Discussion Question 4

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + f(u) \qquad \text{no-flux BCs.}$$

1. $u_t = f(u)$ $\quad u_0$

$w = u - u_0$

$\Rightarrow \quad w_t = f_u(u_0) w$

$f_u(u_0) < 0$

2. $u_t = D\frac{\partial^2 u}{\partial x^2} + f(u)$

$$\Rightarrow \quad w_t = D\frac{\partial^2 w}{\partial x^2} + f_u(u_0) w \quad \text{— (A)}$$

$$w(x,t) = \sum_k c_k \underline{e^{\lambda_k t} W_k(x)}$$

$$D\frac{\partial^2 W_k}{\partial x^2} + k^2 W_k = 0 \quad \text{— (B)}$$

$$\frac{\partial}{\partial t}\left(e^{\lambda_k t} W_k(x)\right) = f_u(u_0) e^{\lambda_k t} W_k(x) + D\frac{\partial^2}{\partial x^2}\left(e^{\lambda_k t} W_k(x)\right)$$

$$\lambda_k e^{\lambda_k t} W_k(x) = f_u(u_0) e^{\lambda_k t} W_k(x) + \underbrace{D\frac{\partial^2 W_k(x)}{\partial x^2}}_{= -k^2 W_k(x)} e^{\lambda_k t}$$

$$\lambda_k = f_u(u_0) + -k^2$$

$$\mathrm{Re}(\lambda_k) > 0 \quad \Rightarrow \quad f_u(u_0) - k^2 > 0$$

$$\Rightarrow \quad k^2 < f_u(u_0) < 0 \quad ⚡$$

∂_x^2

For convenience, let

$$\hat{g}(x,t) = g(x,t) - g_i$$

and drop the ^ notation.

Also, yeast moves very slowly compared to the rate of glucose diffusion, so let us assume that yeast is non motile, i.e. $D_0 = 0$.

So, we rewrite the equations as

$$\frac{\partial n}{\partial t} = kng$$

$$\frac{\partial g}{\partial t} = D\frac{\partial^2 g}{\partial x^2} - ckng$$

Let $N(z) = n(x,t)$ and $G(z) = g(x,t)$ where $z = x - vt$.

Then,

$$-v\frac{dN}{dz} = kNG \qquad (1)$$

$$-v\frac{dG}{dz} = D\frac{d^2G}{dz^2} - ckNG \qquad (2)$$

We could turn this into a 3-D system, or we could take $c \cdot (1) + (2)$

$$\Rightarrow \quad -vc\frac{dN}{dz} - v\frac{dG}{dz} = D\frac{d^2G}{dz^2}$$

Integrating, we get

$$\int_{-\infty}^{z}\left(-vc\frac{dN}{dz} - v\frac{dG}{dz}\right)dz = \int_{-\infty}^{z} D\frac{d^2G}{dz^2}\,dz$$

$$-vcN - vG\Big|_{-\infty}^{z} = D\frac{dG}{dz}\Big|_{-\infty}^{z}$$

$$-vcN(z) - vG(z) + vcN(-\infty) + vG(-\infty) = D\frac{dG(z)}{dz} - D\frac{dG(-\infty)}{dz}.$$

Assume that at $z=-\infty$, $N(z)=N_0$ where N_0 is a limiting density of yeast. (It's limited by available glucose.) Also, assume $G(-\infty)=0$ and

$$\frac{dG(-\infty)}{dz} = 0.$$

So, $$vcN_0 - vcN - vG = D\frac{dG}{dz}$$

So, from this equation and 1, we have

$$\frac{dG}{dz} = -\frac{v}{D}G - \frac{vcN}{D} + \frac{vcN_0}{D}$$

$$\frac{dN}{dz} = -\frac{kNG}{v}$$

N nullcline: $N=0$, $G=0$

G nullcline: $G = c(N_0 - N)$

Steady states: $(\bar{G}, \bar{N}) = (0, N_0)$, $(cN_0, 0)$.

Jacobian $J = \begin{bmatrix} -\frac{v}{D} & -\frac{vc}{D} \\ -\frac{kN}{v} & -\frac{kG}{v} \end{bmatrix}$.

At $(0, N_0)$,

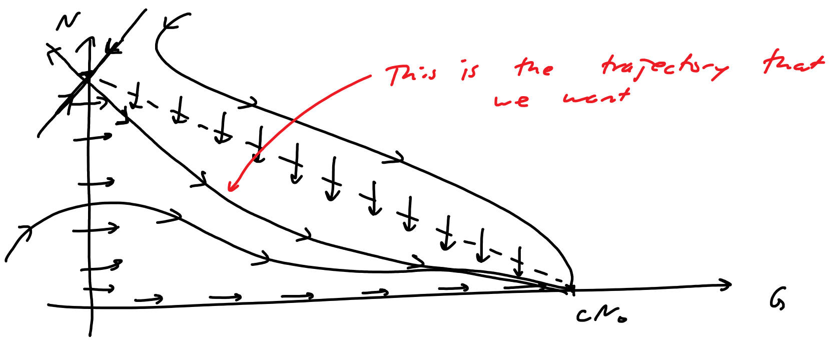
$\det(J - \lambda I) = \lambda^2 + \frac{v}{D}\lambda - \frac{kcN_0}{D} = 0$

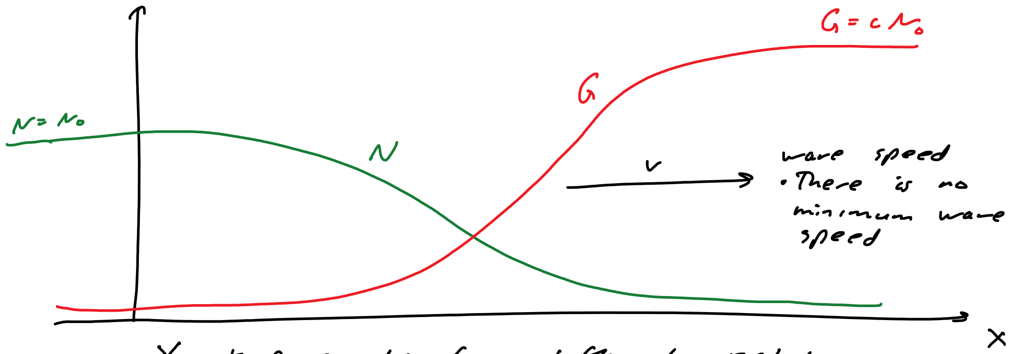
$\Rightarrow \lambda = \dfrac{-\frac{v}{D} \pm \sqrt{\frac{v^2}{D^2} + \frac{4kcN_0}{D}}}{2} \Rightarrow$ saddle

At $(cN_0, 0)$,

$J = \begin{bmatrix} -\frac{v}{D} & -\frac{vc}{D} \\ 0 & -\frac{ckN_0}{v} \end{bmatrix}$

$\Rightarrow \lambda = -\frac{v}{D}, \quad \lambda = \frac{-ckN_0}{v} \Rightarrow$ stable node





wave speed
- There is no minimum wave speed

Yeast propagates from left to right, consuming glucose.

Travelling Pulses

A travelling wave solution that starts and ends at the same steady state of the governing equations.

FitzHugh-Nagumo Equations

$$\varepsilon \frac{\partial v}{\partial t} = \varepsilon^2 \frac{\partial^2 v}{\partial x^2} + f(v,w)$$

$$\frac{\partial w}{\partial t} = g(v,w)$$

where $\varepsilon > 0$ is assumed to be very small.

Apply travelling wave coordinates $z = x - ct$, where $c > 0$.

we get

$$\varepsilon^2 v_{zz} + c\varepsilon v_z + f(v,w) = 0$$

$$c w_z + g(v,w) = 0.$$

Let us examine the simplified, piecewise linear case $f(v,w) = H(v-\alpha) - v - w$

$g(v,w) = v$

where $H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0. \end{cases}$

In this case, we can construct exact solutions. The steady state where $f(v,w) = g(v,w) = 0$ is $(0,0)$.