REALIZATION OF HOMOTOPY INVARIANTS BY PD^3–PAIRS

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Abstract. Up to oriented homotopy equivalence, a PD^3–pair $(X, \partial X)$ with aspherical boundary components is uniquely determined by the Π_1–system $\{\kappa_i : \Pi_1(\partial X_i, *) \to \Pi_1(X, *)\}_{i\in J}$, the orientation character $\omega_X \in \mathrm{H}^1(X; \mathbb{Z}/2\mathbb{Z})$ and the image of the fundamental class $[X, \partial X] \in \mathrm{H}_3(X, \partial X; \mathbb{Z}^\omega)$ under the classifying map [3]. We call the triple $(\{\kappa_i\}_{i\in J}, \omega_X, [X, \partial X])$ the fundamental triple of the PD^3–pair $(X, \partial X)$.

Using Peter Hilton's homotopy theory of modules, Turaev [12] gave a condition for realization in the absolute case of PD^3–complexes X with $\partial X = \emptyset$. Given a finitely presentable group G and $\omega \in \mathrm{H}^1(G; \mathbb{Z}/2\mathbb{Z})$, he defined a homomorphism

$$\nu : \mathrm{H}_3(G; \mathbb{Z}^\omega) \longrightarrow [F, I]$$

where F is some $\mathbb{Z}[G]$–module, $I = \ker \mathrm{aug}$ and $[A, B]$ denotes the group of homotopy classes of $\mathbb{Z}[G]$–morphisms from the $\mathbb{Z}[G]$–module A to the $\mathbb{Z}[G]$–module B. Turaev showed that, given $\mu \in \mathrm{H}_3(G; \mathbb{Z}^\omega)$, the triple (G, ω, μ) is relized by a PD^3–complex X if and only if $\nu(\mu)$ is a class of homotopy equivalences of $\mathbb{Z}[G]$–modules.

Using Turaev's construction of the homomorphism ν, we generalize the condition for realization to the case of PD^3–pairs $(X, \partial X)$, where ∂X is not necessarily empty.

1. Outline

Section 2 is concerned with notation and the existence of Eilenberg–Mac Lane pairs.

Section 3 discusses properties of the relative twisted cap product needed for the formulation of the realization condition and the proof of sufficiency in the Π_1–injective case.

In Section 4 we briefly revise the projective homotopy category of modules over a ring, also called the stable category. The final theorem of this section plays a crucial rôle in the construction of a PD^3–pair from given invariants.

The realization condition is formulated in Section 5 and Section 6 contains the proof of the realization theorem for the Π_1–injective case.

2. Preliminaries

Let G be a group, let $\Lambda := \mathbb{Z}[G]$ be the integral group ring of G and let $\mathrm{aug} : \Lambda \to \mathbb{Z}$ denote the augmentation homomorphism determined by $\mathrm{aug}(g) := 1$ for all $g \in G$. The kernel I of the augmentation homomorphism is called the augmentation ideal.

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Furthermore, take $\omega \in \mathrm{H}^1(G,\mathbb{Z}/2\mathbb{Z})$. Since $\mathrm{H}^1(G,\mathbb{Z}/2\mathbb{Z})$ is naturally isomorphic to $\mathrm{Hom}(G,\mathbb{Z}/2\mathbb{Z})$, the cohomology class ω determines a homomorphism from G to the group $\mathbb{Z}/2\mathbb{Z} = \{0,1\}$. This homomorphism, in turn, gives rise to the anti–isomorphism

$$\bar{\ } : \Lambda \longrightarrow \Lambda;\ \lambda \longmapsto \overline{\lambda}$$

determined by

$$\overline{g} := (-1)^{\omega(g)} g^{-1} \quad \text{for } g \in G.$$

We may associate a left Λ–module with every right Λ–module and vice versa by means of the anti–isomorphism $\bar{\ }$. Namely, given a right Λ–module A and a left Λ–module B, define a left action on the set underlying A and a right action on the set underlying B by

$$\lambda.a := a.\overline{\lambda} \quad \text{for } a \in A,\ \lambda \in \Lambda;$$
$$b.\lambda := \overline{\lambda}.b \quad \text{for } b \in B,\ \lambda \in \Lambda.$$

We denote the modules thus obtained by ${}^{\omega}\!A$ and B^{ω} respectively.

Given a short exact sequence $Q \rightarrowtail P \twoheadrightarrow D$ of augmented chain complexes of left Λ–modules with compatible equivariant diagonals and a "twisting" $\omega \in \mathrm{H}^1(G,\mathbb{Z}/2\mathbb{Z})$, the relative twisted cap products are defined at the chain level by

$$\cap : \mathrm{Hom}_\Lambda(P,{}^{\omega}\!M)_{-k} \otimes (\mathbb{Z}^{\omega} \otimes_\Lambda D)_n \to (M \otimes_\Lambda D)_{n-k}$$
$$\varphi \cap (z \otimes d) := \varphi / (z \otimes \Delta_{\mathrm{rel}}(d))$$

and

$$\cap : \mathrm{Hom}_\Lambda(D,{}^{\omega}\!M)_{-k} \otimes (\mathbb{Z}^{\omega} \otimes_\Lambda D)_n \to (M \otimes_\Lambda P)_{n-k}$$
$$\varphi \cap (z \otimes d) := \varphi / (z \otimes {\Delta'}_{\mathrm{rel}}(d)).$$

for any right Λ–module M [3]. Passing to homology we obtain the relative twisted cap products

$$\cap : \mathrm{H}^k(P,{}^{\omega}\!M) \otimes \mathrm{H}_n(D,\mathbb{Z}^{\omega}) \to \mathrm{H}_{n-k}(D,M)$$

and

$$\cap : \mathrm{H}^k(D,{}^{\omega}\!M) \otimes \mathrm{H}_n(D,\mathbb{Z}^{\omega}) \to \mathrm{H}_{n-k}(P,M).$$

Now let $\{\kappa_i : G_i \to G\}_{i \in J}$ be a family of group homomorphisms and let (X,Y) be a pair of CW–complexes with Π_1–system $\{\kappa_i : G_i \to G\}_{i \in J}$. Put $\Lambda := \mathbb{Z}[G]$ and let $p : \tilde{X} \to X$ be the universal covering of X. Let $C(X)$ denote the cellular chain complex of $\tilde{X}$ viewed as a complex of Λ–modules. We denote the subcomplex of $C(X)$ generated by the cells lying above Y by $C(Y)$ and put $C(X,Y) := C(X)/C(Y)$, so that $C(Y) \rightarrowtail C(X) \twoheadrightarrow C(X,Y)$ is a short exact sequence of left Λ–modules. We call $C(X,Y)$ the relative cellular complex and $C(Y) \rightarrowtail C(X) \twoheadrightarrow C(X,Y)$ the short exact sequence of cellular chain complexes of the pair (X,Y).

Given a family $\{\kappa_i : G_i \to G\}_{i \in J}$ of group homomorphisms we may ask whether there is a pair (X,Y) which has Π_1–system $\{\kappa_i : G_i \to G\}_{i \in J}$. The answer is yes, namely, for $i \in J$ take $K(G_i;1)$ complexes Y_i and a $K(G;1)$ complex X. Then the family $\kappa_i : G_i \to G$ of homomorphisms determines a map $f : \coprod_{i \in J} Y_i \to X$. Let K be the mapping cylinder of f and identify $\coprod_{i\in J} Y_i$ with its image under the inclusion in K. Then (K, Y) is a pair with Π_1–system $\{\kappa_i : G_i \to G\}_{i\in J}$.

As we do not require the homomorphisms κ_i to be injective we will adopt the following non–standard definition for the purpose of this paper.

Definition 2.1. *Let $\{\kappa_i : G_i \to G\}_{i\in J}$ be a family of group homomorphisms. An Eilenberg–Mac Lane pair of type $K(\{\kappa_i : G_i \to G\}_{i\in J}; 1)$ is a pair (X, Y) such that X is an Eilenberg–Mac Lane complex of type $K(G; 1)$, the connected components $\{Y_i\}_{i\in J}$ of Y are Eilenberg–Mac Lane complexes of type $K(G_i; 1)$ and the Π_1–system of (X, Y) is isomorphic to $\{\kappa_i : G_i \to G\}_{i\in J}$.*

In the standard definition of Eilenberg–Mac Lane pairs given by Bieri–Eckmann in [1] the homomorphisms κ_i are required to be injective.

An Eilenberg–Mac Lane pair of type $(G, \{G_i\}_{i\in J}; 1)$ is determined up to homotopy of pairs and we write $K(G, \{G_i\}_{i\in J}; 1)$ for any such pair. With this definition we proved the following lemma.

Lemma 2.2. *Let $\{\kappa_i : G_i \to G\}_{i\in J}$ be a family of group homomorphisms. Then there is an Eilenberg–Mac Lane pair (X, Y) of type $(G, \{G_i\}_{i\in J}; 1)$.*

3. Properties of the Relative Twisted Cap Products

First note that, given a Λ–bimodule M, there is a left action of Λ on $M\otimes_\Lambda B$ and a right action of Λ on $\mathrm{Hom}_\Lambda(B, M)$ for any left Λ–module B defined by

$$\lambda.(m\otimes b) := (\lambda.m)\otimes b \quad \text{and} \quad (\varphi.\lambda)(b) := \varphi(b).\lambda$$

for $\lambda \in \Lambda, b \in B, m \in M$ and $\varphi \in \mathrm{Hom}_\Lambda(B, M)$. In particular, $\mathrm{Hom}_\Lambda(B, \Lambda)$ is a right Λ–module. Thus any left Λ–module A gives rise to the functor $\mathrm{Hom}_\Lambda({}^\omega\mathrm{Hom}_\Lambda(-, \Lambda), A))$ from the category ${}_\Lambda\mathcal{M}$ of left Λ–modules to the category $\mathcal{Ab}$ of abelian groups. This is related to the functor $A^\omega\otimes_\Lambda -$, by the following lemma.

Lemma 3.1. *There is a natural transformation*

$$\eta_B : A^\omega\otimes_\Lambda B \longrightarrow \mathrm{Hom}_\Lambda({}^\omega\mathrm{Hom}_\Lambda(B, \Lambda), A)$$

given by

$$\eta_B(a\otimes b) : {}^\omega\mathrm{Hom}_\Lambda(B, \Lambda) \longrightarrow A, \quad \varphi \longmapsto \overline{\varphi(b)}a$$

for every left Λ–module B.

Observation 3.2. When we restrict the functors $A^\omega\otimes_\Lambda -$ and $\mathrm{Hom}_\Lambda({}^\omega\mathrm{Hom}_\Lambda(-, \Lambda), A)$ to the category of free left Λ–modules, the natural transformation η becomes a natural equivalence as both $A^\omega\otimes_\Lambda \Lambda^n$ and $\mathrm{Hom}_\Lambda({}^\omega\mathrm{Hom}_\Lambda(\Lambda^n, \Lambda), A)$ are isomorphic to A^n as abelian groups.

If M is a Λ–bimodule, then so is ${}^\omega M^\omega$. Hence ${}^\omega M^\omega\otimes_\Lambda B$ carries a left Λ–module structure and $\mathrm{Hom}_\Lambda(B, M)$ carries a right Λ–module structure for every left Λ–module B. Thus ${}^\omega M^\omega\otimes_\Lambda -$ and ${}^\omega\mathrm{Hom}_\Lambda({}^\omega\mathrm{Hom}_\Lambda(-, \Lambda), M)$ are functors from the category of left Λ–modules to itself.

Observation 3.3. The natural transformation η of Lemma 3.1 respects the additional left Λ–module structure when $A = M$ is a Λ–bimodule. In other words, given a Λ–bimodule M, the natural transformation η is in fact a natural transformation from ${}^{\omega}M^{\omega}\otimes_{\Lambda}-$ to ${}^{\omega}\mathrm{Hom}_{\Lambda}({}^{\omega}\mathrm{Hom}_{\Lambda}(-,\Lambda), M)$ as functors from the category of left Λ–modules to itself. In particular, for $M = \Lambda$, we may identify the left Λ–module B with ${}^{\omega}\Lambda^{\omega}\otimes_{\Lambda}B$ by means of the isomorphism ${}^{\omega}\Lambda^{\omega}\otimes_{\Lambda}B \to B, \lambda\otimes b \mapsto \overline{\lambda}b$. Then η is the evaluation homomorphism from B to its double dual ${}^{\omega}\mathrm{Hom}_{\Lambda}({}^{\omega}\mathrm{Hom}_{\Lambda}(-,\Lambda),\Lambda)$.

The next lemma shows that the chain map given by taking the cap product with a cycle is almost chain homotopic to its dual. To be more precise, there is a diagram involving this chain map and its dual which commutes up to chain homotopy.

Lemma 3.4. *Let $1\otimes x \in \mathbb{Z}^{\omega}\otimes_{\Lambda}D_n$ be a cycle. Then the diagram*

$$\begin{tikzcd}
{}^{\omega}\mathrm{Hom}_{\Lambda}(D_k,\Lambda) \arrow[r, "\theta"] \arrow[d, "\cap 1\otimes x"'] & {}^{\omega}\mathrm{Hom}_{\Lambda}({}^{\omega}\Lambda^{\omega}\otimes_{\Lambda}D_k,\Lambda) \arrow[d, "(\cap 1\otimes x)^*"] \\
{}^{\omega}\Lambda^{\omega}\otimes_{\Lambda}P_{n-k} \arrow[r, "\eta_{P_{n-k}}"'] & {}^{\omega}\mathrm{Hom}_{\Lambda}({}^{\omega}\mathrm{Hom}_{\Lambda}(P_{n-k},\Lambda),\Lambda)
\end{tikzcd}$$

commutes up to chain homotopy, where η is the natural equivalence of Observation 3.2 and the isomorphism θ is given by $\theta(\varphi)(\lambda\otimes d) := \overline{\lambda}\varphi(d)$ for $\varphi \in {}^{\omega}\mathrm{Hom}_{\Lambda}(D_k,\Lambda), d \in D_k$ and $\lambda \in \Lambda$.

Proof. Suppose $x = \pi(y)$ and $\Delta(y) = \sum y_i\otimes y'_j$. Take $\varphi \in {}^{\omega}\mathrm{Hom}_{\Lambda}(D_k,\Lambda)$ and $\psi \in {}^{\omega}\mathrm{Hom}_{\Lambda}(P_{n-k},\Lambda)$. Then

$$\begin{aligned}
\big((\cap 1\otimes x)^*(\theta(\varphi))\big)(\psi) &= \theta(\varphi)(\psi\cap 1\otimes x)\\
&= \theta(\varphi)(\psi(y_{n-k})\otimes\pi(y'_k))\\
&= \overline{\psi(y_{n-k})}\varphi(\pi(y'_k))\\
&= \eta\big(\varphi(\pi(y'_k))\otimes y_{n-k}\big)(\psi)\\
&= \eta\big(/(\mathrm{id}\otimes\mathrm{id}\otimes((\pi\otimes\mathrm{id})\circ T\circ\Delta))(\varphi\otimes 1\otimes x)\big)(\psi)
\end{aligned}$$

where $T : P\otimes P \to P\otimes P$ is defined by $T(\sum_{i+j=n} y_i\otimes y'_j) = \sum_{i+j=n} y'_j\otimes y_i$. But $T\circ\Delta$ is again a diagonal on P and hence (see [11], p.250) chain homotopic to Δ. As $1\otimes x$ is a cycle, we obtain

$$\begin{aligned}
(\cap 1\otimes x)^*\circ\theta &= \eta\circ\big(/\circ\big(\mathrm{id}\otimes\mathrm{id}\otimes((\pi\otimes\mathrm{id})\circ T\circ\Delta)\big)\big)(-\otimes 1\otimes x)\\
&\simeq \eta\circ\big(/\circ\big(\mathrm{id}\otimes\mathrm{id}\otimes((\pi\otimes\mathrm{id})\circ\Delta)\big)\big)(-\otimes 1\otimes x)\\
&\simeq \eta\circ(\cap 1\otimes x).
\end{aligned}$$

□

Suppose that $Q \overset{\iota}{\rightarrowtail} P \overset{\pi}{\twoheadrightarrow} D$ is a short exact sequence of augmented chain complexes of free Λ–modules with compatible diagonals. Then $Q \overset{\iota}{\rightarrowtail} P \overset{\pi}{\twoheadrightarrow} D$ splits and stays split short exact when we tensor or apply the Hom_{Λ}–functor. Given a right Λ–module M, we denote the connecting homomorphisms of $\mathbb{Z}^\omega\otimes_\Lambda Q \rightarrowtail \mathbb{Z}^\omega\otimes_\Lambda P \twoheadrightarrow \mathbb{Z}^\omega\otimes_\Lambda D$, $M\otimes_\Lambda Q \rightarrowtail M\otimes_\Lambda P \twoheadrightarrow M\otimes_\Lambda D$ and ${}^\omega\mathrm{Hom}_\Lambda(D,{}^\omega M) \rightarrowtail {}^\omega\mathrm{Hom}_\Lambda(P,{}^\omega M) \twoheadrightarrow {}^\omega\mathrm{Hom}_\Lambda(Q,{}^\omega M)$ by δ_*, δ'_* and δ^* respectively.

Proposition 3.5. *Take* $x\in H^k(D,{}^\omega M)$, $y\in H_n(D,\mathbb{Z}^\omega)$, $z\in H^l(P,{}^\omega M)$ *and* $u\in H^{k-1}(Q,{}^\omega M)$. *Then*

(i) $(\mathrm{id}\otimes\pi)_*(x\cap y)=(\pi^*x)\cap y$;
(ii) $\delta'_*(z\cap y)=(\iota^*z)\cap\delta_*y$;
(iii) $(\mathrm{id}\otimes\iota)_*(u\cap\delta_*y)=(-1)^k(\delta^*u)\cap y$.

Proof. (i) Take a cocycle $\varphi\in{}^\omega\mathrm{Hom}_\Lambda(D_k,{}^\omega M)$ and a cycle $n\otimes d\in\mathbb{Z}^\omega\otimes_\Lambda D_n$ representing x and y respectively. Furthermore take $p\in P$ with $n\otimes d=n\otimes\pi(p)$ and and suppose $\Delta(p)=\sum p_i\otimes p'_j$. Then

$$\begin{aligned}(\mathrm{id}\otimes\pi)(\varphi\cap n\otimes d) &= (\mathrm{id}\otimes\pi)(\varphi/n\otimes\Delta_{\mathrm{rel}}d)=(\mathrm{id}\otimes\pi)(\varphi/n\otimes\sum\pi(p_i)\otimes p'_j)\\ &= (\mathrm{id}\otimes\pi)(n\varphi(\pi(p_k))\otimes p'_{n-k})=n\varphi(\pi(p_k))\otimes\pi(p'_{n-k})\\ &= \varphi\circ\pi/n\otimes\sum p_i\otimes\pi(p'_j)=\pi^*(\varphi)\cap n\otimes d.\end{aligned}$$

As $(\mathrm{id}\otimes\pi)(\varphi\cap n\otimes d)$ represents $(\mathrm{id}\otimes\pi)_*(x\cap y)$ and $\pi^*(\varphi)\cap n\otimes d$ represents $(\pi^*x)\cap y$, we have thus proved (i).

(ii) Take a cocycle $\varphi\in{}^\omega\mathrm{Hom}_\Lambda(P_l,{}^\omega M)$ and a cycle $n\otimes d\in\mathbb{Z}^\omega\otimes_\Lambda D_n$ representing z and y respectively. Furthermore take $p\in P$ and $q\in Q$ such that $n\otimes d=n\otimes\pi(p)$ and $n\otimes\partial p=n\otimes\iota(q)$ and suppose $\Delta(q)=\sum q_i\otimes q'_j$. Then $\iota^*z\cap\delta_*y$ is represented by

$$\varphi\circ\iota/(n\otimes\Delta q)=\varphi\circ\iota/(n\otimes\sum q_i\otimes q'_j)=n\varphi(\iota(q_l))\otimes q'_{n-l}$$

and

$$\begin{aligned}(\mathrm{id}\otimes\iota)\big(\varphi\circ\iota/(n\otimes\Delta q)\big) &= n\varphi(\iota(q_l))\otimes\iota(q'_{n-l})=\varphi/(n\otimes(\iota\otimes\iota)\Delta q)\\ &= \varphi/(n\otimes\Delta\iota(q))=\varphi/(n\otimes\Delta\partial p)\\ &= \varphi/(n\otimes\partial\Delta p)=\partial(\varphi/(n\otimes\Delta p))\end{aligned}$$

as φ is a cocycle. Furthermore $z\cap y$ is represented by

$$\varphi/(n\otimes\Delta_{\mathrm{rel}}d)=\varphi/\big(n\otimes(\mathrm{id}\otimes\pi)\Delta p\big),$$

so that $\delta'_*(z\cap y)$ is represented by $n\otimes a$ where

$$(\mathrm{id}\otimes\iota)(n\otimes a)=\partial\big(\varphi/(n\otimes\Delta p)\big).$$

As $(\mathrm{id}\otimes\iota)$ is a monomorphism we may conclude that $\delta'_*(z\cap y)=\iota^*z\cap\delta_*y$.

(iii) Take $\varphi\in{}^\omega\mathrm{Hom}_\Lambda(Q_{k-1},{}^\omega M)$ and $n\otimes d\in\mathbb{Z}^\omega\otimes_\Lambda D_n$ representing u and y respectively. Take $\psi\in{}^\omega\mathrm{Hom}_\Lambda(P_{k-1},{}^\omega M)$ with $\varphi=\iota^*\psi$ and $\eta\in{}^\omega\mathrm{Hom}_\Lambda(D_k,{}^\omega M)$ with $\pi^*\eta=\partial^*\psi$. Then δ^*u is represented by η. Further, take $p\in P_n$ with $\pi p=d$ and $q\in Q_{n-1}$ with $\iota q=\partial p$, so that δ'_*y is represented by $n\otimes q$, and suppose $\Delta p=\sum p_i\otimes p'_j$ and $\Delta q=\sum q_i\otimes q'_j$. Then

$(\mathrm{id}\otimes\iota)_*(u\cap\delta_* y)$ is represented by

$$\begin{aligned}(\mathrm{id}\otimes\iota)\big(\varphi\cap n\otimes q\big) &= (\mathrm{id}\otimes\iota)\big(\varphi/n\otimes\Delta q\big) = (\mathrm{id}\otimes\iota)\big(\varphi/n\otimes\sum q_i\otimes q'_j\big)\\ &= (\mathrm{id}\otimes\iota)\big(n\varphi(q_{k-1})\otimes q'_{n-k-1}\big) = n\varphi(q_{k-1})\otimes\iota(q'_{n-k-1})\\ &= n\iota^*\psi(q_{k-1})\otimes\iota(q'_{n-k-1}) = \psi/n\otimes(\iota\otimes\iota)\Delta q\\ &= \psi/n\otimes\Delta\iota(q) = \psi/n\otimes\Delta\partial p\\ &= \psi/n\otimes\partial\Delta p.\end{aligned}$$

Since / is a chain map, we obtain

$$\partial(\psi/n\otimes\Delta p) = (\partial^*\psi)/n\otimes\Delta p + (-1)^{k-1}\psi/n\otimes\partial\Delta p.$$

On the other hand

$$\begin{aligned}\partial^*\psi/n\otimes\Delta p &= \pi^*\eta/n\otimes\Delta p = \pi^*\eta/n\otimes\sum p_i\otimes p'_j\\ &= n\eta(\pi(p_k))\otimes p'_{n-k} = \eta/n\otimes\Delta'_{\mathrm{rel}}d\\ &= \eta\cap n\otimes d,\end{aligned}$$

which shows that $\partial^*\psi/n\otimes\Delta p$ represents $(\delta^* u)\cap y$. As $\partial(\psi/n\otimes\Delta p)$ is a boundary, we may conclude that

$$(\mathrm{id}\otimes\iota)_*(u\cap\delta_* y) = (-1)^k(\delta^* u)\cap y.$$

□

Proposition 3.5 allows us to prove commutativity of a diagram, also called a cap product ladder, which involves long exact homology and co–homology sequences arising from $Q\rightarrowtail P\twoheadrightarrow D$ and the cap product with a homology class $y\in\mathrm{H}_n(D;\mathbb{Z}^\omega)$.

Theorem 3.6 (Cap Product Ladder). *Let $Q\overset{\iota}{\rightarrowtail} P\overset{\pi}{\twoheadrightarrow} D$ be a short exact sequence of augmented chain complexes of free Λ–modules with compatible diagonals. Then, given $y\in H_n(D;\mathbb{Z}^\omega)$, the diagram*

$$\begin{tikzcd}\cdots \arrow[r] & H^r(D,{}^\omega M) \arrow[r, "\pi^*"] \arrow[d, "\cap y"] & H^r(P,{}^\omega M) \arrow[r, "\iota^*"] \arrow[d, "\cap y"] & H^r(Q,{}^\omega M) \arrow[r, "\delta^*"] \arrow[d, "\cap\delta_* y"] & H^{r+1}(D,{}^\omega M) \arrow[r] \arrow[d, "\cap y"] & \cdots\\ \cdots \arrow[r] & H_{n-r}(P,M) \arrow[r] & H_{n-r}(D,M) \arrow[r] & H_{n-r-1}(Q,M) \arrow[r] & H_{n-r-1}(P,M) \arrow[r] & \cdots\end{tikzcd}$$

commutes, up to sign.

Proof. Given $x\in\mathrm{H}^r(D,{}^\omega M)$, Property (i) of Proposition 3.5 implies $(\pi^* x)\cap y = (\mathrm{id}\otimes\pi)_*(x\cap y)$. For $z\in\mathrm{H}^r(P,{}^\omega M)$ we have $\iota^* z\cap\delta^* y = \delta'_*(z\cap y)$ by (ii). Finally, (iii) yields $(\mathrm{id}\otimes\iota)_*(u\cap\delta_* y) = (-1)^k(\delta^* u)\cap y$ for $u\in\mathrm{H}^r(Q,{}^\omega M)$. Hence the first two squares commute and the third commutes up to sign. □

4. Projective Homotopy Theory of Modules

In this section Λ may be any ring with unit. Unless otherwise specified, $A, B, \ldots$ will denote left Λ–modules and $\varphi, \psi, \ldots$ will denote Λ–morphisms.

Definition 4.1. *The Λ–morphism $\varphi : A \to B$ is nullhomotopic, written as $\varphi \simeq 0$, if there is a commutative diagram*

$$\begin{tikzcd}
A \arrow[rr, "\varphi"] \arrow[dr] & & B \\
& P \arrow[ur] &
\end{tikzcd} \tag{1}$$

where P is a projective Λ–module.

As every projective Λ–module is a direct summand of a free Λ–module the existence of Diagram (1) is equivalent to the existence of a diagram of the form

$$\begin{tikzcd}
A \arrow[rr, "\varphi"] \arrow[dr] & & B \\
& \Lambda^m. \arrow[ur] &
\end{tikzcd}$$

If $\varepsilon : PA \twoheadrightarrow A$ is an epimorphism and PA is projective then PA is called a path space over A (in analogy to topological homotopy theory). Since the category of left Λ–modules has enough projectives, every Λ–module A has a path space. It is not difficult to show that a Λ–morphism $\varphi : A \to B$ is nullhomotopic if and only if it factors through a given path space $\varepsilon : PB \twoheadrightarrow B$ of B, that is, if and only if there is a commutative diagram

$$\begin{tikzcd}
A \arrow[rr, "\varphi"] \arrow[dr] & & B \\
& PB. \arrow[ur, two heads] &
\end{tikzcd}$$

Thus, if $\varphi : A \to B$ factors through one particular path space of B, it factors through any path space of B. Hence

$$\mathrm{Nhom}_\Lambda(\mathrm{A}, \mathrm{B}) := \{\varphi : \mathrm{A} \to \mathrm{B} \mid \varphi \simeq 0\ \}$$

is a subgroup of $\mathrm{Hom}_\Lambda(\mathrm{A}, \mathrm{B})$.

Definition 4.2. *Two Λ–morphisms φ and ψ are homotopic if $\varphi - \psi \simeq 0$. Furthermore the group*

$$[A, B] := \mathrm{Hom}_\Lambda(\mathrm{A}, \mathrm{B})/\mathrm{Nhom}_\Lambda(\mathrm{A}, \mathrm{B})$$

of homotopy classes of Λ–morphisms is called the homotopy group from A to B.

It is not difficult to show that homotopy respects composition of Λ–morphisms. Thus we obtain a category, called the projective homotopy category (PHOM) or the stable category whose objects are left Λ–modules and whose morphisms are homtopy classes of Λ–morphisms. Furthermore $[A, B]$ is functorial in both variables and preserves direct products.

As in topological homotopy theory, we say that $\varphi : A \to B$ is a homotopy equivalence if and only if there is a Λ–morphism $\psi : B \to A$ such that $\varphi\psi \simeq \mathrm{id}_B$ and $\psi\varphi \simeq \mathrm{id}_A$. If there is a homotopy equivalence $\varphi : A \to B$ then A and B are said to be homotopy equivalent and we denote the set of homotopy equivalences from A to B by $\mathrm{Equi}(A, B)$.

Lemma 4.3. *A Λ–module A is projective if and only if $[X, A] = 0$ for every Λ–module X.*

Proof. We only need to show that $[X, A] = 0$ for every Λ–module X implies that A is projective. So assume that $[X, A] = 0$ for every Λ–module X. Then $[A, A] = 0$ which implies $\mathrm{id}_A \simeq 0$, that is, id_A factors through a path space $PA \twoheadrightarrow A$ of A. Thus there is a commutative diagram

$$\begin{tikzcd}
A \arrow[rr, "\mathrm{id}_A"] \arrow[dr, tail, "\iota"'] & & A \\
& PA. \arrow[ur, two heads, "\pi"'] &
\end{tikzcd}$$

Now let $\varphi : A \to B$ be a Λ–morphism and let $\varepsilon : C \twoheadrightarrow B$ be an epimorphism. Since PA is projective there is a Λ–morphism $\psi : PA \to C$ such that $\varepsilon\psi = \varphi\pi$. Hence $\varepsilon\psi\iota = \varphi\pi\iota = \varphi$, showing that A is projective.

$$\begin{tikzcd}
& & C \arrow[d, two heads, "\varepsilon"] \\
PA \arrow[r, two heads, "\pi"] \arrow[urr, "\psi"] & A \arrow[r, "\varphi"] \arrow[ur, "\psi\iota"] & B
\end{tikzcd}$$

□

Given a path space $\varepsilon : PB \twoheadrightarrow B$, any $\varphi : A \to B$ factors as

$$A \overset{\iota}{\rightarrowtail} A \oplus PB \overset{\varphi'}{\twoheadrightarrow} B,$$

where φ' is defined by $\varphi'(a, p) = \varphi(a) + \varepsilon(p)$ for $a \in A$ and $p \in PB$.

The statement as well as the proof of the following theorem are dual to Theorem 13.7 in [8] and its proof.

Theorem 4.4. *A homotopy equivalence $\varphi : A \to B$ factors as*

$$A \overset{\iota}{\rightarrowtail} A \oplus P \overset{\tilde{\varphi}}{\twoheadrightarrow} B \oplus Q \overset{\pi}{\twoheadrightarrow} B$$

where P and Q are projective and ι and π are the natural inclusion and projection respectively.

Proof. First assume that φ is an epimorphism. Let $\psi : B \to A$ be a homotopy inverse of φ and let $\varepsilon : PA \twoheadrightarrow A$ be a path space of A. Then $\varphi\varepsilon : PA \twoheadrightarrow B$ is a path space of B and hence $\varphi\psi - \mathrm{id}_B \simeq 0$ implies that there is a Λ–morphism $\eta : B \to PA$ such that the diagram

$$\begin{tikzcd}
PA \arrow[r, two heads, "\varepsilon"] & A \arrow[r, two heads, "\varphi"] & B \\
B \arrow[u, "\eta"] \arrow[urr, "\varphi\psi - \mathrm{id}_B"'] & &
\end{tikzcd}$$

commutes. Put $\tilde{\psi} := \psi - \varepsilon\eta$. Then $\tilde{\psi} \simeq \psi$ and

$$\varphi\tilde{\psi} = \varphi(\psi - \varepsilon\eta) = \varphi\psi - \varphi\varepsilon\eta = \varphi\psi - \varphi\psi + \mathrm{id}_B = \mathrm{id}_B.$$

Hence $\tilde{\psi}$ is a monomorphism and the short exact sequence

$$\begin{tikzcd}
B \arrow[r, tail, "\tilde{\psi}"] & A \arrow[r, two heads, "\pi'"] & \mathrm{coker}\tilde{\psi}
\end{tikzcd}$$

splits so that $A = \tilde{\psi}(B) \oplus Q$ where $Q = \mathrm{coker}\tilde{\psi}$. In order to show that Q is projective it is enough to show that $[X, Q] = 0$ for all X. So take any Λ–module X. Then

$$\begin{tikzcd}
{[X,B]} \arrow[r, tail, two heads, "\psi_*"] & {[X, \tilde{\psi}(B) \oplus Q]} \arrow[r, tail, two heads] & {[X, \tilde{\psi}(B)] \oplus [X,Q]} \arrow[r, two heads] & {[X,Q]}
\end{tikzcd}$$

is onto. But what does this homomorphism do to the homotopy class of a Λ–morphism $\nu : X \to B$?

$$[\nu] \mapsto [\psi\nu] = [\tilde{\psi}\nu] \mapsto [\pi'\tilde{\psi}\nu] = 0.$$

Hence $[X, Q] = 0$ showing that Q is projective.

Thus φ factors as

$$\begin{tikzcd}
A = \tilde{\psi}(B) \oplus Q \arrow[r, tail, two heads] & B \oplus Q \arrow[r, two heads] & B.
\end{tikzcd}$$

Given an arbitrary homotopy equivalence $\varphi : A \to B$ we obtain

$$\begin{tikzcd}
A \arrow[r, tail, "\iota"] & A \oplus PB \arrow[rr, two heads, "\varphi'"] \arrow[dr, two heads] & & B \\
& & B \oplus Q \arrow[ur, two heads] & .
\end{tikzcd}$$

□

Observation 4.5. If the Λ–modules A and B in Theorem 4.4 are finitely generated, then the projective Λ–modules P and Q are also finitely generated. Thus there is a finitely generated projective Λ–module $\tilde{P}$ such that $P \oplus \tilde{P} \cong \Lambda^n$ for some $n \in \mathbb{N}$. Hence φ factors as

$$\begin{tikzcd}
A \arrow[r, tail] & A \oplus (P \oplus \tilde{P}) \arrow[r, two heads] & B \oplus (Q \oplus \tilde{P}) \arrow[r, two heads] & B
\end{tikzcd}$$

or

$$\begin{tikzcd}
A \arrow[r, tail] & A \oplus \Lambda^n \arrow[r, two heads] & B \oplus \tilde{Q} \arrow[r, two heads] & B
\end{tikzcd}$$

where $\tilde{Q} = Q \oplus \tilde{P}$ is finitely generated projective.

5. Formulation of the Realization Condition

We have seen in [3] that, up to oriented homotopy equivalence, a PD^3–pair $(X, \partial X)$ with aspherical boundary components is uniquely determined by the Π_1–system $\{\kappa_i : \Pi_1(\partial X_i, *) \to \Pi_1(X, *)\}_{i \in J}$, the orientation character $\omega_X \in \mathrm{H}^1(X; \mathbb{Z}/2\mathbb{Z})$ and the image of the fundamental class $[X, \partial X] \in \mathrm{H}_3(X, \partial X; \mathbb{Z}^\omega)$ under the classifying map

$$c : (X, \partial X) \longrightarrow K(\{\kappa_i\}_{i \in J}; 1).$$

In other words, the triple $(\{\kappa_i\}_{i \in J}, \omega_X, c_*([X, \partial X]))$ forms a complete set of homotopy invariants for PD^3–pairs, also called the *fundamental triple* of $(X, \partial X)$. We say that $(X, \partial X)$ *realizes* $(\{\kappa_i\}_{i \in J}, \omega, \mu)$.

Question 5.1. Given a Π_1–system $\{\kappa_i : G_i \to G\}_{i \in J}, \omega \in \mathrm{H}^1(G; \mathbb{Z}/2\mathbb{Z})$ and a homology class $\mu \in \mathrm{H}_3(G, \{G_i\}_{i \in J}; \mathbb{Z}^\omega)$, is there a PD^3–pair $(X, \partial X)$ realizing $(\{\kappa_i\}_{i \in J}, \omega, \mu)$?

Turaev [12] gave a condition for realization in the absolute case of PD^3–complexes X with $\partial X = \emptyset$. Given a finitely presentable group G and $\omega \in \mathrm{H}^1(G; \mathbb{Z}/2\mathbb{Z})$, he defined a homomorphism

$$\nu : \mathrm{H}_3(G; \mathbb{Z}^\omega) \longrightarrow [F, I]$$

where F is some $\mathbb{Z}[G]$–module, $I = \ker \mathrm{aug}$ and $[A, B]$ denotes the group of homotopy classes of $\mathbb{Z}[G]$–morphisms from the $\mathbb{Z}[G]$–module A to the $\mathbb{Z}[G]$–module B. Turaev showed that, given $\mu \in \mathrm{H}_3(G; \mathbb{Z}^\omega)$, the triple (G, ω, μ) is relized by a PD^3–complex X if and only if $\nu(\mu)$ is a class of homotopy equivalences of $\mathbb{Z}[G]$–modules.

Using Turaev's construction of the homomorphism ν, we generalize the condition for realization to the case of PD^3–pairs $(X, \partial X)$, where ∂X is not necessarily empty.

First we introduce two functors from the category of left Λ–modules to itself, where Λ is the integral group ring of the group H.

We take $\omega \in \mathrm{H}^1(H, \mathbb{Z}/2\mathbb{Z})$ and use the notation of Chapter 1.

Given a chain complex $\ldots \to C_{r+1} \xrightarrow{\partial_r} C_r \to \ldots$ of left Λ–modules, put

$$\mathrm{G}_r(C) := \mathrm{coker}\partial_r = C_r / \mathrm{im}\partial_r.$$

If $f : C \to D$ is a chain map then $f_r(\mathrm{im}\partial_r^C) \subseteq \mathrm{im}\partial_r^D$. Hence there is an induced Λ–morphism of cokernels $\mathrm{G}_r(f) : \mathrm{G}_r(C) \to \mathrm{G}_r(D)$ such that the diagram

$$\begin{tikzcd}
\mathrm{im}\partial_r^C \arrow[r, tail] \arrow[d] & C_r \arrow[r, two heads] \arrow[d, "f_r"] & \mathrm{G}_r(C) \arrow[d, "\mathrm{G}_r(f)"] \\
\mathrm{im}\partial_r^D \arrow[r, tail] & D_r \arrow[r, two heads] & \mathrm{G}_r(D)
\end{tikzcd}$$

commutes. It is not difficult to check that $G = G_*$ is a functor from the category of chain complexes of left Λ–modules to itself.

Following Turaev we write C^* for ${}^\omega\mathrm{Hom}_\Lambda(C, \Lambda)$ and compose the two functors G and ${}^\omega\mathrm{Hom}_\Lambda(-, \Lambda)$ to obtain the functor F (see [12] p.265) given by

$$F^r(C) = G_{-r}(C^*) = C^r / \mathrm{im}\partial_{r-1}^*. \tag{2}$$

The following lemma allows us to pass from the category of chain complexes of left Λ–modules to the stable category, that is, the category of left Λ–modules and homotopy classes of Λ–morphisms.

Lemma 5.2. *Let $f, g : C \to D$ be chain homotopic maps of chain complexes over Λ. If D_n is projective, then $G_n(f) \simeq G_n(g)$ as Λ–morphisms.*

Proof. Let χ be a chain homotopy from f to g. Observe that, for all $n \in \mathbb{Z}$, the boundary operators ∂^C_{n-1} and ∂^D_{n-1} factor as

$$\begin{tikzcd}
C_n \arrow[r, "\partial^C_{n-1}"] \arrow[d, "\sigma^C_{n-1}"'] & C_{n-1} & \text{and} & D_n \arrow[r, "\partial^D_{n-1}"] \arrow[d, "\sigma^D_{n-1}"'] & D_{n-1} \\
G_n(C) \arrow[ur, "\rho^C_{n-1}"'] & & & G_n(D) \arrow[ur, "\rho^D_{n-1}"'] &
\end{tikzcd}$$

respectively. Then

$$\begin{aligned}
\sigma^D_{n-1}(f_n - g_n) &= \sigma^D_{n-1}(\chi_{n-1}\partial^C_{n-1} + \partial^D_n \chi_n) \\
&= \sigma^D_{n-1}\chi_{n-1}\rho^C_{n-1}\sigma^C_{n-1} + \sigma^D_{n-1}\partial^D_n \chi_n \\
&= \sigma^D_{n-1}\chi_{n-1}\rho^C_{n-1}\sigma^C_{n-1}
\end{aligned}$$

Thus the diagram

$$\begin{tikzcd}
C_{n+1} \arrow[rrr, "f_{n+1}-g_{n+1}"] \arrow[d, "\partial^C_n"'] & & & D_{n+1} \arrow[d, "\partial^D_n"'] \\
C_n \arrow[rrr, "f_n-g_n"] \arrow[d, "\sigma^C_{n-1}"'] & & & D_n \arrow[d, "\sigma^D_{n-1}"'] \\
G_n(C) \arrow[r, "\rho^C_{n-1}"] & C_{n-1} \arrow[r, "\chi_{n-1}"] & D_n \arrow[r, "\sigma^D_{n-1}"] & G_n(D)
\end{tikzcd}$$

commutes. As the induced map of cokernels is uniquely determined, this implies

$$G_n(f) - G_n(g) = G_n(f-g) = \sigma^D_{n-1}\chi_{n-1}\rho^C_{n-1} \simeq 0$$

as D_n is projective. ☐

Corollary 5.3. *Let $f : C \to D$ be a homotopy equivalence of chain complexes over Λ. If C_n and D_n are projective, then $G_n(f)$ is a homotopy equivalence of Λ–modules.*

Corollary 5.3 is crucial for the formulation of the condition for realization.

Observation 5.4. Lemma 5.2 shows that we may view G_n as a functor from the category of chain complexes of projective left Λ–modules and homotopy classes of chain maps to the stable category.

Lemma 5.5. *Let (X, Y) be a pair of CW–complexes with X connected and $\omega \in H^1(X; \mathbb{Z}/2\mathbb{Z})$ such that $H_n(X, Y; \mathbb{Z}^\omega) \cong \mathbb{Z}$ with generator $[1\otimes x]$. Then there is a chain $w_1 \in C_1(X)$ such that the Λ–morphism $\cap 1\otimes x : C^*(X, Y) \to {}^{\omega}\Lambda^{\omega}\otimes_\Lambda C(X) \cong C(X)$ is given by*

$$\varphi \cap 1\otimes x = \overline{\varphi(x)}.(1 + \partial_0 w_1)$$

for every cocycle $\varphi \in C^*(X,Y)$*, where we identify* $\lambda \otimes c \in {}^{\omega}\Lambda^{\omega} \otimes_{\Lambda} C(X)$ *with* $\overline{\lambda}.c \in C(X)$*.*

Proof. Take $y \in C_n(X)$ with $\pi(y) = x$, where $\pi : C(X) \twoheadrightarrow C(X, \partial X)$ is the natural projection, and assume $\Delta y = \sum y_i \otimes z_{n-i}$ with $y_i, z_i \in C_i(X)$. Then $(\mathrm{id} \otimes \varepsilon)\Delta(y) = y$ implies $y_n.\varepsilon(z_0) = y$. As $[1 \otimes x]$ is a generator, x and thus y are indivisible so that $y = y_n$ and $\varepsilon(z_0) = 1$ up to sign. As X is connected, we may assume $C_0(X) = \Lambda$ and identify $\mathrm{im}\partial_1$ with $I = \ker \varepsilon$. Then $\varepsilon(z_0) = 1$ implies $z_0 = 1 + w_0$ where $w_0 \in I$, and hence $z_0 = 1 + \partial_0 w_1$ for some $w_1 \in C_1(X)$. Hence

$$\begin{aligned} \varphi \cap 1 \otimes x &= \varphi / 1 \otimes (\pi \otimes \mathrm{id})(\textstyle\sum y_i \otimes z_{n-i}) = \varphi(\pi(y_n)) \otimes z_0 \\ &= \overline{\varphi(\pi(y_n))}.z_0 = \overline{\varphi(x)}.(1 + \partial_0 w_1). \end{aligned}$$

□

Now let $(X, \partial X)$ be a PD^3–pair and take a cycle $1 \otimes x \in \mathbb{Z}^{\omega} \otimes_{\Lambda} C_3(X, \partial X)$ representing $[X, \partial X]$. Then

$$\cap 1 \otimes x : C^*(X,Y) \to {}^{\omega}\Lambda^{\omega} \otimes_{\Lambda} C(X) \cong C(X)$$

is a chain homotopy equivalence. As both $C_2^*(X,Y)$ and $C_1(X)$ are free and hence projective, Corollary 5.3 implies that

$$G_{-2}(\cap 1 \otimes x) : F^2(C(X, \partial X)) = G_{-2}(C^*(X,Y)) \to G_1(C(X))$$

is a homotopy equivalence of Λ–modules.

Since $C(X)$ is the cellular chain complex of the universal covering space of X, $\mathrm{H}_1(C(X)) = 0$ so that

$$G_1(C(X)) = C_1(X)/\mathrm{im}\partial_1 = C_1(X)/\ker \partial_0 \cong \mathrm{im}\partial_0 = \ker \mathrm{aug} = I,$$

that is, there is an isomorphism

$$\vartheta : G_1(C(X)) \to I \quad \text{given by} \quad \vartheta([c]) := \partial_0(c).$$

Then $\vartheta \circ G_{-2}(\cap 1 \otimes x)$ is also a homotopy equivalence of Λ–modules, and the fact that $\cap 1 \otimes x$ is a chain map together with Lemma 5.5 yields

$$\begin{aligned} (\vartheta \circ G_{-2}(\cap 1 \otimes x))([\varphi]) &= \vartheta([\varphi \cap 1 \otimes x]) = \partial_0(\varphi \cap 1 \otimes x) \\ &= (\partial_2^* \varphi) \cap 1 \otimes x = \overline{(\partial_2^* \varphi)(x)}.(1 + \partial_0 w_1) \\ &= \overline{(\partial_2^* \varphi)(x)} + \overline{(\partial_2^* \varphi)(x)}\partial_0 w_1 \end{aligned}$$

for $\varphi \in C_2^*(X, \partial X)$ and some $w_1 \in C_1(X)$. Observe that the Λ–morphism

$$F^2(C(X, \partial X)) \longrightarrow I, \ [\varphi] \longmapsto \overline{(\partial_2^* \varphi)(x)}\partial_0 w_1$$

is null–homotopic since it factors through the Λ–module $C_1(X)$, namely as

$$[\varphi] \longmapsto \overline{(\partial_2^* \varphi)(x)} w_1 \longmapsto \partial_0(\overline{(\partial_2^* \varphi)(x)} w_1) = \overline{(\partial_2^* \varphi)(x)}\partial_0 w_1.$$

Thus

$$F^2(C(X, \partial X)) \longrightarrow I, \ [\varphi] \longmapsto \overline{(\partial_2^* \varphi)(x)} \tag{3}$$

is a homotopy equivalence of Λ–modules.

Now attach cells of dimension three and larger to $(X, \partial X)$ in order to obtain an Eilenberg–Mac Lane pair $(K, \partial X)$ of type $K(\{\kappa_i : \Pi_1(\partial X_i, *) \to \Pi_1(X, *)\}_{i \in J}; 1)$. Then the classifying map $\iota : (X, \partial X) \to (K, \partial X)$ is cellular and we may identify the cellular chain complexes of the pair $(X, \partial X)$ with their image under the chain map induced by ι. In particular, we obtain $C_i(K) = C_i(X), C_i(K, \partial X) = C_i(X, \partial X)$ for $i = 0, 1, 2$ and $[1 \otimes x] = [X, \partial X] = \iota_*([X, \partial X])$. Thus (3) yields

Lemma 5.6. *The* Λ*–morphism*

$$F^2(C(K, \partial X)) \longrightarrow I,\ \ [\varphi] \longmapsto \overline{(\partial_2^* \varphi)(x)}. \tag{4}$$

is a homotopy equivalence of Λ*–modules.*

Given a chain complex C of free left Λ–modules, Turaev constructed a group homomorphism

$$\nu_{C,r} : \mathrm{H}_{r+1}(\mathbb{Z}^\omega \otimes_\Lambda C) \longrightarrow [F^r, I]$$

such that $\nu_{C(X,\partial X),2}([1 \otimes x]) = \nu_{C(K,\partial X),2}(\iota_*([X, \partial X]))$ is the homotopy class of the homotopy equivalence (4).

We revise Turaev's construction and some of its properties. Given a chain complex C of free left Λ–modules, note that $\overline{I}$ is the kernel of the Λ–morphism $\Lambda \to \mathbb{Z}^\omega \otimes_\Lambda \Lambda,\ \lambda \mapsto 1 \otimes \lambda$, so that $\overline{I} \rightarrowtail \Lambda \twoheadrightarrow \mathbb{Z}^\omega \otimes_\Lambda \Lambda$ is short exact. As C is free, the sequence $\overline{I}C \rightarrowtail C \twoheadrightarrow \mathbb{Z}^\omega \otimes_\Lambda C$ of chain complexes is also short exact, yielding the connecting homomorphism

$$\delta : \mathrm{H}_{r+1}(\mathbb{Z}^\omega \otimes_\Lambda C) \longrightarrow \mathrm{H}_r(\overline{I}C). \tag{5}$$

Identifying $c \in C_r$ with $1 \otimes c \in \Lambda^\omega \otimes_\Lambda C$, the natural equivalence η of Lemma 3.1 yields the Λ–morphism

$$\eta : C_r \longrightarrow (C_r^*)^*,\ c \longmapsto \eta(c)$$

given by

$$\eta(c)(\varphi) = \overline{\varphi(c)}.$$

For a cycle $c \in C_r$ we obtain

$$\eta(c)(\partial_{r-1}^* \varphi) = \overline{(\partial_{r-1}^* \varphi)(c)} = \varphi(\partial_{r-1} c)) = 0$$

for every $\varphi \in C_{r-1}^*$. Thus $\eta(c)$ factors through the cokernel $F^r(C)$ of ∂_{r-1}^*, that is, there is a Λ–morphism $\widetilde{\eta(c)}$ such that

$$\begin{tikzcd}
C_r^* \arrow[r, "\eta(c)"] \arrow[d] & \Lambda \\
F^r(C) \arrow[ur, "\widetilde{\eta(c)}"'] &
\end{tikzcd}$$

commutes. If $c = \overline{\lambda}.d \in \overline{I}C$ is a cycle with $\lambda \in I$ and $d \in C_r$, then

$$\begin{aligned}
\mathrm{aug}\big(\eta(c)([\varphi])\big) &= \mathrm{aug}(\overline{\varphi(c)}) = \mathrm{aug}(\overline{\varphi(\overline{\lambda}.d)}) \\
&= \mathrm{aug}(\overline{\varphi(d)}.\lambda) = \overline{\varphi(d)}\mathrm{aug}(\lambda) \\
&= 0
\end{aligned}$$

for every $[\varphi] \in F^r(C)$. Hence the image of $\widetilde{\eta(c)}$ is contained in I and there is a well–defined Λ–morphism

$$\widehat{\eta(c)} : F^r(C) \longrightarrow I,\ [\varphi] \longmapsto \overline{\varphi(c)}.$$

Given a boundary $c = \partial_r(\overline{\mu}.e) \in \overline{I}C$ with $\mu \in I$ and $e \in C_{r+1}$, the Λ–morphism $\widehat{\eta(c)}$ is null–homotopic since it factors through Λ, namely as

$$\begin{tikzcd} F^r(C) \arrow[r] & \Lambda \arrow[r] & I \\ & \lambda \arrow[r, mapsto] & \overline{\mu}.\lambda. \end{tikzcd}$$

Thus the homotopy class of $\widehat{\eta(c)}$ depends on the homology class of the cycle $c \in \overline{I}C$ only and the homomorphism

$$\mathrm{H}(\overline{I}C) \longrightarrow [F^r(C), I],\ [c] \longmapsto [\widehat{\eta(c)}] \tag{6}$$

is well–defined. Composing (6) with the connecting homomorphism (5), Turaev obtains the homomorphism

$$\nu_{C,r} : \mathrm{H}_{r+1}(\mathbb{Z}^\omega \otimes_\Lambda C) \longrightarrow [F^r(C), I] \tag{7}$$

given by

$$\nu_{C,r}([1\otimes c]) := [\widehat{\eta(c)}].$$

Lemma 5.7. *Given* $[1\otimes x] \in H_{r+1}(\mathbb{Z}^\omega \otimes_\Lambda C)$ *and* $[\varphi] \in F^r(C)$, *the homotopy class* $\nu_{C,r}([1\otimes x])$ *is represented by the* Λ*–morphism*

$$F^r(C) \longrightarrow I,\ [\varphi] \longmapsto \overline{\varphi(\partial_r(x))}.$$

Proof. Take $[1\otimes x] \in \mathrm{H}_{r+1}(\mathbb{Z}^\omega \otimes_\Lambda C)$ and $[\varphi] \in F^r(C)$. Then $\delta([1\otimes x]) = \partial_r x$ and $\nu_{C,r}([1\otimes x])$ is represented by

$$\widehat{\eta(\partial_r x)} : F^r(C) \longrightarrow I,\ [\varphi] \longmapsto \overline{\varphi(\partial_r(x))}.$$

□

Lemma 5.8. *Let* $f : C \to D$ *be a chain map of chain complexes of* Λ*–modules. Then the diagram*

$$\begin{tikzcd} F^r(D) \arrow[r, "\nu_{D,r}(f_*\mu)"] \arrow[d, "F^r(f)"'] & I \arrow[d, equal] \\ F^r(C) \arrow[r, "\nu_{C,r}(\mu)"] & I \end{tikzcd}$$

commutes for every $\mu \in H_{r+1}(\mathbb{Z}^\omega \otimes_\Lambda C)$.

Proof. Take $\mu \in \mathrm{H}_{r+1}(\mathbb{Z}^\omega \otimes_\Lambda C)$ and $x \in C_r$ with $\mu = [1\otimes x]$. Then, for $[\varphi] \in F^r(C)$,

$$\begin{aligned} \nu_{C,r}(\mu)\big(F^r(f)([\varphi])\big) &= \nu_{C,r}(\mu)\big([\varphi \circ f]\big) = \overline{\varphi \circ f\big(\partial_r(x)\big)} \\ &= \overline{\varphi\big(\partial_r(f(x))\big)} = \nu_{D,r}(f_*\mu)\big([\varphi]\big). \end{aligned}$$

□

Lemma 5.9. *Suppose that C is a chain complex of free left Λ–modules such that C_r is finitely generated and $H_r(C) = H_{r+1}(C) = 0$. Then $\nu_{C,r}$ is an isomorphism.*

Proof. Cf. [12], Lemma 2.5. □

We are now able to provide a necessary condition for a Π_1–system $\{\kappa_i : G_i \to G\}_{i\in J}, \omega \in \mathrm{H}^1(G;\mathbb{Z}/2\mathbb{Z})$ and $\mu \in \mathrm{H}_3(G,\{G_i\}_{i\in J};\mathbb{Z}^\omega)$ to be realized by a PD^3–pair $(X,\partial X)$.

Theorem 5.10. *Given a Π_1–system $\{\kappa_i : G_i \to G\}_{i\in J}$, $\omega \in H^1(G;\mathbb{Z}/2\mathbb{Z})$ and $\mu \in H_3(G,\{G_i\}_{i\in J};\mathbb{Z}^\omega)$, let $(K,\partial K)$ be an Eilenberg–Mac Lane pair of type $K(\{\kappa_i\}_{i\in J};1)$. If $(\{\kappa_i\}_{i\in J},\omega,\mu)$ is the fundamental triple of a PD^3–pair, then $\nu_{C(K,\partial K),2}(\mu)$ is a homotopy equivalence of Λ–modules.*

Proof. Take a Π_1–system $\{\kappa_i : G_i \to G\}_{i\in J}, \omega \in \mathrm{H}^1(G;\mathbb{Z}/2\mathbb{Z}), \mu \in \mathrm{H}_3(G,\{G_i\}_{i\in J};\mathbb{Z}^\omega)$ and suppose $(\{\kappa_i\}_{i\in J},\omega,\mu)$ is the fundamental triple of the PD^3–pair $(X,\partial X)$. Attaching cells of dimension three and larger to X we obtain an Eilenberg–Mac Lane pair $(K,\partial X)$ of type $K(\{\kappa_i\}_{i\in J};1)$. Take $1\otimes x \in \mathbb{Z}^\omega\otimes_\Lambda C_3(X,\partial X) \subseteq \mathbb{Z}^\omega\otimes_\Lambda C_3(K,\partial X)$ with $[1\otimes x] = \mu$. Then

$$F^2(C(K,\partial X)) \longrightarrow I,\ [\varphi] \longmapsto \overline{\varphi(\partial_2(x))}$$

is a homotopy equivalence of Λ–modules by Lemma 5.6 and represents $\nu_{C(K,\partial X),2}(\mu)$ by Lemma 5.7.

It remains to show that $\nu_{C(L,\partial L),2}(\mu)$ is a homotopy equivalence of Λ–modules for any Eilenberg–Mac Lane pair $(L,\partial L)$. But given any Eilenberg–Mac Lane pair $(L,\partial L)$ of type $K(\{\kappa_i\}_{i\in J};1)$, there is a homotopy equivalence $f : (K,\partial X) \to (L,\partial L)$ of pairs of CW complexes inducing a chain homotopy equivalence $g : C(K,\partial X) \longrightarrow C(L,\partial L)$. Hence $g^* : C^*(K,\partial X) \longrightarrow C^*(L,\partial L)$ is also a chain homotopy equivalence and Corollary 5.3 implies that $F^2(g) = G_{-2}(g^*)$ is a homotopy equivalence of Λ–modules. By Lemma 5.8, the diagram

$$\begin{tikzcd}
F^2(C(L,\partial L)) \arrow[r, "\nu_{C(L,\partial L),2}(f_*\mu)"] \arrow[d, "F^2(g)"'] & I \arrow[d, equal] \\
F^2(C(K,\partial K)) \arrow[r, "\nu_{C(K,\partial K),2}(\mu)"] & I
\end{tikzcd}$$

commutes and hence $\nu_{C(L,\partial L),2}(f_*\mu)$ is a homotopy equivalence of Λ–modules if and only if $\nu_{C(K,\partial K),2}(\mu)$ is one. □

In the final section of this paper we show that the necessary condition of Theorem 5.10 is sufficient in the Π_1–injective case.

6. The Π_1–Injective Case

For $\{\kappa_i : G_i \to G\}_{i\in J}$ to be the Π_1–system of a PD^3–pair $(X,\partial X)$, the groups G_i must be surface groups for all $i \in J$ as the components of ∂X are PD^2–complexes by definition and thus homotopy equivalent to closed surfaces. Furthermore, G must be finitely presentable, as X must, by definition, be dominated by a finite CW complex. Now we restrict attention to Π_1–systems $\{\kappa_i : G_i \to G\}_{i\in J}$ which are Π_1–*injective*, that is, κ_i is injective for every $i \in J$.

So let $\{\kappa_i : G_i \to G\}_{i\in J}$ be a Π_1–system such that G is finitely presentable, G_i is a surface group and κ_i is injective for every $i \in J$. Then there is an Eilenberg–Mac Lane pair $(K, \partial X)$ of type $K(\{\kappa_i\}_{i\in J}; 1)$ and by the mapping cylinder construction we may assume that the components ∂X_i of ∂X are all surfaces. Since G is finitely presentable, we may also assume that K has finite 2–skeleton $K^{[2]}$.

Take $\omega \in \mathrm{H}^1(K; \mathbb{Z}/2\mathbb{Z})$ and $\mu \in \mathrm{H}_3(K, \partial X; \mathbb{Z}^\omega)$ such that $\nu_{C(K,\partial X),2}(\mu)$ is a class of homotopy eqivalences and $\delta_*\mu = [\partial X]$ where $[\partial X]$ is the fundamental class of the PD^2–complex ∂X and δ_* is the connecting homomorphism of $C(\partial X) \rightarrowtail C(K) \twoheadrightarrow C(K, \partial X)$.

Following Turaev's construction in the absolute case, we now construct a PD^3–pair realizing $(\{\kappa_i\}_{i\in J}, \omega, \mu)$.

Since we have assumed that K has finite 2–skeleton $K^{[2]}$, the Λ–modules $C_2(K, \partial X)$ and thus $F^2(C(K, \partial X))$ are finitely generated. Let $h : F^2(C(K, \partial X)) \to I$ be a Λ–morphism representing $\nu_{C(K,\partial X),2}(\mu)$. Then h is a homotopy equivalence of Λ–modules and thus factors as

$$\begin{tikzcd} F^2(C(K,\partial X)) \arrow[r, tail] & F^2(C(K,\partial X)) \oplus \Lambda^m \arrow[r, tail, two heads, "j"] & I \oplus P \arrow[r, two heads] & I \end{tikzcd}$$

where P is finitely generated and projective, by Theorem 4.4.

Let $B = (e^0 \vee e^2) \cup e^3$ be a three dimensional ball. If we replace K by $K \vee (\vee_{i=1}^m B)$, then $K^{[2]}$ is replaced by $K^{[2]} \vee (\vee_{i=1}^m e^2)$ and $F^2(C(K, \partial X))$ is replaced by $F^2(C(K, \partial X)) \oplus \Lambda^m$. Thus we may assume without loss of generality that h factors as

$$\begin{tikzcd} F^2(C(K,\partial X)) \arrow[r, tail, two heads, "j"] & I \oplus P \arrow[r, two heads] & I \end{tikzcd} \tag{8}$$

where P is finitely generated and projective.

First we consider the case where P is free, that is, $P \cong \Lambda^n$ for some $n \in \mathbb{N}$. Let $\pi : C^2(K, \partial X) \twoheadrightarrow F^2(C(K, \partial X))$ and $\iota : I \rightarrowtail \Lambda$ be the natural projection and inclusion respectively and use the natural equivalence η to identify $(A^*)^*$ with A for a left Λ–module A. Consider the Λ–morphism

$$\begin{tikzcd} \varphi : C^2(K,\partial X) \arrow[r, two heads, "\pi"] & F^2(C(K,\partial X)) \arrow[r, tail, two heads, "j"] & I \oplus P \arrow[r, tail, "\begin{bmatrix} \iota & 0 \\ 0 & 1 \end{bmatrix}"] & \Lambda \oplus P. \end{tikzcd} \tag{9}$$

It follows from the definition of φ that $\varphi \circ \partial_1^* = 0$. Hence $(\partial_1 \circ \varphi^*)^* = \varphi \circ \partial_1^* = 0$ so that $\mathrm{im}\varphi^* \subseteq \ker \partial_1$.

Let $p : \tilde{K} \to K$ be the universal covering. Since κ_i is injective for every $i \in J$, the components of $p^{-1}(\partial X)$ are universal covering spaces of Eilenberg–Mac Lane complexes, so that $\mathrm{H}_2(p^{-1}(\partial X)) = \mathrm{H}_1(p^{-1}(\partial X)) = 0$. Thus the long exact homology sequence of the pair $(p^{-1}(K^{[2]}), p^{-1}(\partial X))$ yields

$$\mathrm{H}_2(p^{-1}(K^{[2]})) \cong \mathrm{H}_2(p^{-1}(K^{[2]}), p^{-1}(\partial X)).$$

The Hurewicz Isomorphism Theorem implies $\Pi_2(p^{-1}(K^{[2]})) \cong \mathrm{H}_2(p^{-1}(K^{[2]}))$ and thus

$$\begin{aligned} \mathrm{im}\varphi^* \subseteq \ker \partial_1 &= \mathrm{H}_2(p^{-1}(K^{[2]}), p^{-1}(\partial X)) \\ &\cong \mathrm{H}_2(p^{-1}(K^{[2]})) \\ &\cong \Pi_2(p^{-1}(K^{[2]})). \end{aligned}$$

We may thus attach $(n+1)$ three–dimensonal cells to $K^{[2]}$ to obtain a pair $(X, \partial X)$ of CW–complexes whose relative cellular chain complex is given by

$$D : 0 \longrightarrow (\Lambda \oplus P)^* \xrightarrow{\varphi^*} C_2(K, \partial X) \longrightarrow C_1(K, \partial X) \longrightarrow C_0(K, \partial X).$$

As $\Pi_2(K) = 0$, the inclusion $(K^{[2]}, \partial X) \to (K, \partial X)$ extends to a map

$$f : (X, \partial X) \longrightarrow (K, \partial X) \tag{10}$$

which induces an isomorphism of Π_1–systems. Thus we may view ω as an element of $\mathrm{H}^1(X; \mathbb{Z}/2\mathbb{Z})$.

Proposition 6.1. *$(X, \partial X)$ is a PD^3–pair realizing $(\{\kappa_i\}_{i\in J}, \omega, \mu)$.*

Proof. We must show that

(i) $\mathrm{H}_3(X, \partial X; \mathbb{Z}^\omega) \cong \mathbb{Z}$;

(ii) $f_*([X, \partial X]) = \mu$ where $[X, \partial X]$ generates $\mathrm{H}_3(X, \partial X; \mathbb{Z}^\omega)$;

(iii) $\delta_*[X, \partial X] = [\partial X]$ where $[\partial X]$ is the fundamental class of the PD^2–complex ∂X and δ_* is the connecting homomorphism of the short exact sequence $C(\partial X) \rightarrowtail C(X) \twoheadrightarrow C(X, \partial X)$;

(iv) $\cap[X, \partial X] : \mathrm{H}^r(X; {}^\omega\Lambda^\omega) \to \mathrm{H}_{r-3}(X, \partial X; \Lambda)$ is an isomorphism for every $r \in \mathbb{Z}$.

(i) As $C(X, \partial X)$ is a chain complex of free Λ–modules, $\mathbb{Z}^\omega \otimes_\Lambda C(X, \partial X) \cong \mathrm{Hom}_\Lambda(C^*(X, \partial X), \mathbb{Z})$ by Observation 3.2 and $C^*(X, \partial X) \cong C(X, \partial X)$. Thus

$$\begin{aligned} \mathrm{H}_3(X, \partial X; \mathbb{Z}^\omega) &= \mathrm{H}_3(\mathbb{Z}^\omega \otimes_\Lambda C(X, \partial X)) \\ &\cong \mathrm{H}^3({}^\omega\mathrm{Hom}_\Lambda(C^*(X, \partial X), \mathbb{Z})) \\ &\cong \ker\left((\varphi^*)^*\right)^\dagger \\ &\cong \ker \varphi^\dagger \end{aligned}$$

where $\varphi^\dagger$ arises by applying $\mathrm{Hom}_\Lambda(-, \mathbb{Z})$. Recall that $\varphi = \begin{bmatrix} \iota & 0 \\ 0 & 1 \end{bmatrix} \circ j \circ \pi$. As π and j are surjective, $\pi^\dagger$ and $j^\dagger$ are injective. Hence $\ker \varphi^\dagger = \ker \begin{bmatrix} \iota^\dagger & 0 \\ 0 & 1 \end{bmatrix} = \ker \iota^\dagger$. But I is generated by elements $1 - g, g \in G$, and $\psi \circ \iota(1 - g) = \psi(1) - g\psi(1) = 0$ for every $\psi \in C^2(K, \partial X)$, so that $\ker \iota^\dagger = \mathrm{Hom}_\Lambda(\Lambda, \mathbb{Z}) \cong \mathbb{Z}$. Thus

$$\mathrm{H}_3(X, \partial X; \mathbb{Z}^\omega) \cong \ker \varphi^\dagger \cong \ker \iota^\dagger \cong \mathbb{Z}.$$

(ii) $\mathrm{H}_3(X, \partial X; \mathbb{Z}^\omega) \cong \mathbb{Z}$ is generated by $[X, \partial X] = [1 \otimes x]$ where $x = (1, 0) \in \Lambda^* \bigoplus P^* = (\Lambda \bigoplus P)^* = C_3(X, \partial X)$ is the projection onto the first factor. By Lemma 5.7, $\nu_{C(X,\partial X),2}([1 \otimes x])$ is represented by

$$F^2(C(X, \partial X)) \longrightarrow I, \ [\psi] \longmapsto \overline{\psi(\partial_2(x))}.$$

But, again identifying free Λ–modules and Λ–morphisms between them with their double dual, we obtain, for $\psi \in C^2(X, \partial X) = C^2(K, \partial X)$,

$$\begin{aligned}\overline{\psi(\partial_2(x))} &= \overline{\psi(\varphi^*(x))} = \overline{\psi \circ \varphi^*(x)} = \overline{(\varphi^*)^*(\psi)(x)} \\ &= x(\varphi(\psi)) = x \circ \begin{bmatrix} \iota & 0 \\ 0 & 1 \end{bmatrix} \circ j \circ \pi(\psi) = h([\psi]).\end{aligned}$$

Thus $\nu_{C(X,\partial X),2}([X, \partial X])$ is the homotopy class of h, so that $\nu_{C(K,\partial X),2}(\mu) = \nu_{C(X,\partial X),2}([X, \partial X])$. Lemma 5.8 implies $\nu_{C(K,\partial X),2}(\mu) = \nu_{C(X,\partial X),2}([X, \partial X]) = \nu_{C(K,\partial X),2}(f_*[X, \partial X])$. As $\nu_{C(K,\partial X),2}$ is injective by Lemma 5.9, we may conclude $\mu = f_*[X, \partial X]$.

(iii) The map $f : (X, \partial X) \to (K, \partial X)$ gives rise to the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{H}_3(C(X, \partial X); \mathbb{Z}^\omega) \arrow[r, "\delta_*"] \arrow[d, "f_*"'] & \mathrm{H}_2(C(\partial X); \mathbb{Z}^\omega) \arrow[r] \arrow[d, "f_* = \mathrm{id}"'] & \cdots \\
\cdots \arrow[r] & \mathrm{H}_3(C(K, \partial X); \mathbb{Z}^\omega) \arrow[r, "\delta_*"] & \mathrm{H}_2(C(\partial X); \mathbb{Z}^\omega) \arrow[r] & \cdots .
\end{tikzcd}$$

Hence $\delta_*([X, \partial X]) = \delta_*(f_*([X, \partial X])) = \delta_*(\mu) = [\partial X]$.

(iv) First observe that the definition of $(X, \partial X)$ implies

$$\mathrm{H}^2(X, \partial X; {}^{\omega}\Lambda^{\omega}) = \mathrm{H}_{-2}({}^{\omega}\mathrm{Hom}_\Lambda(C(X, \partial X); {}^{\omega}\Lambda^{\omega})) = 0.$$

Since $\mathrm{H}_1(X, \Lambda) = \mathrm{H}_1(C(X)) = 0$ as well, the homomorphism

$$\cap[X, \partial X] : \mathrm{H}^2(X, \partial X; {}^{\omega}\Lambda^{\omega}) \to \mathrm{H}_1(X; \Lambda)$$

is an isomorphism.

As $\Lambda \otimes P$ is free, we may use the natural transformation η to identify ${}^{\omega}\mathrm{Hom}_\Lambda((\Lambda \oplus P)^*, {}^{\omega}\Lambda^{\omega})$ with $\Lambda \oplus P$ and $(\varphi^*)^*$ with φ. Then

$$\begin{aligned}\mathrm{H}^3(X, \partial X; {}^{\omega}\Lambda^{\omega}) &= H_{-3}({}^{\omega}\mathrm{Hom}_\Lambda(C(X, \partial X), {}^{\omega}\Lambda^{\omega})) \\ &= {}^{\omega}\mathrm{Hom}_\Lambda((\Lambda \oplus P)^*, {}^{\omega}\Lambda^{\omega})/\mathrm{im}(\varphi^*)^* \\ &\cong (\Lambda \oplus P)/\mathrm{im}\varphi \\ &\cong \Lambda/I \cong \mathbb{Z}.\end{aligned}$$

Clearly, $\mathrm{H}^3(X, \partial X; {}^{\omega}\Lambda^{\omega})$ is generated by $\psi = (1, 0) \in (\Lambda^*)^* \oplus (P^*)^* = C_3^*(X, \partial X) = C_3^*(X)$. By Lemma 5.5,

$$[\psi] \cap [X, \partial X] = [\psi] \cap [1 \otimes x] = \overline{\psi(x)} = 1,$$

that is, $\cap[X, \partial X]$ maps ψ to a generator of $\mathrm{H}_0(X; \Lambda)$. Hence

$$\cap[X, \partial X] : \mathrm{H}^3(X, \partial X; {}^{\omega}\Lambda^{\omega}) \to \mathrm{H}_0(X; \Lambda)$$

is an isomorphism. Since ∂X is a PD^2–complex,

$$\cap[\partial X] : \mathrm{H}^r(\partial X; {}^{\omega}\Lambda^{\omega}) \longrightarrow \mathrm{H}_{2-r}(\partial X; \Lambda)$$

is an isomorphism for every $r \in \mathbb{Z}$. Thus the Cap Product Ladder (cf. 3.6) of $(X, \partial X)$ with $y = [X, \partial X]$ and the Five Lemma imply that

$$\cap[X, \partial X] : \mathrm{H}^r(X; {}^{\omega}\Lambda^{\omega}) \to \mathrm{H}_{r-3}(X, \partial X; \Lambda)$$

is an isomorphism for $r = 2$ and $r = 3$. Therefore $\cap 1{\otimes}x$ gives rise to the chain homotopy equivalence

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathrm{im}\partial_1^* \arrow[r, tail] \arrow[d] & {}^{\omega}\mathrm{Hom}_\Lambda(C_2(X), {}^{\omega}\Lambda^{\omega}) \arrow[r] \arrow[d, "\cap 1\otimes x"] & {}^{\omega}\mathrm{Hom}_\Lambda(C_3(X), {}^{\omega}\Lambda^{\omega}) \arrow[r] \arrow[d, "\cap 1\otimes x"] & 0 \arrow[d] \\
0 \arrow[r] & \Lambda\otimes_\Lambda \mathrm{im}\partial_1 \arrow[r, tail] & \Lambda\otimes_\Lambda C_1(X, \partial X) \arrow[r] & \Lambda\otimes_\Lambda C_0(X, \partial X) \arrow[r] & 0
\end{tikzcd}$$

of chain complexes of left Λ–modules. Identifying $\Lambda\otimes_\Lambda A$ with A for left Λ–modules A and applying ${}^{\omega}\mathrm{Hom}_\Lambda(-, {}^{\omega}\Lambda^{\omega})$ we obtain the chain homotopy equivalence

$$\begin{tikzcd}
0 & {}^{\omega}\mathrm{Hom}_\Lambda(\mathrm{im}\partial_1^*, {}^{\omega}\Lambda^{\omega}) \arrow[l] & {}^{\omega}\mathrm{Hom}_\Lambda({}^{\omega}\mathrm{Hom}_\Lambda(C_2(X), {}^{\omega}\Lambda^{\omega}), {}^{\omega}\Lambda^{\omega}) \arrow[l] & {} \arrow[l] \\
0 & {}^{\omega}\mathrm{Hom}_\Lambda(\mathrm{im}\partial_1, {}^{\omega}\Lambda^{\omega}) \arrow[l] \arrow[u] & {}^{\omega}\mathrm{Hom}_\Lambda(C_1(X, \partial X), {}^{\omega}\Lambda^{\omega}) \arrow[l] \arrow[u, "(\cap 1\otimes x)^*"'] & {} \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
{}^{\omega}\mathrm{Hom}_\Lambda({}^{\omega}\mathrm{Hom}_\Lambda(C_3(X), {}^{\omega}\Lambda^{\omega}), {}^{\omega}\Lambda^{\omega}) & 0 \arrow[l] \\
{}^{\omega}\mathrm{Hom}_\Lambda(C_0(X, \partial X), {}^{\omega}\Lambda^{\omega}) \arrow[u, "(\cap 1\otimes x)^*"'] & 0, \arrow[l] \arrow[u]
\end{tikzcd}$$

which shows that $(\cap[1{\otimes}x])^*$ induces homology isomorphisms. But Lemma 3.4 shows that $\cap(1{\otimes}x)$ induces isomorphisms in homology if and only if $(\cap 1{\otimes}x)^*$ does. Thus

$$\cap[X, \partial X] = \cap[1{\otimes}x] : \mathrm{H}^k(X, \partial X; {}^{\omega}\Lambda^{\omega}) \longrightarrow \mathrm{H}_{3-k}(X; \Lambda)$$

is an isomorphism for $k = 0$ and $k = 1$.

The Cap Product Ladder of $(X, \partial X)$ with $y = [X, \partial X]$ and the Five Lemma imply that

$$\cap[X, \partial X] : \mathrm{H}^r(X; {}^{\omega}\Lambda^{\omega}) \longrightarrow \mathrm{H}_{3-k}(X, \partial X; \Lambda)$$

is an isomorphism for $r = 0$ and $r = 1$ and hence for every $r \in \mathbb{Z}$. □

It remains to investigate the general case where the module P in the factorization (8) of the homotopy equivalence h is finitely generated projective, but not necessarily free. Then there is a finitely generated projective Λ–module Q such that $P^* \oplus Q = \Lambda^n$ and we may attach infinitely many 3–cells to $K^{[2]} \vee \left(\vee_{i=1}^{\infty} e^2\right)$ in order to obtain a pair $(X, \partial X)$ of CW–complexes whose relative cellular chain complex is given by

$$\begin{tikzcd}
D : 0 \arrow[r] & (\Lambda\otimes P)^* \oplus \Lambda^{\infty} \arrow[r, "\begin{bmatrix} \varphi^* & 0 \\ 0 & 1 \end{bmatrix}"] & C_2(K, \partial X) \oplus \Lambda^{\infty}
\end{tikzcd}$$

$$\begin{tikzcd}
{} \arrow[r, "\begin{bmatrix} \partial_1 & 0 \end{bmatrix}"] & C_1(K, \partial X) \arrow[r] & C_0(K, \partial X).
\end{tikzcd}$$

As $(\Lambda \oplus P)^* \oplus \Lambda^\infty \cong \Lambda^* \oplus P^* \oplus (Q \oplus P^* \oplus Q \oplus \ldots) \cong \Lambda^\infty$ is free, the proof that $(X, \partial X)$ realizes $(\{\kappa_i\}_{i \in J}, \omega, \mu)$ is analogous to the proof of Proposition 6.1. It only remains to verify that X is in fact dominated by a finite cell–complex.

We follow Turaev's argument for the absolute case which uses Wall's results on finiteness conditions for CW–complexes. Since X is a finite dimensional cell–complex (of dimension three), Theorem F together with Theorems A and E of [14] imply that in order to show that X is finitely dominated, it is sufficient to show that X is homotopy equivalent to a CW–complex with finite skeleta.

Consider the cellular chain complex of X,

$$\begin{tikzcd}
C(X) : 0 \arrow[r] & (\Lambda \otimes P)^* \oplus \Lambda^\infty \arrow[r] & C_2(K, \partial X) \oplus \Lambda^\infty \oplus C_2(\partial X) \\
\arrow[r] & C_1(K, \partial X) \oplus C_1(\partial X) \arrow[r] & C_0(K, \partial X) \oplus C_0(\partial X),
\end{tikzcd}$$

and note that it is chain homotopy equivalent to the chain complex

$$\begin{tikzcd}
E : \ldots \arrow[r] & \Lambda^n \arrow[r, "\mathrm{pr}"] & \Lambda^n \arrow[r, "\mathrm{pr}'"] & \Lambda^n \\
\arrow[r, "q"] & (\Lambda \oplus P)^* \oplus Q \arrow[r, "\begin{bmatrix} \varphi^* & 0 \\ 0 & 0 \end{bmatrix}"] & C_2(K, \partial X) \oplus C_2(\partial X) \arrow[r] & C_1(K, \partial X) \oplus C_1(\partial X) \\
\arrow[r] & C_0(K, \partial X) \oplus C_0(\partial X), & &
\end{tikzcd}$$

where $\mathrm{pr} : \Lambda^n = P^* \oplus Q \to Q$ and $\mathrm{pr}' : \Lambda^n = P^* \oplus Q \to P^*$ are the canonical projections and $q(x) = (0, 0, \mathrm{pr}(x)) \in (\Lambda \oplus P)^* \oplus Q$ for $x \in \Lambda^n$. By Theorem 2 of [15], there is a CW–complex Y with cellular chain complex E which is homotopy equivalent to X. Clearly, Y has finite skeleta and we may conclude that X is finitely dominated.

Theorem 6.2. *Let $\{\kappa_i : G_i \to G\}_{i \in J}$ be a Π_1–system such that G is finitely presentable, G_i is a surface group and κ_i is injective for every $i \in J$. Let $(K, \partial X)$ be an Eilenberg–Mac Lane pair of type $K(\{\kappa_i\}_{i \in J}; 1)$ such that the components ∂X_i of ∂X are all surfaces. Take $\omega \in H^1(K; \mathbb{Z}/2\mathbb{Z})$ and $\mu \in H_3(K, \partial X; \mathbb{Z}^\omega)$ such that $\delta_* \mu = [\partial X]$ where $[\partial X]$ is the fundamental class of the PD^2–complex ∂X and δ_* is the connecting homomorphism of $C(\partial X) \rightarrowtail C(X) \twoheadrightarrow C(X, \partial X)$. Then $(\{\kappa_i\}_{i \in J}, \omega, \mu)$ is realized by a PD^3–pair $(X, \partial X)$ if and only if $\nu_{C(K, \partial X), 2}(\mu)$ is a class of homotopy eqivalences.*

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