HARISH-CHANDRA MODULES FOR YANGIANS

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Abstract. We study Harish-Chandra representations of the Yangian $\mathrm{Y}(\mathfrak{gl}_2)$ which admit a decomposition with respect to a natural maximal commutative subalgebra Γ and satisfy a polynomial condition. We prove an analogue of Kostant theorem showing that the restricted Yangian $\mathrm{Y}_p(\mathfrak{gl}_2)$ is a free module over Γ and show that every character of Γ defines a finite number of irreducible Harish-Chandra modules. We study the categories of generic Harish-Chandra modules, describe their simple modules and indecomposable modules in tame blocks.

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Contents

1. Introduction

Throughout the paper we fix an algebraically closed field $\Bbbk$ of characteristic 0.

The notion of a Harish-Chandra module with respect to a certain subalgebra is one of the most important in the representation theory of Lie algebras ([Di]). For example, weight modules are Harish-Chandra modules with respect to a Cartan subalgebra. Also the Gelfand-Tsetlin modules ([DFO1]) over the universal enveloping algebra $U(\mathfrak{gl}_n)$ of the general linear Lie algebra $\mathfrak{gl}_n$ are Harish-Chandra modules with respect to a subalgebra generated by the centers of $U(\mathfrak{gl}_k)$, $k = 1, \ldots, n$ where $\mathfrak{gl}_1 \subset \ldots \subset \mathfrak{gl}_n$. In [DFO2] a general setting has been developed for Harish-Chandra modules over associative algebras. Let U be an associative $\Bbbk$-algebra, $U-\mathrm{mod}$ be the category of finitely generated left U-modules and $\Gamma \subset U$ be a subalgebra. Denote by $\mathrm{cfs}(\Gamma)$ a cofinite spectrum of Γ, i.e. the set of maximal two-sided ideals of Γ of finite codimension. A module $M \in U-\mathrm{mod}$ is called Harish-Chandra module (with respect to Γ) if $M = \oplus_{\mathbf{m}\in \mathrm{cfs}\,\Gamma} M(\mathbf{m})$, where

$$M(\mathbf{m}) \;=\; \{x \in M \mid \text{ there exists } k \geqslant 0, \text{ such that } \mathbf{m}^k x = 0\}.$$

A key problem in the classification of all irreducible Harish-Chandra modules is to study the liftings from a given $\mathbf{m} \in \mathrm{cfs}(\Gamma)$ to irreducible Harish-Chandra modules M with $M(\mathbf{m}) \neq 0$. When such lifting is unique then irreducible Harish-Chandra modules are parametrized by the elements of $\mathrm{cfs}(\Gamma)$. In the case of Gelfand-Tsetlin modules over $\mathfrak{gl}_n$ it was shown in [Ov] that the number of nonisomorphic irreducible modules defined by a given $\mathbf{m} \in \mathrm{cfs}(\Gamma)$ is always nonzero and finite.

In this paper we begin a systematic study of Harish-Chandra modules over the Yangians.

The *Yangian for* $\mathfrak{gl}_n$ is a unital associative algebra $\mathrm{Y}(\mathfrak{gl}_n)$ over $\Bbbk$ with countably many generators $t_{ij}^{(1)},\ t_{ij}^{(2)},\dots$ where $1 \leq i, j \leq n$, and the defining relations

$$(u-v)\,[t_{ij}(u), t_{kl}(v)] = t_{kj}(u)\,t_{il}(v) - t_{kj}(v)\,t_{il}(u), \tag{1.1}$$

where

$$t_{ij}(u) = \delta_{ij} + t_{ij}^{(1)} u^{-1} + t_{ij}^{(2)} u^{-2} + \cdots \tag{1.2}$$

and u, v are formal variables. This algebra originally appeared in the works on the *quantum inverse scattering method*; see e.g. Takhtajan–Faddeev [TF], Kulish–Sklyanin [KS]. The term "Yangian" and generalizations of $\mathrm{Y}(\mathfrak{gl}_n)$ to an arbitrary simple Lie algebra were introduced by Drinfeld [D1]. He then classified finite-dimensional irreducible modules over the Yangians in [D2] using earlier results of Tarasov [T1, T2] for the $\mathfrak{sl}_2$ case. An explicit construction of all such modules over $\mathrm{Y}(\mathfrak{sl}_2)$ is given in those papers by Tarasov and also in the work by Chari and Pressley [CP]. Apart from this case, the structure of a general Yangian representation remains unknown. In the case of $\mathrm{Y}(\mathfrak{gl}_n)$ a description of "generic" modules was given in [M1] via Gelfand–Tsetlin bases. A more general class of "tame" representations of $\mathrm{Y}(\mathfrak{gl}_n)$ was introduced and explicitly constructed by Nazarov and Tarasov [NT]. An important role in these works is played by the *Drinfeld generators* [D2]

$$a_i(u), \quad i = 1, \dots, n, \qquad b_i(u), \quad c_i(u), \quad i = 1, \dots, n-1 \tag{1.3}$$

of the algebra $\mathrm{Y}(\mathfrak{gl}_n)$ which are defined as certain *quantum minors* of the matrix $T(u) = \big(t_{ij}(u)\big)$. The coefficients of the series $a_i(u)$, $i = 1, \dots, n$ form a commutative subalgebra of $\mathrm{Y}(\mathfrak{gl}_n)$ which can be regarded as an analogue of a Gelfand-Tsetlin subalgebra of the universal enveloping algebra of $\mathfrak{gl}_n$ [DFO1] We shall call a representation of $\mathrm{Y}(\mathfrak{gl}_n)$ *Harish-Chandra* if it is a Harish-Chandra module with respect to this subalgebra. In particular, finite-dimensional Harish-Chandra modules are precisely the tame modules of [NT]. Note that Harish-Chandra modules for $\mathrm{Y}(\mathfrak{gl}_n)$ are analogs of Gelfand-Tsetlin modules for $\mathfrak{gl}_n$ [DFO1].

In this paper we are concerned with Harish-Chandra representations of the Yangian $\mathrm{Y}(\mathfrak{gl}_2)$. Recall that every irreducible finite-dimensional $\mathrm{Y}(\mathfrak{gl}_2)$-module contains a unique vector ξ annihilated by $t_{12}(u)$ and which is an eigenvector for the Drinfeld generators $a_1(u)$ and $a_2(u)$ defined by

$$a_1(u) = t_{11}(u)\,t_{22}(u-1) - t_{21}(u)\,t_{12}(u-1), \qquad a_2(u) = t_{22}(u); \tag{1.4}$$

see [T1, T2] and [CP]. Moreover, there exists an automorphism $t_{ij}(u) \mapsto c(u)\, t_{ij}(u)$ of $\mathrm{Y}(\mathfrak{gl}_2)$, where $c(u) \in 1 + u^{-1}\,\Bbbk[[u^{-1}]]$, such that the eigenvalues of ξ become polynomials in u^{-1} under the twisted action of the Yangian. This prompts the introduction of the class of *Harish-Chandra polynomial* modules over $\mathrm{Y}(\mathfrak{gl}_2)$, i.e., such Harish-Chandra modules where the operators $a_1(u)$ and $a_2(u)$ are polynomials. More precisely, by (1.4) it is natural to require that for some positive integer p the polynomials $a_1(u)$ and $a_2(u)$ have degrees $2p$ and p, respectively. Note that $a_1(u)$ is the *quantum determinant* of the matrix $T(u)$ [IK], [KS]. Its coefficients are algebraically independent generators of the center of $\mathrm{Y}(\mathfrak{gl}_2)$.

We can interpret the definition of Harish-Chandra polynomial modules using the algebra $\mathrm{Y}_p(\mathfrak{gl}_2)$ called the *Yangian of level* p; see Cherednik [C1, C2]. It is defined as the quotient of $\mathrm{Y}(\mathfrak{gl}_2)$ by the ideal generated by the elements $t_{ij}^{(r)}$ with $r \geq p+1$. A Harish-Chandra polynomial module over $\mathrm{Y}(\mathfrak{gl}_2)$ is just a Harish-Chandra module over $\mathrm{Y}_p(\mathfrak{gl}_2)$ for some positive integer p.

For another interpretation consider the *Yangian for* $\mathfrak{sl}_2$ which is the subalgebra $\mathrm{Y}(\mathfrak{sl}_2)$ of $\mathrm{Y}(\mathfrak{gl}_2)$ generated by the coefficients of the series $e(u)$, $f(u)$ and $h(u)$ [D2] defined by

$$\begin{aligned} e(u) &= t_{22}(u)^{-1} t_{12}(u), \\ f(u) &= t_{21}(u)\, t_{22}(u)^{-1}, \\ h(u) &= t_{11}(u)\, t_{22}(u)^{-1} - t_{21}(u)\, t_{22}(u)^{-1} t_{12}(u)\, t_{22}(u)^{-1}. \end{aligned} \tag{1.5}$$

Note that the series $h(u)$ can also be given by

$$h(u) = a_1(u)\, a_2(u)^{-1} a_2(u-1)^{-1} \tag{1.6}$$

so that the coefficients of $h(u)$ form a commutative subalgebra of $\mathrm{Y}(\mathfrak{sl}_2)$. Therefore, the restriction of a Harish-Chandra $\mathrm{Y}(\mathfrak{gl}_2)$-module to $\mathrm{Y}(\mathfrak{sl}_2)$ admits an eigenbasis for this subalgebra. We also point out that both the above interpretations extend to an arbitrary Yangian $\mathrm{Y}(\mathfrak{gl}_n)$.

In this paper we study Harish-Chandra polynomial modules over $\mathrm{Y}(\mathfrak{gl}_2)$. We consider the class of modules admitting a central character so that the coefficients of $a_1(u)$ act as scalars. This class contains all irreducible Harish-Chandra polynomial modules. We study the properties of the subalgebra Γ of $\mathrm{Y}(\mathfrak{gl}_2)$ generated by the coefficients of $a_1(u)$ and $a_2(u)$. In particular we show that $\mathrm{Y}(\mathfrak{gl}_2)$ is free as a left and as a right Γ-module (Theorem 1) which is an analogue of Kostant theorem [K]. Moreover, we show that Γ is a Harish-Chandra subalgebra (Theorem 3) in the sense of [DFO2] and that each character of Γ extends to a finitely many non-isomorphic irreducible $\mathrm{Y}(\mathfrak{gl}_2)$-modules (Theorem 4). This gives an equivalence between the category $\mathbb{H}(\mathrm{Y}(\mathfrak{gl}_2), \Gamma)$ of Harish-Chandra polynomial modules and the category of finitely generated modules over a certain category $\mathcal{A}$ whose objects are the maximal ideals of Γ. A full subcategory $\mathbb{H}W(\mathrm{Y}(\mathfrak{gl}_2), \Gamma)$ consisting of weight polynomial Harish-Chandra modules, when the action of $a_2(u)$ is diagonalizable, is equivalent to the category of finitely generated modules over a certain quotient category of $\mathcal{A}$ (see Section 2.1 for details). An important role in our study is played by certain universal weight polynomial Harish-Chandra modules (Section 3, Theorem 2) such that every irreducible module in $\mathbb{H}W(\mathrm{Y}(\mathfrak{gl}_2), \Gamma)$ is a quotient of the corresponding universal module. In section 7 we study a full subcategory in $\mathbb{H}W(\mathrm{Y}(\mathfrak{gl}_2), \Gamma)$ of generic modules, this imposes a certain integrability condition on the eigenvalues of $a_2(u)$ while those of $a_1(u)$ are arbitrary. In particular, we give a complete description of irreducible modules (Theorem 5) and indecomposable modules in tame blocks of this category (Theorem 6).

2. Preliminaries

2.1. **Harish-Chandra subalgebras.** In the setting of [DFO2] the subalgebra Γ need not to be commutative. But in this paper we will only deal with the commutative case, hence $\mathrm{cfs}(\Gamma)$ coincides with the set $\operatorname{Specm}\Gamma$ of all maximal ideals in Γ.

When for all $\mathbf{m} \in \operatorname{Specm}\Gamma$ and all $x \in M(\mathbf{m})$ holds $\mathbf{m}x = 0$ such Harish-Chandra module M is called weight (with respect to Γ).

All Harish-Chandra modules (with respect to Γ) form a full abelian subcategory in the category of $U-\mathrm{mod}$ which we will denote by $\mathbb{H}(U,\Gamma)$. A full subcategory of $\mathbb{H}(U,\Gamma)$ consisting of weight modules we denote by $\mathbb{H}W(U,\Gamma)$. The support of a Harish-Chandra module M is a set $\operatorname{Supp} M \subset \operatorname{Specm}\Gamma$ consisting of such $\mathbf{m}$ that $M(\mathbf{m}) \neq 0$. For $D \subset \operatorname{Specm}\Gamma$ denote by $\mathbb{H}(U,\Gamma,D)$ the full subcategory in $\mathbb{H}(U,\Gamma)$ formed by M such that $\operatorname{Supp} M \subset D$. For a given $\mathbf{m} \in \operatorname{Specm}\Gamma$ let $\chi_{\mathbf{m}} : \Gamma \to \Gamma/\mathbf{m}$ be a character of Γ. If there exists an irreducible Harish-Chandra module M with $M(\mathbf{m}) \neq 0$ then we say that $\chi_{\mathbf{m}}$ extends to M.

The notion of a Harish-Chandra subalgebra ([DFO2]) gives an effective tool for the study of the category $\mathbb{H}(U,\Gamma)$. A commutative subalgebra $\Gamma \subset U$ is called a Harish-Chandra subalgebra in U if for any $a \in U$ the $\Gamma-$bimodule $\Gamma a \Gamma$ is finitely generated as left and as right $\Gamma-$module. In this case for a finite-dimensional $\Gamma-$module X the module $U \otimes_\Gamma X$ is a Harish-Chandra module.

For $a \in U$ let

$$X_a \;=\; \{(\mathbf{m},\mathbf{n}) \in \operatorname{Specm}\Gamma \times \operatorname{Specm}\Gamma \mid \Gamma/\mathbf{n} \quad \text{is a subquotient of} \quad \Gamma a\Gamma/\Gamma a\mathbf{m}\}.$$

Equivalently, $(\mathbf{m},\mathbf{n}) \in X_a$ if and only if $(\Gamma/\mathbf{n}) \otimes_\Gamma \Gamma a \Gamma \otimes_\Gamma (\Gamma/\mathbf{m}) \neq 0$. Denote by Δ the minimal equivalence on $\operatorname{Specm}\Gamma$ containing all X_a, $a \in U$ and by $\Delta(A,\Gamma)$ the set of the $\Delta-$equivalence classes on $\operatorname{Specm}\Gamma$. Then for any $a \in U$ and $\mathbf{m} \in \operatorname{Specm}\Gamma$ holds

$$aM(\mathbf{m}) \subset \sum_{(\mathbf{m},\mathbf{n}) \in X_a} M(\mathbf{n}), \quad \mathbb{H}(U,\Gamma) = \bigoplus_{D \in \Delta(U,\Gamma)} \mathbb{H}(U,\Gamma,D). \tag{2.7}$$

Define a category $\mathcal{A} = \mathcal{A}_{U,\Gamma}$ with $\operatorname{Ob}\mathcal{A} = \Gamma$ and the space of morphisms from $\mathbf{m}$ to $\mathbf{n}$ being

(2.8)
$$\mathcal{A}(\mathbf{m},\mathbf{n}) \;=\; \lim_{\substack{\longleftarrow \\ n,m}} U/(\mathbf{n}^n U + U\mathbf{m}^m) \quad (\text{ equivalently } \lim_{\substack{\longleftarrow \\ n,m}} \Gamma/\mathbf{n}^n \otimes_\Gamma U \otimes_\Gamma \Gamma/\mathbf{m}^m).$$

Then we have $\mathcal{A} = \bigoplus_{D \in \Delta(U,\Gamma)} \mathcal{A}(D)$, where $\mathcal{A}(D)$ is the restriction of $\mathcal{A}$ on D. The category $\mathcal{A}$ is endowed with the topology of the inverse limit and the category of $\Bbbk$-vector spaces ($\Bbbk-\mathrm{mod}$) with the discrete topology. Consider the category $\mathcal{A}-\mathrm{mod}_d$ of continuous functors $M : \mathcal{A} \longrightarrow \Bbbk-\mathrm{mod}$ (discrete modules in [DFO2], 1.5). For any discrete $\mathcal{A}-$module N define a Harish-Chandra $U-$module $\mathbb{F}(N) = \oplus_{\mathbf{m} \in \operatorname{Specm}\Gamma} N(\mathbf{m})$ and for $x \in N(\mathbf{m})$ and $a \in U$ define $ax = \sum_{\mathbf{n} \in \operatorname{Specm}\Gamma} a_{\mathbf{n}} x$ where $a_{\mathbf{n}}$ is the image of a in $\mathcal{A}(\mathbf{m},\mathbf{n})$. If $f : M \longrightarrow N$ is a morphism in $\mathcal{A}-\mathrm{mod}_d$ then define $\mathbb{F}(f) = \oplus_{\mathbf{m} \in \operatorname{Specm}\Gamma} f(\mathbf{m})$. Hence we have a functor $\mathbb{F} : \mathcal{A}-\mathrm{mod}_d \longrightarrow \mathbb{H}(U,\Gamma)$.

Proposition 2.1. *(*[DFO2]*, Theorem 17) The functor $\mathbb{F}$ is an equivalence.*

We will identify a discrete $\mathcal{A}$-module N with the corresponding Harish-Chandra module $\mathbb{F}(N)$. Let $\Gamma_{\mathbf{m}} = \lim\limits_{\leftarrow m} \Gamma/\mathbf{m}^m$ be the completion of Γ by $\mathbf{m} \in \operatorname{Specm} \Gamma$. Then the space $\mathcal{A}(\mathbf{m}, \mathbf{n})$ has a structure of $\Gamma_{\mathbf{n}} - \Gamma_{\mathbf{m}}$-bimodule.

For $\mathbf{m} \in \operatorname{Specm} \Gamma$ denote by $\hat{\mathbf{m}}$ a completion of $\mathbf{m}$. Consider a two-sided ideal $I \subset \mathcal{A}$ generated by $\hat{\mathbf{m}}$ for all $\mathbf{m} \in \operatorname{Specm} \Gamma$ and set $\mathcal{A}_W = \mathcal{A}/I$. Then Proposition 2.1 implies the following statement.

Corollary 1. *The categories $\mathbb{H}W(U, \Gamma)$ and $\mathcal{A}_W - \mathrm{mod}$ are equivalent.*

The subalgebra Γ is called big in $\mathbf{m} \in \operatorname{Specm} \Gamma$ if $\mathcal{A}(\mathbf{m}, \mathbf{m})$ is finitely generated as $\Gamma_{\mathbf{m}}-$module.

Lemma 2.1. *(*[DFO2]*, Corollary 19) If Γ is big in $\mathbf{m} \in \operatorname{Specm} \Gamma$ then there exist finitely many non-isomorphic irreducible Harish-Chandra $U-$modules M such that $M(\mathbf{m}) \neq 0$. For any such module $\dim M(\mathbf{m}) < \infty$.*

2.2. **Special PBW algebras.** Let U be an associative algebra over $\Bbbk$ endowed with an increasing filtration $\{U_i\}_{i \in \mathbb{Z}}$, $U_{-1} = \{0\}$, $U_0 = \Bbbk$, $U_i U_j \subset U_{i+j}$. For $u \in U_i \setminus U_{i-1}$ set $\deg u = i$. Let $\overline{U} = \operatorname{gr} U$ be the associated graded algebra $\overline{U} = \bigoplus_{i=0}^{\infty} U_i/U_{i-1}$. For $u \in U$ denote by $\overline{u}$ its image in $\overline{U}$ and for a subset $S \subset U$ set $\overline{S} = \{\overline{s} \mid s \in S\} \subset \overline{U}$. The algebra U is called a *special* PBW algebra if any element of U can be written uniquely as a linear combination of ordered monomials in some fixed generators of U and if $\overline{U}$ is a polynomial algebra. Such algebras were introduced in [FO].

Let $\Lambda = \Bbbk[X_1, \dots, X_n]$ be a polynomial algebra. For $g_1, \dots, g_t \in \Lambda$ denote by $\mathsf{V}(g_1, \dots, g_t)$ a set of all zeroes of the ideal generated by the elements $g_1, \dots, g_t$. A sequence $g_1, \dots, g_t \in \Lambda$ is called regular (in Λ) if the class of g_i in $\Lambda/(g_1, \dots, g_{i-1})$ is non-invertible and is not a zero divisor for any $i = 1, \dots, t$.

Next proposition contains the basic properties of regular sequences which can be easily checked or can be found in [BH].

Proposition 2.2.
1. *The sequence $X_1, \dots, X_r, G_1, \dots G_t$ with $G_1, \dots, G_t \in \Lambda$ is regular in Λ if and only if the sequence $g_1, \dots, g_t$ is regular in $\Bbbk[X_{r+1}, \dots, X_n]$, where $g_i(X_{r+1}, \dots, X_n) = G_i(0, \dots, 0, X_{r+1}, \dots, X_n)$.*
2. *A sequence $g_1, \dots g_t$ is regular in Λ if and only if the variety $V(g_1, \dots, g_t)$ is equidimensional of dimension $n - t$.*
3. *A sequence $g_1 g_1', g_2, \dots, g_t$ is regular if and only if the sequences $g_1, g_2, \dots, g_t$ and $g_1', g_2, \dots, g_t$ are regular.*

The following analogue of Kostant theorem ([K]) is valid for special PBW algebras.

Proposition 2.3. *(*[FO]*) Let U be a special PBW algebra and let $g_1, \dots, g_t \in U$ be mutually commuting elements such that $\overline{g}_1, \dots, \overline{g}_t$ is a regular sequence in $\overline{U}$, $\Gamma = \Bbbk[g_1, \dots, g_t]$. Then U is a free left (right) $\Gamma-$module. Moreover Γ is a direct summand of U.*

3. Freeness of $\mathrm{Y}_p(\mathfrak{gl}_2)$ over its commutative subalgebra

Let p be a positive integer. The *level p Yangian* $\mathrm{Y}_p(\mathfrak{gl}_2)$ for the Lie algebra $\mathfrak{gl}_2$ [C2] can be defined as the algebra over $\Bbbk$ with generators $t_{ij}^{(1)}, \dots, t_{ij}^{(p)}$, $i, j = 1, 2$, subject to the relations

$$[T_{ij}(u), T_{kl}(v)] = \frac{1}{u-v}(T_{kj}(u)T_{il}(v) - T_{kj}(v)T_{il}(u)), \tag{3.9}$$

where u, v are formal variables and

$$T_{ij}(u) = \delta_{ij}\, u^p + \sum_{k=1}^{p} t_{ij}^{(k)}\, u^{p-k} \in \mathrm{Y}_p(\mathfrak{gl}_2)[u]. \tag{3.10}$$

Explicitly, (3.9) reads

$$[t_{ij}^{(r)}, t_{kl}^{(s)}] = \sum_{a=1}^{\min(r,s)} \big(t_{kj}^{(a-1)} t_{il}^{(r+s-a)} - t_{kj}^{(r+s-a)} t_{il}^{(a-1)}\big), \tag{3.11}$$

where $t_{ij}^{(0)} = \delta_{ij}$ and $t_{ij}^{(r)} = 0$ for $r \geq p+1$. Note that the level 1 Yangian $\mathrm{Y}_1(\mathfrak{gl}_2)$ coincides with the universal enveloping algebra $\mathrm{U}(\mathfrak{gl}_2)$. Set $\deg t_{ij}^{(k)} = k$ for $i, j,\ k = 1, \dots, p$. This defines a natural filtration on the Yangian $\mathrm{Y}_p(\mathfrak{gl}_2)$. The corresponding graded algebra will be denoted by $\overline{\mathrm{Y}}_p(\mathfrak{gl}_2)$. We have the following analog of the Poincaré–Birkhoff–Witt theorem for the algebra $\mathrm{Y}_p(\mathfrak{gl}_2)$.

Proposition 3.1. *(*[C2]*; see also* [M2]*) Given an arbitrary linear ordering on the set of the generators $t_{ij}^{(k)}$, any element of the algebra $\mathrm{Y}_p(\mathfrak{gl}_2)$ is uniquely written as a linear combination of ordered monomials in these generators. Moreover, the algebra $\overline{\mathrm{Y}}_p(\mathfrak{gl}_2)$ is a polynomial algebra in generators $\bar{t}_{ij}^{(k)}$.*

Proposition 3.1 implies that $\mathrm{Y}_p(\mathfrak{gl}_2)$ is a special PBW algebra. Denote by $D(u)$ the *quantum determinant*

$$\begin{aligned} D(u) &= T_{11}(u)T_{22}(u-1) - T_{21}(u)T_{12}(u-1) \\ &= T_{11}(u-1)T_{22}(u) - T_{12}(u-1)T_{21}(u). \end{aligned} \tag{3.12}$$

It was shown in [C1, C2] (see also [M2] for a different proof) that the coefficients of the polynomial $D(u)$ are algebraically independent generators of the center of the algebra $\mathrm{Y}_p(\mathfrak{gl}_2)$.

Denote by Γ the subalgebra of $\mathrm{Y}_p(\mathfrak{gl}_2)$ generated by the coefficients of $D(u)$ and $t_{22}^{(k)}$, $k = 1, \dots, p$. This algebra is obviously commutative. We will show later (Corollary 3) that Γ is a Harish-Chandra subalgebra in $\mathrm{Y}_p(\mathfrak{gl}_2)$.

Lemma 3.1. *The sequence $\bar{t}_{22}^{(1)}, \dots, \bar{t}_{22}^{(p)}, \bar{d}_1, \dots, \bar{d}_{2p}$ of the images of the generators of Γ is regular in $\overline{\mathrm{Y}}_p(\mathfrak{gl}_2)$.*

Proof. Denote $t_i = \bar{t}_{11}^{(i)} + \bar{t}_{22}^{(i)}$, $i = 1, \dots, p$, $\Delta_{i,j} = \bar{t}_{11}^{(i)}\bar{t}_{22}^{(j)} - \bar{t}_{21}^{(i)}\bar{t}_{12}^{(j)}$, $i, j = 1, \dots, p$, $i \neq j$. It follows from 3.12 that

$$\overline{D}(u) = u^{2p} + \sum_{i=1}^{2p} \bar{d}_i u^{2p-i},$$

where $\overline{d}_i = t_i + \sum_{j=1}^{i-1} \Delta_{j,i-j}$ for $i = 1, \ldots, p$ and $\overline{d}_i = \sum_{j=i-p}^{p} \Delta_{i,i-j}$ for $i = p+1, \ldots, 2p$. Hence we need to show that the sequence

$$\overline{t}_{22}^{(1)}, \ldots, \overline{t}_{22}^{(p)}, t_1, t_2 + \Delta_{11}, \ldots, t_p + \sum_{i=1}^{p-1} \Delta_{i,p-i}, \sum_{i=1}^{p} \Delta_{i,p+1-i}, \ldots, \Delta_{pp}$$

is regular. We will denote by ∇_i the result of the substitution $\overline{t}_{22}^{(1)} = \ldots = \overline{t}_{22}^{(p)} = 0$ in $\overline{d}_i$, $i = 1, \ldots, 2p$. By Proposition 2.2, (1) we only need to show the regularity of the sequence

$$\nabla_1, \ldots, \nabla_{2p}.$$

Consider the following triangular automorphism ϕ of $\overline{\mathrm{Y}}_p(\mathfrak{gl}_2)/I$: $\overline{t}_{11}^{(i)} \mapsto \overline{t}_{11}^{(i)} + \sum_{j=1}^{i-1} \Delta_{i,i-j}$, $\overline{t}_{21}^{(i)} \to \overline{t}_{21}^{(i)}$, $\overline{t}_{12}^{(i)} \to \overline{t}_{12}^{(i)}$, $i = 1, \ldots, p$, where I is an ideal generated by $\overline{t}_{22}^{(1)}, \ldots, \overline{t}_{22}^{(p)}$. Applying ϕ^{-1} to the sequence $\nabla_1, \ldots, \nabla_{2p}$ we see that it is enough to show the regularity of the sequence

$$\overline{t}_{11}^{(1)}, \ldots, \overline{t}_{11}^{(p)}, \nabla_{p+1}, \ldots, \nabla_{2p}.$$

Again by Proposition 2.2, (1) this is equivalent to the regularity of the sequence $\nabla_{p+1}, \ldots, \nabla_{2p}$. For each pair i, j, $i, j = 1, \ldots, p$, $i+j \geq p+1$ consider the following elements of $\mathbb{k}[\overline{t}_{12}^{(i)}, \overline{t}_{21}^{(i)} \,|\, i, j = p+1, \ldots, 2p]$ arranged in the table s_{ij} below

$$\begin{pmatrix} \overline{t}_{21}^{(i)}\overline{t}_{12}^{(j)} \\ \overline{t}_{21}^{(i-1)}\overline{t}_{12}^{(j)} + \overline{t}_{21}^{(i)}\overline{t}_{12}^{(j-1)} \\ \overline{t}_{21}^{(i-2)}\overline{t}_{12}^{(j)} + \overline{t}_{21}^{(i-1)}\overline{t}_{12}^{(j-1)} + \overline{t}_{21}^{(i)}\overline{t}_{12}^{(j-2)} \\ \vdots \\ \overline{t}_{21}^{(p+1-j)}\overline{t}_{12}^{(j)} + \overline{t}_{21}^{(p-j)}\overline{t}_{12}^{(j+1)} + \ldots\ldots + \overline{t}_{21}^{(i+1)}\overline{t}_{12}^{(p-i)} + \overline{t}_{21}^{(i)}\overline{t}_{12}^{(p+1-i)} \end{pmatrix}$$

Note that when $i = j = p$ the rows of the table are exactly the elements ∇_i, $i = p+1, \ldots, 2p$. We will show by induction on $i + j$ that the rows of this table form a regular sequence. Let $i + j = p + 1$. Then s_{ij} consists of the unique element $\overline{t}_{21}^{(i)}\overline{t}_{12}^{(j)}$ and the corresponding variety is obviously equidimensional. Hence the statement follows from Proposition 2.2, (2). Applying Proposition 2.2, (3) to the table above we obtain the following two tables s'_{ij} and s''_{ij}

$$\begin{pmatrix} \overline{t}_{21}^{(i)} \\ \overline{t}_{21}^{(i-1)}\overline{t}_{12}^{(j)} + \overline{t}_{21}^{(i)}\overline{t}_{12}^{(j-1)} \\ \vdots \\ \overline{t}_{21}^{(p+1-j)}\overline{t}_{12}^{(j)} + \ldots + \overline{t}_{21}^{(i)}\overline{t}_{12}^{(p+1-i)} \end{pmatrix}; \begin{pmatrix} \overline{t}_{12}^{(j)} \\ \overline{t}_{21}^{(i-1)}\overline{t}_{12}^{(j)} + \overline{t}_{21}^{(i)}\overline{t}_{12}^{(j-1)} \\ \vdots \\ \overline{t}_{21}^{(p+1-j)}\overline{t}_{12}^{(j)} + \ldots + \overline{t}_{21}^{(i)}\overline{t}_{12}^{(p+1-i)} \end{pmatrix}$$

Next we apply Proposition 2.2, (1) substituting $\overline{t}_{21}^{(i)} = 0$ in s'_{ij} and $\overline{t}_{12}^{(i)} = 0$ in s''_{ij}. It is easy to see that after the substitution we obtain the tables s_{i-1j} and s_{ij-1}. Applying the induction to these sequences we conclude their regularity which implies the regularity of the sequence s_{ij} for all $i, j = 1, \ldots, p$, $i + j \geq p + 1$ by

Proposition 2.2, (3). In particular, the sequence s_{pp} is regular which completes the proof. □

We immediately obtain the following

Corollary 2. *The generators* $t_{22}^{(1)}, \dots, t_{22}^{(p)}, d_1, \dots, d_{2p}$ *of* Γ *are algebraically independent.*

We will denote by $K(\Gamma)$ the field of fractions of Γ.

Combining Lemma 3.1 with Proposition 2.3 we obtain the following

Theorem 1.
(1) $\mathrm{Y}_p(\mathfrak{gl}_2)$ *is free as a left (right) module over* Γ*. Moreover* Γ *is a direct summand of* $\mathrm{Y}_p(\mathfrak{gl}_2)$.
(2) *For any* $\mathbf{m} \in \operatorname{Specm}\Gamma$ *the character* $\chi_{\mathbf{m}}$ *extends to an irreducible* $\mathrm{Y}_p(\mathfrak{gl}_2)$*-module.*

For a subset $P \subset \mathrm{Y}_p(\mathfrak{gl}_2)$ denote by $\mathbb{D}(P)$ the set of all $x \in \mathrm{Y}_p(\mathfrak{gl}_2)$ such that there exists $z \in \Gamma$, $z \neq 0$ for which $zx \in P$.

Corollary 3. *Let* $P \subset \mathrm{Y}_p(\mathfrak{gl}_2)$ *be a finitely generated left* $\Gamma-$*module then* $\mathbb{D}(P)$ *is a finitely generated left* $\Gamma-$*module.*

Proof. Since Γ is a domain then $\mathbb{D}(P)$ is a Γ-submodule in $\mathrm{Y}_p(\mathfrak{gl}_2)$. Using the fact that $\mathrm{Y}_p(\mathfrak{gl}_2)$ is a free left $\Gamma-$module we conclude that $\mathrm{Y}_p(\mathfrak{gl}_2) \simeq F_P \oplus F$ where F_P and F are free left $\Gamma-$modules, F_P has a finite rank and $P \subset F_P$. Then $\mathbb{D}(P) \subset F_P$ and hence it is finitely generated as a module over a noetherian ring. □

4. Harish-Chandra modules for $\mathfrak{gl}(2)$ Yangians

Let L be a polynomial algebra in variables $b_1, \dots, b_p, g_1, \dots g_{2p}$. Define a $\Bbbk$-monomorphism $\imath : \Gamma \to \mathrm{L}$ such that $\imath(t_{22}^{(k)}) = \sigma_{k,p}(b_1, \dots, b_p)$, $\imath(d_i) = \sigma_{i,2p}(g_1, \dots, g_{2p})$ where $\sigma_{i,j}$ is the i-th elementary symmetric polynomial in j variables. We will identify the elements of Γ with their images in L and treat them as polynomials in variables $b_1, \dots, b_p, g_1, \dots g_{2p}$ invariant under the action of the group $S_p \times S_{2p}$. Set $\mathcal{L} = \operatorname{Specm}\mathrm{L}$. We will identify $\mathcal{L}$ with $\Bbbk^{3p}$. If $\beta = (\beta_1, \dots, \beta_p)$, $\gamma = (\gamma_1, \dots, \gamma_{2p})$ and $\ell = (\beta_1, \dots, \beta_p, \gamma_1, \dots, \gamma_{2p})$ then we will write $\ell = (\beta, \gamma)$. The map $\imath$ induces an epimorphism $\imath^* : \mathcal{L} \to \operatorname{Specm}\Gamma$. If $\ell \in \mathcal{L}$ and $\mathbf{m} = \imath^*(\ell)$ then $D(\ell)$ will denote the equivalence class of $\mathbf{m}$ in $\Delta(\mathrm{Y}_p(\mathfrak{gl}_2), \Gamma)$.

Let $\mathcal{L}_0 \subset \mathcal{L}$, $\mathcal{L}_0 \simeq \mathbb{Z}^p$, be a lattice generated by $\delta_i \in \Bbbk^{3p}$, $i = 1, \dots, p$, where $\delta_i = (\delta_i^1, \dots, \delta_i^{3p})$, $\delta_i^j = \delta_{ij}$, $j = 1, \dots, 3p$. Then $\mathcal{L}_0$ acts on $\mathcal{L}$ by shifting $\delta_i(\ell) := \ell + \delta_i$. Also the group $S_p \times S_{2p}$ acts on $\mathcal{L}$ by permutations. Thus the semidirect product $\mathbb{W}$ of the groups $S_p \times S_{2p}$ and $\mathcal{L}_0$ acts on $\mathcal{L}$ and L. Denote by S a multiplicative set in L generated by the elements $b_i - b_j - m$ for all $i \neq j$ and all $m \in \mathbb{Z}$ and by $\mathbb{L}$ the localization of L by S. Note that S is invariant under the action of $\mathbb{W}$ and hence $\mathbb{W}$ acts on $\mathbb{L}$.

Let $\mathcal{L}_1 = \operatorname{Specm}\mathbb{L} \subset \mathcal{L}$, i.e. $\mathcal{L}_1$ consists of *generic* $3p$-tuples $\ell = (\beta, \gamma)$ such that $\beta_i - \beta_j \notin \mathbb{Z}$ for all $i \neq j$. If $\ell \in \mathcal{L}_1$ then the modules from the category $\mathbb{H}(\mathrm{Y}_p(\mathfrak{gl}_2), \Gamma, D(\ell))$ are called *generic* Harish-Chandra modules.

Fix $\ell = (\beta, \gamma) \in \mathcal{L}$. Let I_ℓ be the left ideal of $\mathrm{Y}_p(\mathfrak{gl}_2)$ generated by the coefficients of the polynomials $T_{22}(u) - \beta(u)$ and $D(u) - \gamma(u)$. Define the corresponding quotient module over $\mathrm{Y}_p(\mathfrak{gl}_2)$ by

$$M(\ell) = \mathrm{Y}_p(\mathfrak{gl}_2)/I_\ell.$$

It follows from Theorem 1 that I_ℓ is a proper ideal of $\mathrm{Y}_p(\mathfrak{gl}_2)$ and so $M(\ell)$ is a non-trivial module. Therefore, the image of 1 in $M(\ell)$ is nonzero. We shall denote it by ξ. The next proposition shows the universality of the module $M(\ell)$.

Proposition 4.1. *Let $\ell = (\beta, \gamma) \in \mathcal{L}$ and let V be a weight $\mathrm{Y}_p(\mathfrak{gl}_2)$-module with a central character γ generated by a nonzero $\eta \in V_\beta$. Then V is a homomorphic image of $M(\ell)$.*

Proof. Indeed, there is a homomorphism $f : M(\ell) \to V$ which maps ξ to η. Since η generates V the statement follows. □

4.1. **Weight modules.** For $\ell = (\beta, \gamma) \in \mathcal{L}$ the category $\mathbb{H}W(\mathrm{Y}_p(\mathfrak{gl}_2),\ \Gamma,\ D(\ell))$ consists of finitely generated weight modules V with central character γ and with $\operatorname{Supp} V \subset D(\ell)$. For simplicity we will denote it by R_ℓ. If $\ell \in \mathcal{L}_1$ then the modules from R_ℓ will be called *generic* weight modules.

Let $\ell = (\beta, \gamma) \in \mathcal{L}$, $\beta = (\beta_1, \dots, \beta_p)$, $\gamma = (\gamma_1, \dots, \gamma_{2p})$, $\beta(u) = (u + \beta_1) \dots (u + \beta_p)$, $\gamma(u) = (u + \gamma_1) \dots (u + \gamma_{2p})$.

A $\mathrm{Y}_p(\mathfrak{gl}_2)$-module V is an object of R_ℓ if V is a direct sum of its *weight* subspaces:

$$V = \bigoplus_{\ell \in \mathcal{L}} V_\ell, \text{ where } V_\ell = \{\eta \in V \mid T_{22}(u)\eta = \beta(u)\eta, \quad D(u)\eta = \gamma(u)\eta\}. \tag{4.13}$$

If $V \in R_\ell$ then we shall simply write V_β instead of V_ℓ and identify $\operatorname{Supp} V$ with the set of all β such that the subspace V_β is nonzero.

Lemma 4.1. *(compare with (2.7)) Let V be a generic weight $\mathrm{Y}_p(\mathfrak{gl}_2)$-module and let $\beta = (\beta_1, \dots, \beta_p) \in \operatorname{Supp} V$. Then*

$$T_{21}(u)V_\beta \subseteq \sum_{i=1}^{p} V_{\beta+\delta_i} \qquad \textit{and} \qquad T_{12}(u)V_\beta \subseteq \sum_{i=1}^{p} V_{\beta-\delta_i} \tag{4.14}$$

where $\beta \pm \delta_i = (\beta_1, \dots, \beta_i \pm 1, \dots, \beta_p)$.

Proof. First we show that $T_{21}(-\beta_i)V_\beta \subseteq V_{\beta+\delta_i}$ for all $i = 1, \dots, p$. Since

$$T_{22}(u-1)T_{21}(u) = T_{21}(u-1)T_{22}(u)$$

we have

$$T_{22}(-\beta_i - 1)T_{21}(-\beta_i)\,\eta = T_{21}(-\beta_i - 1)T_{22}(-\beta_i)\,\eta = 0$$

for all $\eta \in V_\beta$. Also,

$$\begin{aligned} T_{22}(-\beta_j)T_{21}(-\beta_i)\,\eta = {} & (\beta_i - \beta_j)^{-1}(T_{21}(-\beta_i)T_{22}(-\beta_j) - T_{21}(-\beta_j)T_{22}(-\beta_i))\,\eta \\ & + T_{21}(-\beta_i)T_{22}(-\beta_j)\,\eta = 0 \end{aligned}$$

since $T_{22}(-\beta_k)\,\eta = 0$ for all $k = 1, \dots, p$. Using the fact that $\beta_i - \beta_j \notin \mathbb{Z}$ we conclude that $T_{21}(-\beta_i)V_\beta \subseteq V_{\beta+\delta_i}$ for all $i = 1, \dots, p$. Since $T_{21}(u)$ is a polynomial of degree $p - 1$ in u and $\beta_i \neq \beta_j$ if $i \neq j$, we have that $T_{21}(u)V_\beta \subseteq \sum_{i=1}^{p} V_{\beta+\delta_i}$. The case of $T_{12}(u)$ is treated analogously using the identity $T_{22}(u)T_{12}(u-1) = T_{12}(u)T_{22}(u-1)$. □

Corollary 4. *If V is indecomposable generic weight module over $\mathrm{Y}_p(\mathfrak{gl}_2)$ and $\beta \in \operatorname{Supp} V$ then $\operatorname{Supp} V \subseteq \beta + \mathbb{Z}^p$.*

Lemma 4.2. *If V is a generic weight $\mathrm{Y}_p(\mathfrak{gl}_2)$-module with central character $\gamma(u)$ then for any $\beta=(\beta_1,\dots,\beta_p)\in\operatorname{Supp}V$ and any $\eta\in V_\beta$ we have*

$$T_{12}(-\beta_r)T_{21}(-\beta_s)\,\eta=T_{21}(-\beta_s)T_{12}(-\beta_r)\,\eta,$$

if $s\neq r$, and

$$T_{12}(-\beta_i-1)T_{21}(-\beta_i)\,\eta=-\gamma(-\beta_i)\,\eta,$$
$$T_{21}(-\beta_i+1)T_{12}(-\beta_i)\,\eta=-\gamma(-\beta_i+1)\,\eta.$$

Proof. The first equality follows from the defining relations (1.1). The others follow from (3.12). □

Corollary 5. *Let V be a generic weight $\mathrm{Y}_p(\mathfrak{gl}_2)$-module with a central character γ and let $\beta=(\beta_1,\dots,\beta_p)\in\operatorname{Supp}V$.*

(i) *If $\gamma(-\beta_i)\neq 0$ then $\operatorname{Ker}T_{21}(-\beta_i)\cap V_\beta=0$.*
(ii) *If $\gamma(-\beta_i+1)\neq 0$ then $\operatorname{Ker}T_{12}(-\beta_i)\cap V_\beta=0$.*
(iii) *If V is indecomposable and $\gamma(-\beta_i+k)\neq 0$ for all $k\in\mathbb{Z}$ then*

$$\operatorname{Ker}T_{21}(-\psi_i)\cap V_\psi=\operatorname{Ker}T_{12}(-\psi_i)\cap V_\psi=0$$

for all $\psi=(\psi_1,\dots,\psi_p)\in\operatorname{Supp}V$.

Given $(k)=(k_1,\dots,k_p)\in\mathbb{Z}^p$ define the corresponding vector of the module $M(\ell)$ by

$$\xi^{(k)}=\prod_{i,\ k_i>0}T_{21}(-\beta_i-k_i+1)\cdots T_{21}(-\beta_i-1)\,T_{21}(-\beta_i)$$
$$\times\prod_{i,\ k_i<0}T_{12}(-\beta_i-k_i-1)\cdots T_{12}(-\beta_i+1)\,T_{12}(-\beta_i)\,\xi.$$

Theorem 2. *The vectors $\xi^{(k)},(k)\in\mathbb{Z}^p$ form a basis of $M(\ell)$. Moreover, we have the formulas*

$$T_{22}(u)\,\xi^{(k)}=\prod_{i=1}^{p}(u+\beta_i+k_i)\,\xi^{(k)},\tag{4.15}$$

(4.16)

$$T_{21}(u)\,\xi^{(k)}=\sum_{i=1}^{p}A_i(k)\,\frac{(u+\beta_1+k_1)\cdots\wedge_i\cdots(u+\beta_p+k_p)}{(\beta_1-\beta_i+k_1-k_i)\cdots\wedge_i\cdots(\beta_p-\beta_i+k_p-k_i)}\,\xi^{(k+\delta_i)},$$
$$T_{12}(u)\,\xi^{(k)}=\sum_{i=1}^{p}B_i(k)\,\frac{(u+\beta_1+k_1)\cdots\wedge_i\cdots(u+\beta_p+k_p)}{(\beta_1-\beta_i+k_1-k_i)\cdots\wedge_i\cdots(\beta_p-\beta_i+k_p-k_i)}\,\xi^{(k-\delta_i)},$$

where

$$A_i(k)=\begin{cases}1 & \text{if}\quad k_i\geq 0\\ -\gamma(-\beta_i-k_i) & \text{if}\quad k_i<0\end{cases}$$

and

$$B_i(k)=\begin{cases}-\gamma(-\beta_i-k_i+1) & \text{if}\quad k_i>0\\ 1 & \text{if}\quad k_i\leq 0.\end{cases}$$

The action of $T_{11}(u)$ is found from the relation

$$\Big(T_{11}(u)\,T_{22}(u-1)-T_{21}(u)\,T_{12}(u-1)\Big)\,\xi^{(k)}=\gamma(u)\,\xi^{(k)}.\tag{4.17}$$

Proof. We start by proving the formulas for the action of the generators of $\mathrm{Y}_p(\mathfrak{gl}_2)$. Relation (4.15) follows by induction from the defining relations (1.1). By Lemma 4.2 we have: if $k_i > 0$ then

$$T_{21}(-\beta_i - k_i)\,\xi^{(k)} = \xi^{(k+\delta_i)},\qquad T_{12}(-\beta_i - k_i)\,\xi^{(k)} = -\gamma(-\beta_i - k_i + 1)\,\xi^{(k-\delta_i)}; \tag{4.18}$$

if $k_i < 0$ then

$$T_{12}(-\beta_i - k_i)\,\xi^{(k)} = \xi^{(k-\delta_i)},\qquad T_{21}(-\beta_i - k_i)\,\xi^{(k)} = -\gamma(-\beta_i - k_i)\,\xi^{(k+\delta_i)}; \tag{4.19}$$

and if $k_i = 0$ then

$$T_{12}(-\beta_i)\,\xi^{(k)} = \xi^{(k-\delta_i)},\qquad T_{21}(-\beta_i)\,\xi^{(k)} = \xi^{(k+\delta_i)}. \tag{4.20}$$

Applying the Lagrange interpolation formula we obtain the remaining formulas.

To show that the vectors $\xi^{(k)}$ form a basis of $M(\ell)$, denote by $\mathcal{T}_\beta$ the subspace of $\mathrm{Y}_p(\mathfrak{gl}_2)$ spanned by the elements

$$\tau^{(k)} = \prod_{i,\ k_i>0} T_{21}(-\beta_i - k_i + 1)\cdots T_{21}(-\beta_i - 1)\,T_{21}(-\beta_i) \times \prod_{i,\ k_i<0} T_{12}(-\beta_i - k_i - 1)\cdots T_{12}(-\beta_i + 1)\,T_{12}(-\beta_i),$$

where (k) runs over $\mathbb{Z}^p$. It suffices to prove the vector space decomposition

$$\mathrm{Y}_p(\mathfrak{gl}_2) = \mathcal{T}_\ell \oplus I_\ell. \tag{4.21}$$

Due to the formulas proved above, $\mathrm{Y}_p(\mathfrak{gl}_2) = \mathcal{T}_\ell + I_\ell$. We now need to show that the vectors $\tau^{(k)}$ are linearly independent modulo the left ideal I_ℓ. By (4.15) and the genericity assumption, the elements $\tau^{(k)} \mod I_\ell$ are eigenvectors for $T_{22}(u)$ with distinct eigenvalues. So the claim will follow if we demonstrate that each $\tau^{(k)}$ is nonzero modulo I_ℓ. Suppose first that γ is generic: $\gamma(-\beta_i - k) \neq 0$ for all $k \in \mathbb{Z}$ and all i. Then we deduce from (4.18)–(4.20) that $\tau^{(k)} \neq 0 \mod I_\ell$ since $1 \neq 0 \mod I_\ell$ which gives (4.21) for generic γ.

Let now γ be arbitrary. Suppose that a nonzero element τ belongs to the intersection $\mathcal{T}_\ell \cap I_\ell$. Then

$$\tau = \sum_{i=1}^{p} a_i\,(t_{22}^{(i)} - \beta^{(i)}) + \sum_{i=1}^{2p} b_i\,(D^{(i)} - \gamma^{(i)}), \tag{4.22}$$

where $D^{(i)}$, $\beta^{(i)}$ and $\gamma^{(i)}$ are the coefficients of the polynomials $D(u)$, $\beta(u)$ and $\gamma(u)$, respectively, while $a_i, b_i \in \mathrm{Y}_p(\mathfrak{gl}_2)$. Let $\widetilde{\gamma}$ be generic. Then we can rewrite (4.22) as

$$\tau = \sum_{i=1}^{p} a_i\,(t_{22}^{(i)} - \beta^{(i)}) + \sum_{i=1}^{2p} b_i\,(D^{(i)} - \widetilde{\gamma}^{(i)}) + \sum_{i=1}^{2p} b_i\,(\widetilde{\gamma}^{(i)} - \gamma^{(i)}). \tag{4.23}$$

Consider the unique decompositions of the elements b_j in accordance with (4.21) where $\gamma(u)$ is taken to be $\widetilde{\gamma}(u)$:

$$b_j = \tau_j + \sum_{i=1}^{p} a_{ij}\,(t_{22}^{(i)} - \beta^{(i)}) + \sum_{i=1}^{2p} b_{ij}\,(D^{(i)} - \widetilde{\gamma}^{(i)}) \tag{4.24}$$

for some $a_{ij},\, b_{ij} \in \mathrm{Y}_p(\mathfrak{gl}_2)$. Using the decomposition (4.21) for generic $\widetilde{\gamma}(u)$ we must have

$$\tau = \sum_{j=1}^{2p} \tau_j\,(\widetilde{\gamma}^{(j)} - \gamma^{(j)}). \tag{4.25}$$

for all such $\widetilde{\gamma}(u)$. This means that the $\mathcal{T}_\ell$-component of each element $b_j\,(\widetilde{\gamma}^{(j)} - \gamma^{(j)})$ is independent of $\widetilde{\gamma}(u)$. However, due to the formulas (4.15)–(4.17), this is only possible if all b_j are zero. Finally, the elements a_i must be zero too by the decomposition (4.22) with generic γ. So, (4.21) holds for all $\gamma(u)$. □

Remark 1. *Given two monic polynomials $\alpha(u)$ and $\beta(u)$ of degree p define the corresponding Verma module $V(\alpha(u), \beta(u))$ as the quotient of $\mathrm{Y}_p(\mathfrak{gl}_2)$ by the left ideal generated by the coefficients of the polynomials $T_{11}(u) - \alpha(u)$, $T_{22}(u) - \beta(u)$ and $T_{12}(u)$; cf.* [T1, T2]. *Then the same argument as above shows that $V(\alpha(u), \beta(u))$ has a basis $\{\xi^{(k)}\}$ parameterized by p-tuples of nonnegative integers $(k) = (k_1, \ldots, k_p)$. The formulas of Theorem 2 hold for the basis vectors $\xi^{(k)}$, where $\gamma(u)$ should be taken to be $\alpha(u)\,\beta(u-1)$ which defines the central character γ of $V(\alpha(u), \beta(u))$. In fact, $V(\alpha(u), \beta(u))$ is isomorphic to the quotient of the corresponding universal module $M(\ell)$, $\ell = (\beta, \gamma)$ by the submodule spanned by the vectors $\{\xi^{(k)}\}$ such that (k) contains at least one negative component k_i.*

Corollary 6. *Let $\ell = (\beta, \gamma) \in \mathcal{L}_1$.*

1. *The module $M(\ell)$ is a generic weight $\mathrm{Y}_p(\mathfrak{gl}_2)$-module with central character γ,* $\operatorname{Supp} M(\ell) = \mathbb{Z}^p$ *and all weight spaces are 1-dimensional.*
2. *The module $M(\ell)$ has a unique maximal submodule and hence a unique irreducible quotient.*
3. *The equivalence class $D(\ell)$ coincides with the set $\ell + \mathcal{L}_0$.*

Proof. Statement (1) follows immediately from Theorem 2. By Proposition 4.1 the sum of all proper submodules is again a proper submodule. Thus $M(\ell)$ has a unique maximal submodule which implies (2). The statement (3) follows immediately from (1). □

We will denote the unique irreducible quotient of $M(\ell)$ by $L(\ell)$. It follows from Corollary 6 that all weight spaces of $L(\ell)$ are 1-dimensional. Using Proposition 4.1 we can now describe all irreducible generic weight $\mathrm{Y}_p(\mathfrak{gl}_2)$-modules.

Corollary 7. *Let $\ell = (\beta, \gamma) \in \mathcal{L}_1$.*

1. *There exists an irreducible generic weight $\mathrm{Y}_p(\mathfrak{gl}_2)$-module $L(\ell)$ with $L(\ell)_\beta \neq 0$ and with central character γ. Moreover, $\dim L(\ell)_\psi = 1$ for all $\psi \in$* $\operatorname{Supp} L(\ell)$.
2. *Any irreducible weight module over $\mathrm{Y}_p(\mathfrak{gl}_2)$ with central character γ generated by a nonzero vector of weight β is isomorphic to $L(\ell)$.*

5. Properties of Γ as a subalgebra of $\mathrm{Y}_p(\mathfrak{gl}_2)$

In this section we adapt the results from [DFO2] and [Ov] for the Yangians. In particular, we show that Γ is a Harish-Chandra subalgebra.

For any $\ell_0 \in \mathcal{L}_1$ the module $M(\ell_0)$ has a basis $\xi^{(k)}$, $(k) \in \mathbb{Z}^p$ with the action of generators of $\mathrm{Y}(\mathfrak{gl}_2)$ defined by formulas (4.15)–(4.17). Then we can relabel the basis elements of $M(\ell_0)$ by ξ_ℓ, $\ell \in \ell_0 + \mathcal{L}_0$. It follows from Theorem 2 that for every $x \in \mathrm{Y}_p(\mathfrak{gl}_2)$ there exists a finite subset $\mathcal{L}_x \subset \mathcal{L}_0$ consisting of elements δ such that

$$\xi_\ell = \sum_{\delta \in \mathcal{L}_x} \theta(x, \ell, \delta) \xi_{\ell+\delta} \tag{5.26}$$

where $\theta(x, \ell, \delta) = \theta(x, \mathsf{b}, \delta)(\ell)$, $\theta(x, \mathsf{b}, \delta) \in \mathbb{L}$, $\mathsf{b} = (b_1, \ldots, b_p, g_1, \ldots, g_{2p})$. Clearly, the set $\mathcal{L}_x$ is $S_p \times S_{2p}$-invariant. Note that for a given x this formula does not depend on ℓ_0.

Let $\mathrm{M}_{\mathcal{L}_0}(\mathbb{L})$ be the ring of locally finite (with the finite number of non-zero elements in each row and each column) matrices over $\mathbb{L}$ with the entries indexed by the elements of $\mathcal{L}_0$. Any $\ell \in \mathcal{L}_1$ defines the evaluation homomorphism $\chi_\ell : \mathbb{L} \longrightarrow \mathbb{k}$, which induces the homomorphism of matrix algebras $\mathrm{M}_{\mathcal{L}_0}(\ell) : \mathrm{M}_{\mathcal{L}_0}(\mathbb{L}) \longrightarrow \mathrm{M}_{\mathcal{L}_0}(\mathbb{k})$. For $\ell, \ell' \in \mathcal{L}_0$ denote by $e_{\ell\,\ell'}$ the corresponding matrix unit in $\mathrm{M}_{\mathcal{L}_0}(\mathbb{L})$. The group $\mathbb{W}$ acts on $\mathrm{M}_{\mathcal{L}_0}(\mathbb{L})$ as follows: $(w^{-1} \cdot X)_{\ell,\ell'} = w^{-1} \cdot X_{w(\ell) w(\ell')}$ for all $w \in \mathbb{W}$, $X = (X_{\ell\,\ell'})_{\ell,\ell' \in \mathrm{L}_0}$, $\ell, \ell' \in \mathcal{L}_0$. Note that this action induces an action of $S_p \times S_{2p}$ on the free $\mathbb{L}$-module $X_0 = \sum_{\delta \in \mathcal{L}_0} \mathbb{L} e_{\delta, \overline{0}}$ where $\overline{0}$ is a zero element in $\mathcal{L}_0$.

Define a map

$$\mathrm{G} : \mathrm{Y}_p(\mathfrak{gl}_2) \to \mathrm{M}_{\mathcal{L}_0}(\mathbb{L})$$

such that for any $x \in \mathrm{Y}_p(\mathfrak{gl}_2)$ and any $\ell \in \mathcal{L}_0$, $\mathrm{G}(x)_{\ell\,\ell'} = \theta(x, \mathsf{b} + \ell, \delta)$ if $\ell' - \ell = \delta$ and 0 otherwise.

Lemma 5.1. (1) G *is a representation of* $\mathrm{Y}_p(\mathfrak{gl}_2)$.
(2) $\mathrm{G}(x)$ *is* $\mathbb{W}$*-invariant for any* $x \in \mathrm{Y}_p(\mathfrak{gl}_2)$. *In particular,* $\mathrm{G}(x)_{\overline{0}\,\overline{0}} \in K(\Gamma)$.
(3) *If* $x = x(b_1, \ldots, b_p, g_1, \ldots, g_{2p}) \in \Gamma$ *then* $\mathrm{G}(x)_{\ell\ell} = x(b_1 + l_1, \ldots, b_p + l_p, g_1, \ldots, g_{2p})$ *where* $\ell = (l_1, \ldots, l_p, 0, \ldots, 0) \in \mathcal{L}_0$.
(4) $\mathrm{G}(\Gamma)$ *consists of* $\mathbb{W}$*-invariant diagonal matrices* X *such that* $X_{\overline{0}\,\overline{0}} \in \Gamma$. *In particular,* $X_{\overline{0}\,\overline{0}} \in \Gamma$ *determines* X.

Proof. Let T be a free (non-commutative) algebra with generators $t_{ij}^{(k)}$, $i, j = 1, 2$, $k = 1, \ldots, p$, $\pi : \mathrm{T} \to \mathrm{Y}_p(\mathfrak{gl}_2)$, $t_{ij}^{(k)} \longmapsto t_{ij}^{(k)}$, be a canonical projection. Define a homomorphism $g : \mathrm{T} \to \mathrm{M}_{\mathcal{L}_0}(\mathbb{L})$ by $g(t_{ij}^{(k)}) = \mathrm{G}(t_{ij}^{(k)})$ for all suitable i, j, k. To prove (1) it is enough to show that $g(\operatorname{Ker} \pi) = 0$. Let $f \in \operatorname{Ker} \pi$ and suppose that $g(f)_{\ell'\ell''} \in \mathbb{L}$ is nonzero for some $\ell', \ell'' \in \mathcal{L}_0$. Then $\mathrm{M}_{\mathcal{L}_0}(\ell)(g(f)) = 0$ and thus $g(f)_{\ell'\ell''}(\ell) = 0$ for any $\ell \in \mathcal{L}_1$. Since $\mathcal{L}_1$ is dense in $\operatorname{Specm} L$ we conclude that $g(f) = 0$ implying (1).

The image of G is $\mathbb{W}$-invariant since it holds for the generators of $\mathrm{Y}_p(\mathfrak{gl}_2)$ (4.15)–(4.17). For any $\sigma \in S_p \times S_{2p}$, $(\sigma^{-1} \cdot \mathrm{G})(x)_{\overline{0}\,\overline{0}} = \sigma^{-1}(\mathrm{G}(x)_{\sigma(\overline{0})\,\sigma(\overline{0})}) = \sigma^{-1}(\mathrm{G}(x)_{\overline{0}\,\overline{0}})$. Hence $\mathrm{G}(x)_{\overline{0}\,\overline{0}}$ is $S_p \times S_{2p}$-invariant proving (2). The statement (3) follows from (2) if we apply a shift by $\ell \in \mathcal{L}_0$ to an arbitrary $x \in \mathrm{Y}_p(\mathfrak{gl}_2)$. The statement (4) follows immediately from (2) and (3). $\square$

The composition r_ℓ of G and $\mathrm{M}_{\mathcal{L}_0}(\ell)$ defines a representation G_ℓ of $\mathrm{Y}_p(\mathfrak{gl}_2)$. It is easy to see that the corresponding $\mathrm{Y}_p(\mathfrak{gl}_2)-$module coincides with the module $M(\ell)$ from Theorem 2.

Proposition 5.1. *The representation* $\mathrm{G}: \mathrm{Y}_p(\mathfrak{gl}_2) \longrightarrow \mathrm{M}_{\mathcal{L}_0}(\mathbb{L})$ *is faithful.*

Proof. It is clear that $\mathrm{Ker}\,\mathrm{G} \subset \cap_{\ell\in\mathcal{L}_1} \mathrm{Ker}\, r_\ell$. Hence it is enough to prove that

$$\bigcap_{\ell\in\mathcal{L}_1} \mathrm{Ker}\, r_\ell = 0.$$

Let $\ell = (\beta,\gamma)$. Then $\mathrm{Ker}\, r_\ell = \mathrm{Ann}\, M(\ell)$ dy definition. Since $M(\ell) = \mathrm{Y}_p(\mathfrak{gl}_2)/I_\ell$ we have that $\mathrm{Ker}\, r_\ell \subset I_\ell$. Therefore, it is enough to show that $\cap_{\ell\in\mathcal{L}_1} I_\ell = 0$. By Theorem 1, (1) the Yangian $\mathrm{Y}_p(\mathfrak{gl}_2)$ is free as a right module over Γ. Let $x_i, i \in \mathcal{I}$ be a basis of $\mathrm{Y}_p(\mathfrak{gl}_2)$ over Γ. If $x = \sum_{i\in\mathcal{I}} x_i z_i$ for some $z_i \in \Gamma$ then $x \in I_\ell$ if and only if $z_i(\ell) = 0$ for all $i \in \mathcal{I}$. Since $\mathcal{L}_1$ is dense in $\mathcal{L}$ in Zariski topology it follows immediately that if $x \in \cap_{\ell\in\mathcal{L}_1} I_\ell$ then $z_i = 0$ for all $i \in \mathcal{I}$ and thus $x = 0$. This completes the proof. □

Immediately from the proof of the theorem above and the density of $\mathcal{L}_1$ in $\mathcal{L}$ we obtain the following analogue of the Harish-Chandra Theorem for Lie algebras [Di].

Corollary 8. *Let* $x \in \mathrm{Y}_p(\mathfrak{gl}_2)$ *be such that* $xM(\ell) = 0$ *for any* $\ell \in \mathcal{L}_1$. *Then* $x = 0$.

Corollary 9. (1) Γ *is a maximal commutative subalgebra in* $\mathrm{Y}_p(\mathfrak{gl}_2)$.
(2) *If for* $x \in \mathrm{Y}_p(\mathfrak{gl}_2)$ *the matrix* $\mathrm{G}(x)$ *is diagonal then* $x \in \Gamma$.

Proof. Consider an element $x \in \mathrm{Y}_p(\mathfrak{gl}_2)$ which commutes with every $z \in \Gamma$. Suppose there exist $\ell_1,\ \ell_2 \in \mathcal{L}_0,\ \ell_1 \neq \ell_2$ such that $\mathrm{G}(x)_{\ell_1\ell_2} \neq 0$. There exists $z \in \Gamma$ such that $z(\ell_1) \neq z(\ell_2)$ and thus $\mathrm{G}(z)_{\ell_1\ell_1} \neq \mathrm{G}(z)_{\ell_2\ell_2}$ by Lemma 5.1, (3). Then we have $\mathrm{G}(xz)_{\ell\ell'} = \mathrm{G}(x)_{\ell\ell'}\mathrm{G}(z)_{\ell'\ell'} = \mathrm{G}(zx)_{\ell\ell'} = \mathrm{G}(z)_{\ell\ell}\mathrm{G}(x)_{\ell\ell'}$ and therefore $\mathrm{G}(x)$ is diagonal. To conclude the maximality of Γ it is enough to prove the statement (2). By Lemma 5.1, (2), $\mathrm{G}(x)_{\overline{0}\,\overline{0}} = \dfrac{f}{g} \in \mathbb{L}$ where $f, g \in \Gamma$ are relatively prime. Suppose that $g \notin \Bbbk$. By Lemma 5.1, (4) we have that $\mathrm{G}(x)\mathrm{G}(g) = \mathrm{G}(f)$ and $xg = f$ by Proposition 5.1. It implies that $x \in \Gamma$ by Theorem 1, (1). This completes the proof. □

Corollary 10. *Let* $p : \mathrm{M}_{\mathcal{L}_0}(\mathcal{L}) \longrightarrow X_0$ *be the projection. Then the composition* $r : \mathrm{Y}_p(\mathfrak{gl}_2) \xrightarrow{\ \mathrm{G}\ } \mathrm{M}_{\mathcal{L}_0}(\mathbb{L}) \xrightarrow{\ p\ } X_0$ *is a monomorphism of* $\mathrm{Y}_p(\mathfrak{gl}_2)-$*modules. The map* p *commutes with the action of* $S_p \times S_{2p}$ *and in particular,* $r(\mathrm{Y}_p(\mathfrak{gl}_2))$ *is* $S_p \times S_{2p}-$*invariant.*

Proof. Note that for any $x \in \mathrm{Y}_p(\mathfrak{gl}_2)$ the matrix $\mathrm{G}(x) \in \mathrm{M}_{\mathcal{L}_0}(\mathbb{L})$ is determined completely by its column $p(\mathrm{G}(x))$. Thus $r(x) = 0$ implies $\mathrm{G}(x) = 0$ and $x = 0$ by faithfulness of G. Hence r is a monomorphism. Other statements follow immediately from the definitions and Lemma 5.1, (2). □

As in [DFO2], we identify the $(\Gamma - \Gamma)$-bimodule structure on $\mathrm{Y}_p(\mathfrak{gl}_2)$ with the corresponding $\Gamma \otimes_\Bbbk \Gamma-$module structure. Let $\mathsf{b} = (b_1, \ldots, b_p, g_1, \ldots, g_{2p})$. For any $z \in \Gamma$ and any $S \subset \mathcal{L}$ introduce the following polynomial

$$F_{S,z} = \prod_{\delta\in S} (z \otimes 1 - 1 \otimes z(\mathsf{b} + \delta)) = \sum_{i=0}^{|S|} z^i \otimes a_i,\, a_i \in \mathbb{L}.$$

Proposition 5.2. *(*[DFO2]*, Lemma 25). Let S be a finite $S_p\times S_{2p}$-invariant subset in $\mathcal{L}$ and z be any element of Γ, $F_{S,z}=\sum_{i=0}^{|S|} z^i\otimes a_i, a_i\in\mathbb{L}$.*

(1) $a_i\in\Gamma$, $i=0,\dots,|S|$.

(2) *For any $x\in \mathrm{Y}_p(\mathfrak{gl}_2)$ such that $\mathcal{L}_x\subset S$ holds $\sum_{i=0}^{q} z^i x a_i=0$.*

Proof. If S is $S_p\times S_{2p}$-invariant then the coefficients of the polynomial $F_{S,z}$ are $S_p\times S_{2p}$-invariant and hence belong to Γ which proves (1). It is enough to check the statement (2) for $S=\mathcal{L}_x$ since $F_{S,z}=F_{S\setminus\mathcal{L}_x,z}F_{\mathcal{L}_x,z}$. Denote $q=|S|$. Let $\ell\in\mathcal{L}_1$ and let ξ_ℓ be a basis element of $M(\ell)$. Then

$$\sum_{i=0}^{q} z^i x a_i(\xi_\ell)=\sum_{i=0}^{q} z^i x a_i(\ell)(\xi_\ell)=$$

$$\sum_{i=0}^{q} z^i a_i(\ell)\sum_{\delta\in\mathcal{L}_x}\theta(x,\ell,\delta)\xi_{\ell+\delta}=$$

$$\sum_{\delta\in\mathcal{L}_x}\theta(x,\ell,\delta)\sum_{i=0}^{q} a_i(\ell)(z^i\xi_{\ell+\delta})=$$

$$\sum_{\delta\in\mathcal{L}_x}\theta(x,\ell,\delta)\sum_{i=0}^{q} a_i(\ell)z(\ell+\delta)^i\xi_{\ell+\delta}=\sum_{\delta\in\mathcal{L}_x}\theta(x,\ell,\delta)F_{\mathcal{L}_x,z}(z(\ell+\delta),\ell)\xi_{\ell+\delta}=0$$

since $F_{\mathcal{L}_x,z}(z(\ell+\delta),\ell)=0$ for every $\delta\in\mathcal{L}_x$. Applying Corollary 8 we obtain the statement of the proposition. □

The main result of this section is the following

Theorem 3. *Γ is a Harish-Chandra subalgebra of $\mathrm{Y}_p(\mathfrak{gl}_2)$.*

Proof. Following [DFO2], Proposition 8, it is enough to show that a Γ-bimodule $\Gamma\, t_{ij}^{(k)}\,\Gamma$ is finitely generated both as left and as right module for every possible choice of indices i,j,k. It is obvious for $i=j=2$ since $t_{22}^{(k)}\in\Gamma$. We prove it for $i=2, j=1$. Since d_i is central for every $i=1,\dots,2p$ we have $d_it_{21}^{(k)}=t_{21}^{(k)}d_i$. From formulas (4.16) follows that $\mathcal{L}_{t_{21}^{(k)}}=\{\delta_i|i=1,\dots,p\}$. Then

$$F_{\mathcal{L}_{t_{21}^{(k)}},t_{22}^{(i)}}=z^p\otimes 1+\sum_{l=0}^{p-1} z^l\otimes a_l,\ a_l\in\Gamma$$

and

$$(t_{22}^{(i)})^p t_{21}^{(k)}+\sum_{l=0}^{p-1}(t_{22}^{(i)})^l t_{21}^{(k)}a_l=0 \tag{5.27}$$

by Proposition 5.2, (2). Hence the elements $(\prod_{i=1}^{p}(t_{22}^{(i)})^{k_i})t_{21}^{(k)}$, $0\leq k_i<p$ form the generators of $\Gamma t_{21}^{(k)}\Gamma$ as a right Γ-module.

Applying a suitable automorphism we conclude that $\Gamma t_{21}^{(k)}\Gamma$ is finitely generated as a left Γ-module.

The cases $i=1, j=2$ and $i=j=1$ can be treated analogously since $\mathcal{L}_{t_{12}^{(k)}} = \{-\delta_i | i=1,\ldots,p\}$ and $\mathcal{L}_{t_{11}^{(k)}} = \{\delta_i - \delta_j | i,j=1,\ldots,p\}$. Hence $\Gamma t_{ij}^{(k)} \Gamma$ is finitely generated as a right and as a left Γ-module. □

6. Category of Harish-Chandra modules over $\mathrm{Y}_p(\mathfrak{gl}_2)$

Since Γ is a Harish-Chandra subalgebra of $\mathrm{Y}_p(\mathfrak{gl}_2)$ we can apply all the statements from Section 2.1. Denote $\mathcal{A} = \mathcal{A}_{\mathrm{Y}_p(\mathfrak{gl}_2),\Gamma}$. Then by Proposition 1, the categories $\mathcal{A} - \mathrm{mod}_d$ and $\mathbb{H}(\mathrm{Y}_p(\mathfrak{gl}_2),\Gamma)$ are equivalent. Also the full subcategory $\mathbb{H}W(\mathrm{Y}_p(\mathfrak{gl}_2),\Gamma)$ consisting of weight modules is equivalent to the module category $\mathcal{A}_W -$mod. If $\ell \in \mathcal{L}$ then the category R_ℓ is equivalent to the block $\mathcal{A}_W(D(\ell))-$mod of the category $\mathcal{A}_W -$ mod.

We will show that each character of Γ extends to a finite number of irreducible Harish-Chandra modules over $\mathrm{Y}_p(\mathfrak{gl}_2)$. This is an analogue of the corresponding result in the case of a Lie algebra $\mathfrak{gl}_n$ which was conjectured in [DFO1] and proved in [Ov]. In this section we use the techniques of [DFO2] and [Ov].

Lemma 6.1. *For any $x \in \mathrm{Y}_p(\mathfrak{gl}(2))$, $f \in \Gamma\otimes\Gamma$, ℓ, $\ell' \in \mathcal{L}_0$ holds*

$$\mathrm{G}(f\cdot x)_{\ell\ell'} \ = \ f(\mathsf{b}+\ell, \mathsf{b}+\ell')\mathrm{G}(x)_{\ell\ell'}.$$

Proof. Let $f = \sum_i z_i \otimes z_i' \in \Gamma\otimes\Gamma$. Then $\mathrm{G}(f\cdot x) = \sum_i \mathrm{G}(z_i)\mathrm{G}(x)\mathrm{G}(z_i')$ and hence

$$\mathrm{G}(f\cdot x)_{\ell\ell'} = \sum_i \mathrm{G}(z_i)_{\ell\,\ell}\mathrm{G}(x)_{\ell\,\ell'}\mathrm{G}(z_i')_{\ell'\,\ell'} = \mathrm{G}(x)_{\ell\,\ell'}\sum_i \mathrm{G}(z_i)_{\ell\,\ell}\mathrm{G}(z_i')_{\ell'\,\ell'} =$$

$$\mathrm{G}(x)_{\ell\,\ell'}\sum_i z_i(\mathsf{b}+\ell)z_i'(\mathsf{b}+\ell') = \mathrm{G}(x)_{\ell\,\ell'} f(\mathsf{b}+\ell,\mathsf{b}+\ell').$$

□

Lemma 6.2. *(*[DFO2]*, Lemma 25). Let $z\in\Gamma$, $S\subset\mathcal{L}$ be a $S_p\times S_{2p}$-invariant set and $x\in\mathrm{Y}_p(\mathfrak{gl}_2)$ be such that $\mathrm{G}(x)_{\ell\,\ell'}=0$ for all ℓ,ℓ', $\ell-\ell'\notin S$ then $F\cdot x = 0$.*

Proof. Let F in the form $F=\sum_i z^i\otimes a_i$ where $a_i\in L$. If $\ell-\ell'\in S$ then $\mathrm{G}(F\cdot x)_{\ell'\ell} = F(\mathsf{b}+\ell,\mathsf{b}+\ell')\mathrm{G}(x)_{\ell\,\ell'}$ by Lemma 6.1. Then $h = z\otimes 1 - 1\otimes z(\mathsf{b}+\ell-\ell')$ divides F, $h(\mathsf{b}+\ell,\mathsf{b}+\ell')=0$, $F(\mathsf{b}+\ell,\mathsf{b}+\ell')=0$ and $F\cdot x=0$. □

Let $S\subset\mathcal{L}_0$ be a finite $S_p\times S_{2p}-$invariant set. Define $\mathrm{Y}^S=\{x\in\mathrm{Y}_p(\mathfrak{gl}_2)\,|\,\mathcal{L}_x\subset S\}$. Clearly Y^S is a $\Gamma-$subbimodule in $\mathrm{Y}_p(\mathfrak{gl}_2)$. We have the following characterization of the bimodule Y^S.

Lemma 6.3. *Let $x\in\mathrm{Y}_p(\mathfrak{gl}_2)$. Then*

1. *$x\in\mathrm{Y}^S$ if and only if whenever $\mathrm{G}(x)_{\ell,\ell'}\neq 0$, for some ℓ, $\ell'\in\mathcal{L}_0$, implies that $\ell-\ell'\in S$.*
2. *$y = F_{\mathcal{L}_x\setminus S,z}\cdot x\in\mathrm{Y}^S$ for any $z\in\Gamma$.*
3. *Y^S is a finitely generated left (right) $\Gamma-$module and $\mathrm{Y}^S=\mathbb{D}(\mathrm{Y}^S)$.*
4. *$\mathrm{Y}^{\{0\}}=\Gamma$.*

Proof. The statement (1) follows from definitions. Let $F=F_{\mathcal{L}_x\setminus S,z}$. To prove (2) calculate the matrix element $\mathrm{G}(y)_{\ell\ell'}$ provided $\ell-\ell'\notin S$. If $\ell-\ell'\notin\mathcal{L}_x$ then $\mathrm{G}(x)_{\ell\ell'}=0$ and hence $\mathrm{G}(y)_{\ell\ell'}=0$. Suppose that $\ell-\ell'\in\mathcal{L}_x\setminus S$ then by Lemma 6.1, $\mathrm{G}(y)_{\ell\ell'}=\mathrm{G}(F\cdot x)_{\ell\ell'}=F(\mathsf{b}+\ell,\mathsf{b}+\ell')\mathrm{G}(x)_{\ell\ell'}$. But

$$F(\mathsf{b}+\ell,\mathsf{b}+\ell') = \prod_{\delta\in\mathcal{L}_x\setminus S}(z(\mathsf{b}+\ell)-z(\mathsf{b}+\ell'+\delta))$$

which is equal to zero. This proves (2).

Let $x \in \mathbb{D}(\mathrm{Y}^S)$ and $z \in \Gamma$ is such that $z \neq 0$ and $zx \in \mathrm{Y}^S$. Since $\mathrm{G}(zx)_{\ell\ell'} = z(\mathsf{b}+\ell)\mathrm{G}(x)_{\ell\ell'}$ then $\mathrm{G}(zx)_{\ell\ell'} = 0$ if and only if $\mathrm{G}(x)_{\ell\ell'} = 0$ implying that $x \in \mathrm{Y}^S$. Hence $\mathrm{Y}^S = \mathbb{D}(\mathrm{Y}^S)$.

Consider $r(\mathrm{Y}^S)$ as a $\Gamma-$submodule of X_0 where $r : \mathrm{Y}_p(\mathfrak{gl}_2) \longrightarrow X_0$ is defined in Corollary 10. Then $r(\mathrm{Y}^S)$ belongs to a free $\mathbb{L}-$submodule of X_0 of finite rank $\sum_{\ell\in S} \mathbb{L} e_{\bar{0}\ell}$. Hence $\mathbb{L}\cdot r(\mathrm{Y}^S)$ is finitely generated $\mathcal{L}$-module. Without loss of generality we can assume that it is generated by the elements $r(x_1), \dots, r(x_s) \in r(\mathrm{Y}^S)$, i.e. $\mathbb{L}\cdot r(\mathrm{Y}^S) = \sum_{i=1}^{s} \mathbb{L}\cdot r(x_i)$. Since $\mathbb{D}(\mathrm{Y}^S) = \mathrm{Y}^S$ we have that $\mathbb{D}(\sum_{i=1}^{s}\Gamma x_i) \subset \mathrm{Y}^S$. Fix $x \in \mathrm{Y}^S$. Then $r(x) = \sum_{i=1}^{s} t_i r(x_i)$, $t_i \in \mathbb{L}$. Note that for any $y \in \mathrm{Y}^S$ and any $\sigma \in S_p \times S_{2p}$, $\sigma \cdot r(y) = r(y)$. Hence $p!(2p)!r(x) = \sum_{\sigma\in S_p\times S_{2p}} \sigma\cdot r(x) = \sum_{\sigma\in S_p\times S_{2p}} \sum_{i=1}^{s} (\sigma\cdot t_i)\sigma\cdot r(x_i)$ which can be rewritten as follows

$$r(x) = \frac{1}{p!(2p!)}\sum_{i=1}^{s} u_i r(x_i),$$

where $u_i = \sum_{\sigma\in S_p\times S_{2p}} \sigma\cdot t_i$. Since each u_i is $S_p\times S_{2p}$-invariant then it belongs to the field of fractions $K(\Gamma)$ for all $i = 1,\dots,s$. Multiplying both parts of the last equality by the common denominator of u_i we obtain that $x \in \mathbb{D}(\sum_{i=1}^{s}\Gamma x_i)$ and thus $\mathbb{D}(\sum_{i=1}^{s}\Gamma x_i) = \mathrm{Y}^S$. Applying Corollary 3 we conclude that Y^S is finitely generated over Γ. This proves (3). By the definition of Y^S, $x \in \mathrm{Y}^{\{0\}}$ if and only if $\mathrm{G}(x)$ is diagonal. Hence $x \in \Gamma$ by Corollary 9, (2). ☐

Let $\mathbf{m}$, $\mathbf{n} \in \operatorname{Specm}\Gamma$, $\ell_{\mathbf{m}}$, $\ell_{\mathbf{n}} \in \mathcal{L}$ are such that $\imath^*(\ell_{\mathbf{m}}) = \mathbf{m}$ and $\imath^*(\ell_{\mathbf{n}}) = \mathbf{n}$. Denote

$$S(\mathbf{m},\mathbf{n}) \ = \{\sigma_1\ell_{\mathbf{n}} - \sigma_2\ell_{\mathbf{m}} \,|\, \sigma_1,\sigma_2 \in S_p\times S_{2p}\} \cap \mathcal{L}_0.$$

Consider the following subset in $\mathcal{L}$

$$\mathcal{L}_2 = \{\ell\in\mathcal{L} \,|\, \ell_i - \ell_j \notin \mathbb{Z}\setminus\{0\}, i,j = 1,\dots,p\}$$

and set $\Omega = \imath^*(\mathcal{L}_2)$.

Proposition 6.1. (1) *For all* $\mathbf{m}$, $\mathbf{n} \in \operatorname{Specm}\Gamma$ *and all* $m, n \geq 0$ *holds*

$$\mathrm{Y}_p(\mathfrak{gl}_2) = \mathrm{Y}^S + \mathbf{n}^n\mathrm{Y}_p(\mathfrak{gl}_2) + \mathrm{Y}_p(\mathfrak{gl}_2)\mathbf{m}^m,$$

where $S = S(\mathbf{m},\mathbf{n})$.

(2) *For all* $\mathbf{m}$, $\mathbf{n} \in \operatorname{Specm}\Gamma$ *a system of generators of* Y^S *as a left* Γ*-module (as a right* Γ*-module) generates* $\mathcal{A}(\mathbf{m},\mathbf{n})$ *as a left* $\Gamma_{\mathbf{n}}-$*module (as a right* $\Gamma_{\mathbf{m}}-$*module), i.e.* $\mathcal{A}(\mathbf{m},\mathbf{n})$ *is finitely generated as a left* $\Gamma_{\mathbf{n}}$ *and as a right* $\Gamma_{\mathbf{m}}-$*module. In particular, the algebra* Γ *is big in every* $\mathbf{n} \in \operatorname{Ob}\mathcal{A}$.

(3) *If* $S(\mathbf{m},\mathbf{n}) = \varnothing$ *then* $\mathcal{A}(\mathbf{m},\mathbf{n}) = 0$ *(cf.* [DFO2]*, Corollary 27).*

(4) *If* $S(\mathbf{m},\mathbf{n}) = \{0\}$ *then* $\mathcal{A}(\mathbf{m},\mathbf{n})$ *is generated as a left* $\Gamma_{\mathbf{n}}$ *and as a right* $\Gamma_{\mathbf{m}}$*-module by the image of* 1 *in* $\mathcal{A}(\mathbf{m},\mathbf{n})$.
(5) *If* $S(\mathbf{m},\mathbf{m}) = \{0\}$ *then* $\mathbf{m}\in\Omega$, $\mathcal{A}(\mathbf{m},\mathbf{m})$ *is a quotient algebra of* Γ *and* $\chi_{\mathbf{m}}$ *extends uniquely to an irreducible* $\mathrm{Y}_p(\mathfrak{gl}_2)$*-module.*
(6) *If* $\ell_{\mathbf{m}}\in\mathcal{L}_1$ *then* $\mathcal{A}(\mathbf{m},\mathbf{m}) = \Gamma_{\mathbf{m}}$.
(7) *Let* $\ell\in\mathcal{L}_1$, $\mathbf{m}=\imath^*(\ell)$ *and* $\mathbf{n}=\imath^*(\ell+\delta_i)$, $i\in\{1,\dots,p\}$. *Then* $\mathcal{A}(\mathbf{m},\mathbf{n})$ *is a free of rank* 1 *right* $\Gamma_{\mathbf{m}}$*- (left* $\Gamma_{\mathbf{n}}$*-) module.*

Proof. (1) It is enough to show that for any $x\in\mathrm{Y}_p(\mathfrak{gl}_2)$ and any $k\geqslant 1$ there exists $x_k\in\mathrm{Y}^S$ such that

$$x\in x_k+\sum_{i=0}^{k}\mathbf{n}^{k-i}x\mathbf{m}^i. \tag{6.28}$$

The statement will follow if we choose $k=m+n+1$. We will use induction on k. If $\mathcal{L}_x\subset S$ then $x\in\mathrm{Y}^S$ and there is nothing to prove. Note that by the definition of the set S for any $\ell\in\mathcal{L}_x\setminus S$ the $S_p\times S_{2p}$-orbits of $\ell_{\mathbf{n}}$ and $\ell_{\mathbf{m}}+\ell$ are disjoint. Hence there exists $z\in\Gamma$ such that $z(\ell_{\mathbf{n}})\neq z(\ell_{\mathbf{m}}+\ell)$ for any $\ell\in\mathcal{L}_x\setminus S$. Let $F=F_{\mathcal{L}_x\setminus S,z}$. Then $F(\ell_{\mathbf{m}},\ell_{\mathbf{n}})=\prod_{\ell\in\mathcal{L}_x\setminus S}(z(\ell_{\mathbf{n}})-z(\ell_{\mathbf{m}}+\ell))\neq 0$ since every factor F is non-zero. We can assume that $F(\ell_{\mathbf{m}},\ell_{\mathbf{n}})=1$. Hence we obtain that $F=1+u$ where $u\in\mathbf{n}\otimes\Gamma+\Gamma\otimes\mathbf{m}$. It follows from Lemma 6.3, (2) that $x_1=F\cdot x$ belongs to Y^S. Hence we have $x_1=(1+u)\cdot x\in x+\mathbf{n}x\Gamma+\Gamma x\mathbf{m}$ and $x\in x_1+\mathbf{n}x\Gamma+\Gamma x\mathbf{m}$. This proves the base of induction. Assume that 6.28 holds for some $k\geq 1$. Then

$$x\in x_k+\sum_{i=0}^{k}\mathbf{n}^{k-i}(x_k+\sum_{j=0}^{k}\mathbf{n}^{k-j}x\mathbf{m}^j)\mathbf{m}^i\subset x_k+\sum_{i=0}^{k}\mathbf{n}^{k-i}x_k\mathbf{m}^i+\sum_{i=0}^{k+1}\mathbf{n}^{k+1-i}x\mathbf{m}^i.$$

Since Y^S is a Γ-bimodule we conclude that $x_k+\sum_{i=0}^{k}\mathbf{n}^{k-i}x_k\mathbf{m}^i\subset\mathrm{Y}^S$ which implies the statement (1). In particular,

$$x_{k+1}-x_k\in\sum_{i=0}^{k}\mathbf{n}^{k-i}\mathrm{Y}^S\mathbf{m}^i.$$

(2) We prove the statement for the case of left module, the case of the right module can be treated analogously. By (1) the image $\overline{x}$ of every $x\in\mathrm{Y}^S$ in $\mathcal{A}(\mathbf{n},\mathbf{m})$ is the limit of the sequence $(\overline{x}_k)_{k\geqslant 1}$, $x_k\in\mathrm{Y}^S$. Let $y_1,\dots,y_m$ be a finite system of generators of Y^S as a left $\Gamma-$module. Then for every $N>1$ there exists a maximal d_N such that

$$y_i\mathbf{m}^N\subset\sum_{j=1}^{m}\mathbf{n}^{d_N}y_j$$

for all $i=1,\dots,m$. Note that by the proof of (1), $x_{k+1}-x_k\in\sum_{i=0}^{k}\mathbf{n}^{k-i}\mathrm{Y}^S\mathbf{m}^i\subset\mathbf{n}^{R_k}\mathrm{Y}^S$ where $R_k=\min\{[k/2],d_{[k/2]}\}$. Since Y^S is a finitely generated right Γ-module and Γ is noetherian then the intersection $\cap_{k\geq 1}\mathrm{Y}^S\mathbf{m}^k=0$. It follows that

$d_N \to \infty$ while $N \to \infty$. Since

$$\overline{x} = \overline{x}_1 + \sum_{k=1}^{\infty} \overline{(x_{k+1} - x_k)}$$

we have $\overline{x} \in \sum_{k=1}^{\infty} \overline{\mathbf{n}^{R_k} \mathrm{Y}^S} \subset \sum_{l=1}^{m} \Gamma_{\mathbf{n}} \overline{y_l}$. Note that the first sum is well defined since $R_k \to \infty$ when $k \to \infty$. We conclude that $\mathcal{A}(\mathbf{n}, \mathbf{m})$ is finitely generated as left $\Gamma_{\mathbf{n}}-$module. This completes the proof of (2).

(3) If $S = \varnothing$, then $\mathrm{Y}^S = 0$ and the statement follows from (1) and the definition of the category $\mathcal{A}$ (2.8).

(4) By the definition of Y^S for every $x \in \mathrm{Y}^{\{0\}}$ the matrix $\mathrm{G}(x)$ is diagonal. Following Corollary 9, (2) it means $x \in \Gamma$, in particular $\mathcal{A}(\mathbf{m}, \mathbf{n})$ is generated (both as a left and as a right module) by the image of $1 \in \Gamma$.

(5) By (4), $\mathrm{Y}^0 = \Gamma$, i.e. $\mathcal{A}(\mathbf{m}, \mathbf{n})$ is 1-generated as a left $\Gamma_{\mathbf{m}}$-module. Then the $\Bbbk$-algebra homomorphism $\hat{\imath}_{\mathbf{m}} : \Gamma_{\mathbf{m}} \longrightarrow \mathcal{A}(\mathbf{m}, \mathbf{m})$, $z \longmapsto z \cdot \mathbf{1_m}$, where $\mathbf{1_m}$ is a unit morphism, is an epimorphism which shows that $A(\mathbf{m}, \mathbf{m})$ is a quotient algebra of $\Gamma_{\mathbf{m}}$. The uniqueness of extension follows from the uniqueness of the simple $A(\mathbf{m}, \mathbf{m})$-module and [DFO2], Theorem 18.

(6) Let $\ell = \ell_{\mathbf{m}}$. Since $\ell \in \mathcal{L}_1$ then for any $k > 0$ there exists a canonical projection $\pi_k : \mathbb{L} \longrightarrow \mathbb{L}/\ell^k \mathbb{L}$. It induces a homomorphism of the matrix algebras $\pi_k : \mathrm{M}_{\mathcal{L}_0}(\mathbb{L}) \longrightarrow \mathrm{M}_{\mathcal{L}_0}(\mathbb{L}/\ell^k)$ and defines a Harish-Chandra module by the following composition

$$\mathrm{Y}_p(\mathfrak{gl}_2) \xrightarrow{\mathrm{G}} \mathrm{M}_{\mathcal{L}_0}(\mathbb{L}) \xrightarrow{\pi_k} \mathrm{M}_{\mathcal{L}_0}(\mathbb{L}/\ell^k).$$

For any $x \in \Gamma$ there exists $k > 0$, such that $x \notin (\ell)^k$. Then $\pi_k \mathrm{G}(x)_{\overline{0},\overline{0}} = x + (\ell)^k \neq 0$ that completes the proof.

(7) The proof is analogous to the proof of (6). Let $z \in \Gamma$, $z \neq 0$. Suppose $\mathcal{A}(\mathbf{m}, \mathbf{n})z = 0$. Then by the construction of the equivalence $\mathbb{F} : \mathcal{A} - \mathrm{mod}_d \longrightarrow \mathbb{H}(U, \Gamma)$ for any Harish-Chandra module M and any $x \in \mathrm{Y}_p(\mathfrak{gl}_2)$ the linear operator xz on M induces a zero map between $M(\mathbf{m})$ and $M(\mathbf{n})$. It is enough to construct a Harish-Chandra module where this is failed. For $k \geq 1$ consider a natural map $\pi_k : \mathbb{L} \to \mathbb{L}/(\ell)^k$ and a composition $\pi_k \cdot \mathrm{G} : \mathrm{Y}_p(\mathfrak{gl}_2) \to \mathrm{M}_{\mathcal{L}_0}(\mathbb{L}/(\ell)^k)$. It defines a Harish-Chandra module structure on a free $\mathbb{L}/(\ell)^k$-module $\bar{X} = \sum_{\delta \in \mathcal{L}_0} \mathbb{L}/(\ell)^k e_{\delta,\overline{0}}$. Consider $x \in \mathrm{Y}_p(\mathfrak{gl}_2)$ such that $\mathrm{G}(x)_{\delta_i \overline{0}} \neq 0$. Then $\mathrm{G}(xz)_{\delta_i \overline{0}} = \mathrm{G}(x)_{\delta_i \overline{0}} \mathrm{G}(z)_{\overline{0}\overline{0}} = \mathrm{G}(x)_{\delta_i \overline{0}} z \neq 0$. Choose k such that $\mathrm{G}(xz)_{\delta_i \overline{0}} \notin (\ell)^k$. Hence $(\pi_k \cdot \mathrm{G})(xz)_{\delta_i, \overline{0}} \neq 0$ and the linear operator xz induces a non-zero map between $\bar{X}(\mathbf{m}) = \mathbb{L}/(\ell)^k$ and $\bar{X}(\mathbf{n}) = \mathbb{L}/(\ell + \delta_i)^k$. The obtained contradiction shows that $\mathcal{A}(\mathbf{m}, \mathbf{n})z \neq 0$. The case $z\mathcal{A}(\mathbf{m}, \mathbf{n}) = 0$ is treated analogously. □

Now we are in the position to state the main result of this section which follows immediately from Lemma 2.1 and Proposition 6.1, (2).

Theorem 4. *Let* $\mathbf{m} \in \operatorname{Specm} \Gamma$. *Then the left ideal* $\mathrm{Y}_p(\mathfrak{gl}(2))\mathbf{m}$ *is contained in finitely many maximal left ideals of* $\mathrm{Y}_p(\mathfrak{gl}(2))$. *In particular,* $\mathbf{m}$ *extends to a finitely many (up to an isomorphism) irreducible* $\mathrm{Y}_p(\mathfrak{gl}(2))-$*modules and for each such module* M, $\dim M(\mathbf{n}) < \infty$ *for all* $\mathbf{n} \in \operatorname{Specm} \Gamma$.

7. Category of generic Harish-Chandra modules

Lemma 7.1. *Let $\ell \in \mathcal{L}_1$, $\ell = (\beta, \gamma)$, $\mathbf{m} = \imath^*(\ell) \in \operatorname{Specm}\Gamma$, $\mathbf{n} = \imath^*(\ell + \delta_i)$, $i \in \{1, \dots, p\}$. If $\beta_i \notin \{\gamma_1, \dots, \gamma_{2p}\}$ then the objects of $\mathcal{A}$ represented by $\mathbf{m}$ and $\mathbf{n}$ are isomorphic.*

Proof. Choose $z_1, z_2 \in \Gamma$ such that $z_1(\ell+\delta_j) = \delta_{ij}$, $z_2(\ell+\delta_i-\delta_j) = \delta_{ij}$, $j = 1, \dots, p$. Denote $z = z_2 t_{12}^{(1)} z_1 t_{21}^{(1)}$. Then $G(z)$ is diagonal by Lemma 6.1 and hence $z \in \Gamma$ by Corollary 9, (2). We will show that the image of z in $\Gamma_{\mathbf{m}}$ is invertible. Clearly, this is equivalent to the fact that $z(\mathbf{m}) \neq 0$. Note that $z(\mathbf{m}) = z(\ell)$. Thus applying formulas (4.15)–(4.17) we have $z(\mathbf{m}) = \gamma(-\beta_i) \neq 0$ since $\ell \in \mathcal{L}_1$. Denote by T_1 (respectively T_2) the generator of $\hat{\Gamma}$-bimodule $\mathcal{A}(\mathbf{m}, \mathbf{n})$ (respectively $\mathcal{A}(\mathbf{n}, \mathbf{m})$) (Proposition 6.1, (7)). Then $z_2 t_{12}^{(1)} = z_{\mathbf{m}} T_2, z_1 t_{21}^{(1)} = T_1 z'_{\mathbf{m}}$ for some $z_{\mathbf{m}}, z'_{\mathbf{m}} \in \Gamma_{\mathbf{m}}$ and $z = z_{\mathbf{m}} T_2 T_1 z'_{\mathbf{m}}$. Since $z(\mathbf{m}) \neq 0$ it follows that $z'_{\mathbf{m}}(\mathbf{m}) \neq 0, z_{\mathbf{m}}(\mathbf{m}) \neq 0$ and hence $T_2 T_1 = z_{\mathbf{m}}^{-1} z (z'_{\mathbf{m}})^{-1}$ is invertible in $\Gamma_{\mathbf{m}}$. The similar argument shows that $T_1 T_2$ is invertible in $\Gamma_{\mathbf{n}}$. Therefore the objects $\mathbf{m}$ and $\mathbf{n}$ are isomorphic. □

Corollary 11. *Let $\ell \in \mathcal{L}_1$, $\ell = (\beta, \gamma)$, $\beta_i - \gamma_j \notin \mathbb{Z}$ and $\mathbf{m} = \imath^*(\ell) \in \operatorname{Specm}\Gamma$. Then the category $\mathbb{H}(\mathrm{Y}_p(\mathfrak{gl}_2), \Gamma, D(\ell))$ is hereditary. Moreover,*

$$\dim \operatorname{Ext}^1_{\mathbb{H}(\mathrm{Y}_p(\mathfrak{gl}_2),\Gamma,D(\ell))}(L(\ell), L(\ell)) = 3p.$$

Proof. By Lemma 7.1 and our assumptions all objects of the category $\mathcal{A}(D(\ell))$ are isomorphic and hence the category $\mathcal{A}(D(\ell))-\mod_d$ is equivalent to the category of finite-dimensional modules over $\Gamma_{\mathbf{m}}$. Applying Proposition 2.1 we conclude that the category $\mathbb{H}(\mathrm{Y}_p(\mathfrak{gl}_2), \Gamma, D(\ell))$ is hereditary. Since $\Gamma_{\mathbf{m}}$ is an algebra of power series in $3p$ variables the statement about $\dim \operatorname{Ext}^1$ follows. □

7.1. **Category of generic weight modules.** Fix $\ell \in \mathcal{L}_1$, $\mathbf{m} = \imath^*(\ell), \mathbf{n} = \imath^*(\ell + \delta_i) \in \operatorname{Specm}\Gamma$, $i \in \{1, \dots, p\}$. Then $\mathcal{A}_W(\mathbf{m}, \mathbf{m}) \simeq \Gamma_{\mathbf{m}}/\Gamma_{\mathbf{m}}\mathbf{m} \simeq \Bbbk$ by Proposition 6.1, (6) and $\dim \mathcal{A}_W(\mathbf{m}, \mathbf{n}) = 1$ by Proposition 6.1, (7). We will give a direct construction of the category $\mathcal{A}_W(D(\ell))$.

Suppose $\ell = (\beta, \gamma)$, $\beta = (\beta_1, \dots, \beta_p) \in \Bbbk^p$, $\gamma = (\gamma_1, \dots, \gamma_{2p}) \in \Bbbk^{2p}$ and

$$\gamma(u) = \prod_{i=1}^{2p} (u + \gamma_i). \tag{7.29}$$

Since $\ell \in \mathcal{L}_1$ then $\beta_i - \beta_j \notin \mathbb{Z}$ for all $i, j = 1, \dots, p$, $i \neq j$. Consider the following category K_ℓ: $\operatorname{Ob}(K_\ell) = \mathbb{Z}^p$ and the morphisms are generated by

$$f_i(k) : (k) \mapsto (k + \delta_i) \qquad \text{and} \qquad e_i(k) : (k) \mapsto (k - \delta_i), \tag{7.30}$$

where $i = 1, \dots, p$ and $(k) = (k_1, \dots, k_p) \in \mathbb{Z}^p$ with the following relations:

$$\begin{aligned}
&f_j(k + \delta_i)\, f_i(k) = f_i(k + \delta_j)\, f_j(k),\\
&e_j(k - \delta_i)\, e_i(k) = e_i(k - \delta_j)\, e_j(k),\\
&e_i(k + \delta_j)\, f_j(k) = f_j(k - \delta_i)\, e_i(k) \qquad \text{for} \quad i \neq j,\\
&e_i(k + \delta_i)\, f_i(k) = -\gamma(-\beta_i - k_i)\, 1_{(k)},\\
&f_i(k - \delta_i)\, e_i(k) = -\gamma(-\beta_i - k_i + 1)\, 1_{(k)}.
\end{aligned}$$

It follows immediately from Lemmas 4.1 and 4.2 that any module in the category R_ℓ can be naturally viewed as a module over the category K_ℓ which defines a functor $F : R_\ell \to K_\ell{-}\mathrm{mod}$. Consider the cyclic subalgebra $C_\ell(a) = \operatorname{Hom}_{K_\ell}(a, a)$ for any

$a \in \mathbb{Z}^p$. Clearly, $C_\ell(a) \simeq \Bbbk$ for any $a \in \mathbb{Z}^p$ due to the defining relations of K_ℓ. For any $a = (k_1, \ldots, k_p) \in \mathbb{Z}^p$ we can construct a universal module $M(\ell, a) \in K_\ell{-}\mathrm{mod}$. Consider $\Bbbk$ as a $C_\ell(a)$-module with

$$e_i(k + \delta_i)\, f_i(k)\, 1 = -\gamma(-\beta_i - k_i),$$
$$f_i(k - \delta_i)\, e_i(k)\, 1 = -\gamma(-\beta_i - k_i + 1).$$

Let $A_{\ell,a}$ be an algebra of paths in K_ℓ originating in a. Now construct a $\mathbb{Z}^p$-graded K_ℓ-module

$$M(\ell, a) = A_{\ell,a} \otimes_{C_\ell(a)} \Bbbk.$$

Clearly, all graded components of $M(\ell, a)$ are 1-dimensional and $M(\ell, a)_a = 1_a \otimes \Bbbk$. A module $M(\ell, a)$ contains a unique maximal $\mathbb{Z}^p$-graded submodule which intersects $M(\ell, a)_a$ trivially and hence has a unique irreducible quotient $L(\ell, a)$ with $L(\ell, a)_a \simeq \Bbbk$ and $\dim L(\ell, a)_b \le 1$ for all $b \in \mathbb{Z}^p$. If V is another irreducible K_ℓ-module with $V_a \neq 0$ then there exists a non-trivial $C_\ell(a)$-homomorphism from $\Bbbk$ to V_a which can be extended to an epimorphism from $M(\ell, a)$ to V. Since V is irreducible we conclude that $V \simeq L(\ell, a)$.

Obviously, we can view $M(\ell)$ as a module over the category K_ℓ with a natural action of the morphisms of K_ℓ and $F(M(\ell)) = M(\ell, \beta)$. Thus a K_ℓ-module $M(\ell, \beta)$ can be extended to a $\mathrm{Y}_p(\mathfrak{gl}_2)$-module $M(\ell)$. Moreover, the functor F preserves the submodule structure of $M(\ell)$. In particular, $F(L(\ell)) = L(\ell, \beta)$.

Proposition 7.1. *If $\ell \in \mathcal{L}_1$ then the categories $K_\ell{-}\mathrm{mod}$ and R_ℓ are equivalent.*

Proof. Let $\ell = (\beta, \gamma)$. We already have a functor $F : R_\ell \to K_\ell{-}\mathrm{mod}$. Suppose that $V \in K_\ell{-}\mathrm{mod}$. We want to show that V can be extended to a $\mathrm{Y}_p(\mathfrak{gl}_2)$-module. Fix $v \in V_{(k)} \setminus \{0\}$. Let $W \subseteq V$ be a submodule generated by v. Then $W_{(k)} = \Bbbk v$ and there is an epimorphism from $M(\ell, a)$ to W, where $a = (k_1, \ldots, k_p)$, which maps $1_a \otimes 1$ to v. Since $F(M(\ell')) = M(\ell, a)$, where $\ell' = (\beta + a, \gamma)$, then W can be extended to a corresponding quotient of $M(\ell')$. Since v was an arbitrary element of V we conclude that V can be extended to a $\mathrm{Y}_p(\mathfrak{gl}_2)$-module and will denote that module by $G(V)$. Clearly, G defines a functor from $K_\ell{-}\mathrm{mod}$ to R_ℓ (action on morphisms is obvious). One can easily see that the functors F and G define an equivalence between the categories $K_\ell{-}\mathrm{mod}$ and R_ℓ. □

7.2. **Support of irreducible generic weight modules.** To complete the classification of irreducible modules we have to know when two irreducible modules $L(\ell)$ and $L(\ell')$ are isomorphic. For that we need to describe the support $\operatorname{Supp} L(\ell)$.

We shall say that the weight subspaces $M(\ell)_\psi$ and $M(\ell)_{\psi+\delta_i}$ are *strongly isomorphic* if $\gamma(-\psi_i) \neq 0$ where $\psi = (\psi_1, \ldots, \psi_p)$. This implies

$$f_i(\psi_1, \ldots, \psi_p)\, M(\ell)_\psi \neq 0 \quad \text{and} \quad e_i(\psi_1, \ldots, \psi_i + 1, \ldots, \psi_p)\, M(\ell)_{\psi+\delta_i} \neq 0.$$

The statement below follows immediately from the relations in K_ℓ (cf. also Corollary 5).

Lemma 7.2. *If $M(\ell)_\psi$ and $M(\ell)_{\psi+\delta_i}$ are strongly isomorphic, then $M(\ell)_{\psi\pm\delta_j}$ and $M(\ell)_{\psi+\delta_i\pm\delta_j}$ are strongly isomorphic for all $i, j = 1, \ldots, p$, $i \neq j$. Moreover, if*

$$f_i(\psi_1, \ldots, \psi_p)\, M(\ell)_\psi = 0 \qquad \text{or} \qquad e_i(\psi_1, \ldots, \psi_p)\, M(\ell)_\psi = 0$$

then

$$f_i(\psi_1, \ldots, \psi_j \pm 1, \ldots, \psi_p)\, M(\ell)_{\psi\pm\delta_j} = 0 \qquad \text{or}$$
$$e_i(\psi_1, \ldots, \psi_j + 1, \ldots, \psi_p)\, M(\ell)_{\psi\pm\delta_j} = 0,$$

respectively, for all $j \neq i$.

Let $a_i, a_i' \in \mathbb{Z} \cup \{\pm\infty\}, a_i \leq a_i', i \in \{1, \ldots, p\}$. Denote

$$P(a_1, \ldots, a_p, a_1', \ldots, a_p') = \{(x_1, \ldots, x_p) \in \mathbb{Z}^p \mid a_i \leq x_i \leq a_i', \ i = 1, \ldots, p\},$$

a parallelepiped in $\mathbb{Z}^p$. Note that some faces of the parallelepiped can be infinite in some directions. In particular, in the case $a_i = -\infty$, $a_i' = \infty$ for all i, the parallelepiped coincides with $\mathbb{Z}^p$.

Theorem 5. *For any irreducible weight module $L(\ell)$ over $\mathrm{Y}_p(\mathfrak{gl}_2)$ there exist elements $a_i, b_i \in \mathbb{Z} \cup \{\pm\infty\}$, $a_i \leq a_i'$, $i \in \{1, \ldots, p\}$ such that*

$$\operatorname{Supp} L(\ell) = P(a_1, \ldots, a_p, a_1', \ldots, a_p').$$

Proof. Let $\ell = (\beta, \gamma) \in \mathcal{L}_1$. Fix $i \in \{1, \ldots, p\}$. If $\gamma(-\beta_i + k) \neq 0$ for all $k \in \mathbb{Z}$ then $(k_1, \ldots, k_i + m, \ldots, k_p) \in \operatorname{Supp} L(\ell)$ as soon as $(k_1, \ldots, k_p) \in \operatorname{Supp} L(\ell)$. This follows immediately from Lemma 7.2. In this case we set $a_i = -\infty$ and $a_i' = \infty$. Let now $\gamma(-\beta_i + k) = 0$ for some $k \in \mathbb{Z}$. Let $m \geq 0$ be the smallest integer (if exists) such that $\gamma(-\beta_i - m) = 0$ and let $n \leq 0$ be the largest integer (if exists) such that $\gamma(-\beta_i - n + 1) = 0$. It follows from Lemma 7.2 that

$$\operatorname{Supp} L(\ell) \cap \{\beta + k\delta_i \mid k \in \mathbb{Z}\} = \{\beta + n\delta_i, \ldots, \beta, \ldots, \beta + m\delta_i\}.$$

If $\beta + s\delta_j \in \operatorname{Supp} L(\ell)$, $j \neq i$ then

$$\operatorname{Supp} L(\ell) \cap \{\beta + s\delta_j + k\delta_i \mid k \in \mathbb{Z}\} = \{\beta + s\delta_j + n\delta_i, \ldots, \beta + s\delta_j, \ldots, \beta + s\delta_j + m\delta_i\}.$$

In this case we set $a_i = \beta_i + n$ and $a_i' = \beta_i + m$. The statement of the theorem now follows. □

7.3. **Indecomposable generic weight modules.** Fix $\ell = (\beta, \gamma) \in \mathcal{L}_1$. A full subcategory $\mathcal{S} \subseteq K_\ell$ is called a *skeleton* of K_ℓ provided the objects of $\mathcal{S}$ are pairwise non-isomorphic and any object of K_ℓ is isomorphic to some object of $\mathcal{S}$. In this case the categories of $K_\ell-$mod and $\mathcal{S}-$mod are equivalent.

For each $i \in \{1, \ldots, p\}$ consider a set $I_i = \{k \in \mathbb{Z} \mid \gamma(-\beta_i - k) = 0\}$. Define a category S_ℓ as a $\mathbb{k}$-category with the set of objects

$$S_0 = \{0, \ldots, |I_1|\} \times \ldots \times \{0, \ldots, |I_p|\}$$

and with morphisms generated by

$$r^k_{(i_1,\ldots,i_p)} : (i_1, \ldots, i_p) \mapsto (i_1, \ldots, i_k + 1, \ldots, i_p),$$
$$s^k_{(j_1,\ldots,j_p)} : (j_1, \ldots, j_p) \mapsto (j_1, \ldots, j_k - 1, \ldots, j_p),$$

where $k \in \{1, \ldots, p\}$ is such that $I_k \neq \emptyset$, $i_k < |I_k|$, $j_k > 0$, subject to the relations:

$$s^k_{(i_1,\ldots,i_k+1,\ldots,i_p)} r^k_{(i_1,\ldots,i_p)} = r^k_{(i_1,\ldots,i_p)} s^k_{(i_1,\ldots,i_k+1,\ldots,i_p)} = 0$$

and

$$x^k_{(a_1,\ldots,a_p)} y^r_{(e_1,\ldots,e_p)} = y^r_{(c_1,\ldots,c_p)} x^k_{(e_1,\ldots,e_p)}$$

for all $k \neq r$ and all possible $x, y \in \{r, s\}$, $a_i, e_i, c_i, 1 \leq i \leq p$ for which this equality makes sense.

It follows from the construction that S_ℓ is the skeleton of the category K_ℓ. Note that the corresponding algebra is finite-dimensional. In particular, S_ℓ is semisimple when $I_k = \emptyset$ for all $1 \leq k \leq p$, i.e. when $\gamma(-\beta_k + r) \neq 0$ for all $k \in \mathbb{Z}$ and all $i = 1, \ldots, p$. Hence it is enough to describe all indecomposable modules over S_ℓ.

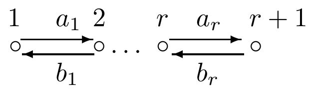
Fix $a \in S_0$ and define a simple S_ℓ-module S_a such that $S_a(b) = \delta_{a,b}\Bbbk$ for all $b \in S_0$ and all morphisms are trivial. Since S_ℓ defines a finite-dimensional algebra we have the following

Proposition 7.2. *Any simple module over S_ℓ is isomorphic to S_a for some $a \in S_0$.*

This is another confirmation of the fact that all weight spaces in any irreducible generic weight $\mathrm{Y}_p(\mathfrak{gl}_2)$-module are 1-dimensional. But this need not to be the case for indecomposable modules. We restrict ourselves to a full subcategory $R_\ell^f \subseteq R_\ell$ which consists of weight modules V with $\dim V_\psi < \infty$ for all $\psi \in \operatorname{Supp} V$. We will establish the representation type of the category R_ℓ^f (finite, tame or wild). For necessary definitions we refer to [Dr].

To establish the representation type of the category R_ℓ^f it is enough to consider the category $S_\ell{-}\mathrm{mod}^f$, of modules over the category S_ℓ with finite-dimensional weight spaces. Denote $X_\ell = \{k \in \{1, \ldots, p\} \mid I_k \neq \emptyset\}$.

7.3.1. *Indecomposable modules in the case $|X_\ell| = 1$.* In this section we describe all indecomposable modules over S_ℓ in the case $|X_\ell| = 1$. Let $X_\ell = \{i\}$ and let $|I_i| = r > 0$. In this case the category S_ℓ has the following quiver $\mathbf{A}$ with relations:



$a_i\, b_i = b_i\, a_i = 0$

We denote by S_i, $i \in \{1, \ldots, r+1\}$, the simple module corresponding to the point i. These modules correspond to all irreducible modules in R_ℓ^f by Proposition 7.2. Now describe remaining indecomposable modules for a quiver above. Fix integers $1 \leq k_1 < k_2 \leq r+1$ and a function $\xi_{k_1,k_2} : \{k_1, k_1+1, \ldots, k_2\} \to \{0,1\}$. Define a module $M = M(k_1, k_2, \xi_{k_1,k_2})$ as follows: $M(i) = \Bbbk e_i$, $k_1 \leq i \leq k_2$, $M(j) = 0$ otherwise, $a_i e_i = e_{i+1}$, $b_i e_{i+1} = 0$ if $\xi_{k_1,k_2}(i) = 1$ and $a_i e_i = 0$, $b_i e_{i+1} = e_i$ if $\xi_{k_1,k_2}(i) = 0$ for all $1 \leq i < k_2$.

The proof of the following proposition is standard; see e.g. [GR].

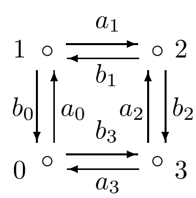
Proposition 7.3. *The modules S_i, $1 \leq i \leq r+1$ and $M(k_1, k_2, \xi_{k_1,k_2})$ with $1 \leq k_1 < k_2 \leq r+1$ and*

$$\xi_{k_1,k_2} : \{k_1, k_1+1, \ldots, k_2\} \to \{0,1\},$$

exhaust all non-isomorphic indecomposable modules for $\mathbf{A}$.

7.3.2. *Indecomposable modules in the case $|X|_\ell = 2$.* In this section we describe the indecomposable modules for S_ℓ when $|X|_\ell = 2$ and $|I_k| = 1$ for each $k \in X_\ell$. Then S_ℓ is isomorphic to the following category $\mathbf{B}$ considered in [BB].

$\mathbf{B}$:



$a_i b_i = b_i a_i = 0, \qquad i = 0, \ldots, 3,$
$a_i a_j = b_l b_m$ for any $i, j, l, m \in \{0, 1, 2, 3\}$,
where possible.

By Proposition 7.2 this category has four non-isomorphic simple modules S_i, $0 \leq i \leq 3$, with a support in a chosen point i. The indecomposable modules were described in [BB]. For the sake of completeness we repeat here this classification.

We will treat the objects of $\mathbf{B}$ as elements of $\mathbb{Z}/4\mathbb{Z}$. Consider the following three families of non-simple indecomposable modules.

Finite family. Fix an $0 \leq i \leq 3$ and define the $\mathbf{B}$-module M_i such that $M_i(j) = \Bbbk e_j$ for each $j = 0, \dots, 3$ and $a_i e_i = e_{i+1}$, $a_{i+1} e_{i+1} = e_{i+2}$, $b_{i-1} e_i = e_{i-2}$, $b_{i-2} e_{i-1} = e_{i-2}$ and $u_j e_k = 0$ for all other cases of $u \in \{a, b\}$ and $j, k = 0, \dots, 3$. Obviously, M_i is indecomposable module for any i.

Infinite discrete families. Let $n \in \mathbb{N}$, $n > 1$, and $j \in \mathbb{Z}_4$. Define a $\mathbf{B}$-module $M_{n,j,1}$ (resp., $M_{n,j,2}$) as follows. Consider n elements $e_1, \dots, e_n$. A $\Bbbk$-basis of the vector space $M_{n,j,1}(l)$ (resp., $M_{n,j,2}(l)$) is the set of e_k such that $j + k - 1 \equiv l (mod\, 4)$. The elements a_l and b_{l-1} act as follows:

$$a_l e_k = \begin{cases} e_{k+1}, & \text{if } l \text{ is even (resp., odd), } k < n \text{ and } j + k - 1 \equiv l (mod\, 4); \\ 0, & \text{otherwise.} \end{cases}$$

$$b_{l-1} e_k = \begin{cases} e_{k-1}, & \text{if } l \text{ is even (resp., odd), } k > 1 \text{ and } j + k - 1 \equiv l (mod\, 4); \\ 0, & \text{otherwise.} \end{cases}$$

All modules $M_{n,j,1}$ and $M_{n,j,2}$, $n > 1$, $0 \leq j \leq 3$ are non-isomorphic indecomposable $\mathbf{B}$-modules.

Infinite continuous families. For each $\lambda \in \Bbbk$, $\lambda \neq 0$, and $d \in \mathbb{Z}$, $d > 0$ define the $\mathbf{B}$-modules $M_{d,\lambda,1}$ and $M_{d,\lambda,2}$ as follows. Set

$$\begin{aligned} &M_{d,\lambda,1}(i) = \Bbbk^d, \\ &M_{d,\lambda,1}(a_0) = M_{d,\lambda,1}(a_2) = M_{d,\lambda,1}(b_1) = \mathbf{I}_d, \\ &M_{d,\lambda,1}(b_0) = M_{d,\lambda,1}(b_2) = M_{d,\lambda,1}(a_1) = M_{d,\lambda,1}(a_3) = 0, \\ &M_{d,\lambda,1}(b_3) = J_{d,\lambda} \end{aligned}$$

and

$$\begin{aligned} &M_{d,\lambda,2}(i) = \Bbbk^d, \\ &M_{d,\lambda,2}(b_0) = M_{d,\lambda,2}(b_2) = M_{d,\lambda,2}(a_1) = \mathbf{I}_d, \\ &M_{d,\lambda,2}(a_0) = M_{d,\lambda,2}(a_2) = M_{d,\lambda,2}(b_1) = M_{d,\lambda,2}(b_3) = 0, \\ &M_{d,\lambda,2}(a_3) = J_{d,\lambda}, \end{aligned}$$

where $J_{d,\lambda}$ is the Jordan cell of dimension d with the eigenvalue λ.

All modules $M_{d,\lambda,k}$, $k = 1, 2$ are indecomposable and corresponding indecomposable modules in R_ℓ^f have all weight spaces of dimension d.

Proposition 7.4. *(*[BB]*, Proposition 3.3.1). The modules S_i, M_i, $M_{n,i,1}$, $M_{n,i,2}$, $M_{d,\lambda,1}$, $M_{d,\lambda,2}$ where $0 \leq i \leq 3$, d is a positive integer, $\lambda \in \Bbbk$, $\lambda \neq 0$, and $n \geq 2$ is an integer, constitute an exhaustive list of pairwise non-isomorphic indecomposable* $\mathbf{B}$*-modules.*

The following theorem which describes the representation type of R_ℓ^f.

Theorem 6. (i) *If $|X_\ell| = 0$ then R_ℓ^f is a semisimple category with a unique indecomposable (=irreducible) module;*

(ii) *If $|X_\ell| = 1$ then R_ℓ^f has finite representation type;*

(iii) *If $|X_\ell| = 2$ then R_ℓ^f has tame representation type if and only if $|I_k| = 1$ for all $k \in X$. Otherwise, R_ℓ^f has wild representation type;*

(iv) *If* $|X_\ell| > 2$ *then* R_ℓ^f *has wild representation type.*

Proof. In the case when $|X_\ell| = 1$ all indecomposable modules for S_ℓ are described in Proposition 7.3. Hence R_ℓ^f has finite representation type. If $|X_\ell| = 2$ and $|I_k| = 1$ for each $k \in X$ then all indecomposable modules for S_ℓ are described in Proposition 7.4. It follows from the definition that R_ℓ^f has tame representation type in this case. If $|I_k| > 1$ for at least one k then it is easy to construct a family of indecomposable modules that depends on two continuous parameters. Hence, in this case R_ℓ^f has wild representation type. Suppose now that $|X_\ell| > 2$. Then S_ℓ contains a full subcategory of wild representation type considered in [BB], Theorem 1. We immediately conclude that R_ℓ^f has wild representation type. This completes the proof. □

Corollary 12. (1) *If* $|X_\ell| = 0$ *then the category* R_ℓ *is a semisimple category with a unique indecomposable module.*
(2) *If* $|X_\ell| = 1$ *then* R_ℓ *has finite representation type with indecomposable modules as in Proposition 7.3.*

Proof. Since cases $|X_\ell| \leq 1$ correspond to finite representation type then the corresponding categories do not admit infinite-dimensional indecomposable modules by [A] and hence every indecomposable module belongs to R_ℓ^f. □

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