_1} + \varepsilon^2 p'(\lambda_1)\alpha - 1 = 0 \qquad (1)$$

and

$$\frac{e^{-\lambda_2 z_1}}{p'(\lambda_2)} + \frac{e^{-\lambda_3 z_1}}{p'(\lambda_3)} + \frac{1}{p'(\lambda_1)} + \varepsilon^2 \alpha = 0$$

Let $s = e^{\lambda_1 z_1}$ and get

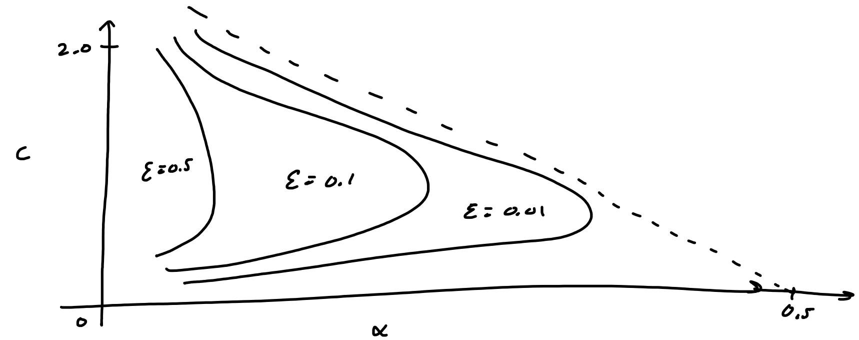
$$h(s) := 2 - s + \frac{p'(\lambda_1)}{p'(\lambda_2)} e^{-\lambda_2 \log(s)/\lambda_1} + \frac{p'(\lambda_1)}{p'(\lambda_2)} e^{-\lambda_3 \log(s)/\lambda_1}$$

$$= 0 \qquad (2)$$

(1) relates $\alpha$ to $z_1$

(2) relates $c$ to $z_1$

So, we can numerically relate wavespeed $c$ to $\alpha$.




For each $\alpha$ and $\varepsilon$ small enough, there are 2 travelling pulses (with 2 different wavespeeds $c$). That means there are 2 travelling pulses, a fast and a slow one.

For large $\alpha$, there are no pulses.

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## A generalised Keller-Segel model of chemotaxis

cell $$u_t = \nabla \cdot \big(\underbrace{\rho(u,v)\nabla u}_{\text{diffusion part}} - \underbrace{\varphi(u,v)\nabla v}_{\text{chemotaxis part}}\big)$$

chemical $$v_t = D_v \nabla^2 v + \mu u - \delta v$$

## Volume-filling mechanism

- cells have a nonzero size
- cell motion is limited in regions occupied by other cells

We introduce a "squeeze probability" $q(u)$.

$q(u)$ = prob. of entering a neighbouring space given a local density $u$

Let $\tilde{u}$ be some finite crowding capacity, i.e., the max density of cells at any point.

We require the following properties:

$q(\tilde{u}) = 0$

$0 \le q(u) \le 1$ for $0 \le u < \tilde{u}$

$q'(u) \le 0$ for $0 \le u < \tilde{u}$

For the generalised model with squeeze probability $q(u)$, we will derive that

$$\boxed{u_t = \nabla \cdot \Big[D_u\big(q(u) - u\,q'(u)\big)\nabla u - \chi u\, q(u)\nabla v\Big].}$$

For notational convenience, let

This is the 1-D case

$u = u(x,t)$,

$u_- = u(x-\Delta x, t)$,

$u_+ = u(x+\Delta x, t)$,

$w = \dfrac{\partial v}{\partial x}(x,t)$,

$w_- = \dfrac{\partial v}{\partial x}(x-\Delta x, t)$,

$w_+ = \dfrac{\partial v}{\partial x}(x+\Delta x, t)$,

$q = q(u)$,

$q_- = q(u_-)$,

$q_+ = q(u_+)$.

At each point $x$, a cell has a prob. $\alpha$ of <u>trying</u> to move left and a prob. $\beta$ of <u>trying</u> to move right in the next time step $\Delta t$.