THE SYMMETRIC GROUP REPRESENTATION ON COHOMOLOGY OF THE REGULAR ELEMENTS OF A MAXIMAL TORUS OF THE SPECIAL LINEAR GROUP

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ABSTRACT. We give a formula for the character of the representation of the Weyl group S_n on each isotypic component of the cohomology of the set of regular elements of a maximal torus of SL_n, with respect to the action of the centre. As a consequence, we find that this representation is induced from a cohomology representation of a wreath product subgroup.

1. INTRODUCTION

For any positive integers r and n, define

$$T(r,n) := \{(z_1, z_2, \cdots, z_n) \in (\mathbb{C}^\times)^n \mid z_1^r, \cdots, z_n^r \text{ distinct}\}.$$

This is an open subvariety of $(\mathbb{C}^\times)^n$, and in fact a hyperplane complement in $\mathbb{C}^n$. Let S_n be the nth symmetric group, μ_r the cyclic group of rth complex roots of 1, and $W(r,n)$ the wreath product $\mu_r \wr S_n = S_n \ltimes \mu_r^n$. We will write the elements of $W(r,n)$ in the form $w(\zeta_1, \cdots, \zeta_n)$ where $w \in S_n$ and $\zeta_i \in \mu_r$, so that the group law is

$$w(\zeta_1, \cdots, \zeta_n)w'(\zeta_1', \cdots, \zeta_n') = ww'(\zeta_{w'(1)}\zeta_1', \cdots, \zeta_{w'(n)}\zeta_n').$$

Clearly $W(r,n)$ acts on $T(r,n)$ via the rules

$$w.(z_1, \cdots, z_n) = (z_{w^{-1}(1)}, \cdots, z_{w^{-1}(n)}),$$
$$(\zeta_1, \cdots, \zeta_n).(z_1, \cdots, z_n) = (\zeta_1 z_1, \cdots, \zeta_n z_n).$$

This action arises naturally: for $r = 1$ it is the action of the Weyl group S_n on the set of regular elements of a maximal torus of $GL_n(\mathbb{C})$, and for $r \geq 2$ it is the action of the reflection group $G(r,1,n) \cong W(r,n)$ on the complement of its reflecting hyperplanes. $W(r,n)$ also acts on $\mathbb{P}T(r,n) \subset \mathbb{P}^{n-1}(\mathbb{C})$, the set of lines lying in $T(r,n)$. Thus the cohomology groups $H^i(T(r,n))$ and $H^i(\mathbb{P}T(r,n))$ – always taken with complex coefficients – are representations of $W(r,n)$. Lehrer has given quite explicit formulas for the characters of these representations –

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see (3.1) and (3.2) below. (Their dimensions, i.e. the Betti numbers of $T(r,n)$ and $\mathbb{P}T(r,n)$, were already known by work of Orlik and Solomon.)

Now consider

$$ST(1,n) := \{(z_i) \in (\mathbb{C}^\times)^n \mid z_1, \cdots, z_n \text{ distinct}, \ z_1 z_2 \cdots z_n = 1\}.$$

This can be identified with the set of regular elements of a maximal torus of $SL_n(\mathbb{C})$. Of course S_n still acts, and there is a commuting action of μ_n (the centre of $SL_n(\mathbb{C})$) by scaling. Thus we have a direct sum decomposition of S_n-modules:

$$H^i(ST(1,n)) \cong \bigoplus_{\chi \in \widehat{\mu_n}} H^i(ST(1,n))_\chi, \tag{1.1}$$

where $H^i(ST(1,n))_\chi$ is the χ-isotypic component of $H^i(ST(1,n))$. In this paper we address the following problem, suggested by Lehrer:

Problem 1.1. Give a formula for the character of the representation of S_n on each $H^i(ST(1,n))_\chi$.

Our solution is given in §3 (see especially (3.6)). The resulting formula for the total representation on $H^i(ST(1,n))$ is essentially equivalent to a result of Fleischmann and Janiszczak (see Remark 3.4).

As a consequence, we will derive the following curious statement. Let $\alpha_{r,m} : W(r,m) \to \mathbb{C}^\times$ be the one-dimensional character taking the following values on generators:

$$\alpha_{r,m}(s_i) = (-1)^{r-1}, \ \alpha_{r,m}((\zeta, 1, \cdots, 1)) = (-1)^{r-1}\zeta,$$

where ζ is a primitive rth root of 1. Note that $W(r,m)$ can be embedded in S_{rm} as the centralizer of the product of m disjoint r-cycles. The character $\alpha_{r,m}$ is the product of the determinant character $\det : W(r,m) \to \mathbb{C}^\times$ (viewing $W(r,m)$ as $G(r,1,m) \subset GL_m(\mathbb{C})$) and the sign character on S_{rm}.

Theorem 1.2. *Let e be the order of $\chi \in \widehat{\mu_n}$. For every i, we have an isomorphism of S_n-modules:*

$$H^i(ST(1,n))_\chi \cong \mathrm{Ind}_{W(e,n/e)}^{S_n}(\alpha_{e,n/e}^{-1} \otimes H^{i-n+n/e}(\mathbb{P}T(e,n/e))).$$

The proof of this Theorem given below merely equates the characters of both sides; I lack a more conceptual understanding of the isomorphism. A small part of Theorem 1.2 is the fact that

$$\dim H^i(ST(1,n))_\chi = \frac{n!}{e^{n/e}(n/e)!} \dim H^{i-n+n/e}(\mathbb{P}T(e,n/e)), \tag{1.2}$$

both sides being nonzero iff $n - n/e \leq i \leq n-1$.

Clearly the quotient of $ST(1,n)$ by μ_n can be identified with $\mathbb{P}T(1,n)$. Thus $H^i(ST(1,n))_\chi \cong H^i(\mathbb{P}T(1,n),\mathcal{L}_\chi)$ where $\mathcal{L}_\chi$ is the local system of rank 1 on $\mathbb{P}T(1,n)$ corresponding to χ. In particular, $H^i(ST(1,n))^{\mu_n} \cong H^i(\mathbb{P}T(1,n))$, which is the $e=1$ case of Theorem 1.2.

The other extreme is the $e=n$ case. Since $W(n,1)\cong\mu_n$ and $\mathbb{P}T(n,1)$ is a point, Theorem 1.2 says that for faithful $\chi\in\widehat{\mu_n}$,

$$H^i(ST(1,n))_\chi \cong \begin{cases} \mathrm{Ind}_{\mu_n}^{S_n}(\alpha_{n,1}), & \text{if } i=n-1,\\ 0, & \text{otherwise.}\end{cases} \tag{1.3}$$

This result for prime n was proved in [1, §4.4]. Note that the character $\alpha_{n,1}\in\widehat{\mu_n}$, which is defined by $\alpha_{n,1}(\zeta)=(-1)^{n-1}\zeta$ for ζ a primitive nth root of 1, is itself faithful precisely when $n\not\equiv 2 \pmod 4$.

2. Equivariant weight polynomials

Suppose X is an irreducible complex variety which is *minimally pure* in the sense that $H_c^i(X)$ is a pure Hodge structure of weight $2i-2\dim X$ for all i (see [1]). Let Γ be a finite group acting on X. We define the *equivariant weight polynomials* of this action by

$$P(\gamma,X,q) := \sum_i (-1)^i \operatorname{tr}(\gamma, H_c^i(X))\, q^{i-\dim X},$$

for all $\gamma\in\Gamma$, where q is an indeterminate ($=t^2$ in the notation of [1]). We also define

$$P^\Gamma(X,q) := \sum_i (-1)^i\, [H_c^i(X)]\, q^{i-\dim X} \in R(\Gamma)[q],$$

where $R(\Gamma)$ is the complexified representation ring of Γ. If Δ is an abelian finite group acting on X whose action commutes with that of Γ, and χ is a character of Δ, we define

$$\begin{aligned} P(\gamma,\chi,X,q) &:= \sum_i (-1)^i \operatorname{tr}(\gamma, H_c^i(X)_\chi)\, q^{i-\dim X}\\ &= \sum_i (-1)^i \frac{1}{|\Delta|}\sum_{\delta\in\Delta} \chi(\delta)^{-1}\operatorname{tr}((\gamma,\delta), H_c^i(X))\, q^{i-\dim X}\\ &= \frac{1}{|\Delta|}\sum_{\delta\in\Delta}\chi(\delta)^{-1} P((\gamma,\delta),X,q), \end{aligned}$$

and similarly

$$P^\Gamma(\chi,X,q) := \sum_i (-1)^i\, [H_c^i(X)_\chi]\, q^{i-\dim X} \in R(\Gamma)[q].$$

If X is nonsingular, we can translate knowledge of $H_c^i(X)$ and $H_c^i(X)_\chi$ into knowledge of $H^i(X)$ and $H^i(X)_\chi$ by Poincaré duality.

Now $T(r,n)$ (respectively, $\mathbb{P}T(r,n)$) is a nonsingular irreducible minimally pure variety of dimension n (respectively, $n-1$); minimal purity is a standard property of hyperplane complements ([1, Example 3.3]). Also, $ST(1,n)$ is clearly a nonsingular irreducible variety of dimension $n-1$. To show that it is minimally pure, one can use [1, Corollary 4.2], or else observe that it is the quotient of $\mathbb{P}T(n,n)$ by the free action of a finite group, as follows. Define a surjective map $\varphi : \mathbb{P}T(n,n) \to ST(1,n)$ by

$$\varphi([x_1 : x_2 : \cdots : x_n]) = \frac{1}{x_1 x_2 \cdots x_n}(x_1^n, x_2^n, \cdots, x_n^n).$$

The fibres of φ are clearly the orbits of the normal subgroup $S\mu_n^n \subset W(n,n)$ defined by

$$S\mu_n^n := \{(\zeta_1, \cdots, \zeta_n) \in \mu_n^n \mid \zeta_1\zeta_2\cdots\zeta_n = 1\}.$$

The action of $S\mu_n^n$ on $\mathbb{P}T(n,n)$ becomes free once one factors out the subgroup $\{(\zeta,\zeta,\cdots,\zeta)\}$, which acts trivially. Thus $ST(1,n)$ is minimally pure, and solving Problem 1.1 amounts to computing the polynomials $P(w,\chi,ST(1,n),q)$, for all $w \in S_n$ and $\chi \in \widehat{\mu_n}$.

Consider the quotient of $T(n,m)$ by $S\mu_n^m$ for arbitrary $m \geq 1$. This can be identified with

$$T^{(n)}(1,m) := \{((z_i),z) \in T(1,m) \times \mathbb{C}^\times \mid z^n = z_1 \cdots z_m\}.$$

The quotient map $\psi : T(n,m) \to T^{(n)}(1,m)$ is given by

$$\psi(x_1, x_2, \cdots, x_m) = ((x_1^n, x_2^n, \cdots, x_m^n), x_1 x_2 \cdots x_m).$$

The group $S_m \times \mu_n \cong W(n,m)/S\mu_n^m$ acts on the quotient $T^{(n)}(1,m)$ in the obvious way: S_m acts on the $T(1,m)$ component, and μ_n acts by scaling z.

When $m = n$, we have an isomorphism

$$T^{(n)}(1,n) \xrightarrow{\sim} ST(1,n) \times \mathbb{C}^\times$$
$$((z_1, \cdots, z_n), z) \mapsto ((z_1 z^{-1}, \cdots, z_n z^{-1}), z),$$

which respects the S_n-action, and transforms the μ_n-action on $T^{(n)}(1,n)$ into the inverse of the μ_n-action on $ST(1,n)$, and a scaling action on $\mathbb{C}^\times$. Since the latter has no effect on cohomology,

$$P(w,\chi,ST(1,n),q) = \frac{1}{q-1} P(w,\chi^{-1},T^{(n)}(1,n),q). \tag{2.1}$$

So we aim to compute $P(w,\chi^{-1},T^{(n)}(1,n),q)$; it turns out to be convenient to compute the polynomials $P(w,\chi^{-1},T^{(n)}(1,m),q)$ for all $m \geq 1$ and $w \in S_m$ together.

Remark 2.1. One can see *a priori* that allowing $m \neq n$ incurs no extra work, thanks to the following neat argument, pointed out to me by Lehrer. If $d = \gcd(m,n)$, the action of $\mu_{n/d} \subset \mu_n$ on $T^{(n)}(1,m)$ is part of the action of the connected group $\mathbb{C}^\times$ defined by

$$t.((z_i),z) = ((t^{n/d}z_i), t^{m/d}z).$$

Hence $\mu_{n/d}$ acts trivially on cohomology, so $P(w,\chi^{-1},T^{(n)}(1,m),q) = 0$ unless $\chi|_{\mu_{n/d}} = 1$, i.e. $\chi^d = 1$, i.e. $e|m$, where e is the order of χ. Moreover, if $e|m$, then writing χ° for the character of μ_e such that $\chi(\zeta) = \chi^\circ(\zeta^{n/e})$ for all $\zeta \in \mu_n$, and χ' for the character of μ_m defined by $\chi'(\zeta) = \chi^\circ(\zeta^{m/e})$ for all $\zeta \in \mu_m$, we have

$$\begin{aligned} P(w,\chi^{-1},T^{(n)}(1,m),q) &= P(w,(\chi^\circ)^{-1},T^{(e)}(1,m),q) \\ &= P(w,(\chi')^{-1},T^{(m)}(1,m),q). \end{aligned}$$

We will not actually use this observation.

The identification of $T^{(n)}(1,m)$ with the quotient of $T(n,m)$ by $S\mu_n^m$ has the following consequence for equivariant weight polynomials:

Proposition 2.2. *For any $w \in S_m$ and $\chi \in \widehat{\mu_n}$,*

$$\begin{aligned} &P(w,\chi^{-1},T^{(n)}(1,m),q) \\ &\qquad = \frac{1}{n^m} \sum_{(\zeta_i)\in\mu_n^m} \chi(\zeta_1\cdots\zeta_m) P(w(\zeta_1,\cdots,\zeta_m),T(n,m),q). \end{aligned}$$

Proof. It is well known that if V is a representation of the finite group G and V^H is the subspace invariant under the normal subgroup H, the character of G/H on V^H is given by

$$\operatorname{tr}(gH,V^H) = \frac{1}{|H|}\sum_{h\in H}\operatorname{tr}(gh,V).$$

Now apply this with $V = H_c^i(T(n,m))$, $G = W(n,m)$, and $H = S\mu_n^m$, so that $V^H \cong H_c^i(T^{(n)}(1,m))$ and $G/H \cong S_m \times \mu_n$. We find that for all $\zeta \in \mu_n$,

$$P((w,\zeta),T^{(n)}(1,m),q) = \frac{1}{n^{m-1}} \sum_{\substack{(\zeta_i)\in\mu_n^m \\ \zeta_1\cdots\zeta_m=\zeta}} P(w(\zeta_1,\cdots,\zeta_m),T(n,m),q).$$

Combining this with the fact that

$$P(w,\chi^{-1},T^{(n)}(1,m),q) = \frac{1}{n}\sum_{\zeta\in\mu_n}\chi(\zeta)P((w,\zeta),T^{(n)}(1,m),q)$$

gives the desired result. □

3. Generating functions

In this section we will compute the sum in Proposition 2.2 using Lehrer's formulas for the equivariant weight polynomials of $T(n,m)$. As is usual in dealing with characters of symmetric groups and wreath products, the computations become easier if one uses suitable "equivariant generating functions".

For any $r \geq 1$, let $\Lambda(r)$ denote the polynomial ring $\mathbb{C}[p_i(\zeta)]$ in countably many independent variables $p_i(\zeta)$ where i is a positive integer and $\zeta \in \mu_r$. Define an $\mathbb{N}$-grading on $\Lambda(r)$ by $\deg(p_i(\zeta)) = i$. Also let $\Lambda(r)[q] := \Lambda(r) \otimes_{\mathbb{C}} \mathbb{C}[q]$, with the $\mathbb{N}$-grading given by the first factor (so $\deg(q) = 0$). Let $\mathbb{A}(r) = \mathbb{C}[\![p_i(\zeta)]\!]$ be the completion of $\Lambda(r)$, and set $\mathbb{A}(r)[q] = \mathbb{A}(r) \otimes \mathbb{C}[q]$.

As in [7, Chapter I, Appendix B], we define an isomorphism $\mathrm{ch}_{W(r,m)} : R(W(r,m)) \xrightarrow{\sim} \Lambda(r)_m$ by

$$\mathrm{ch}_{W(r,m)}([M]) = \frac{1}{r^m m!} \sum_{y \in W(r,m)} \mathrm{tr}(y, M)\, p_y,$$

where $p_y = \prod_{i,\zeta} p_i(\zeta)^{a_i(\zeta)}$ if y has $a_i(\zeta)$ cycles of length i and type ζ. Note that to recover $\mathrm{tr}(y,M)$ from $\mathrm{ch}_{W(r,m)}([M])$ one must multiply the coefficient of $\prod_{i,\zeta} p_i(\zeta)^{a_i(\zeta)}$ by the order of the centralizer of y, which is $\prod_{i,\zeta} a_i(\zeta)!(ri)^{a_i(\zeta)}$. Write $\mathrm{ch}_{W(r,m)}$ also for the induced isomorphism $R(W(r,m))[q] \xrightarrow{\sim} \Lambda(r)[q]_m$.

Lehrer's results on the equivariant weight polynomials for $T(r,m)$ can be conveniently stated in terms of the equivariant generating function $P(r,q) \in \mathbb{A}(r)[q]$ defined by

$$\begin{aligned} P(r,q) &:= 1 + \sum_{m \geq 1} \mathrm{ch}_{W(r,m)}(P^{W(r,m)}(T(r,m),q)) \\ &= 1 + \sum_{m \geq 1} \frac{1}{r^m m!} \sum_{y \in W(r,m)} P(y, T(r,m), q)\, p_y. \end{aligned}$$

In the following result μ denotes the Möbius function.

Theorem 3.1. *If* $R_{r,i,\theta} := \sum_{d|i} |\{\zeta \in \mu_r \mid \zeta^d = \theta\}| \mu(d)(q^{i/d} - 1) \in \mathbb{C}[q]$,

$$P(r,q) = \prod_{\substack{i \geq 1 \\ \theta \in \mu_r}} (1 + p_i(\theta))^{R_{r,i,\theta}/ri}.$$

Proof. Mostly this is a reformulation of results in [4] ($r = 2$) and [6] ($r \geq 2$). See [3, Theorem 8.4] for a proof for all $r \geq 2$ ($T(r,m)$ is the same as what is there called $M(r,m)$). The $r = 1$ case can be proved by the same method (note that $T(1,m)$ is different from the variety $M(1,m)$ considered in [3, Theorem 8.2], since it has the extra condition of nonzero coordinates). $\square$

Recovering the traces of individual elements by the above rule, we get an equivalent statement, closer to Lehrer's in [6, §6]: if $y \in W(r,m)$ has $a_i(\zeta)$ cycles of length i and type ζ,

$$P(y,T(r,m),q) = \prod_{\substack{i\geq 1\\ \zeta\in\mu_r}} R_{r,i,\zeta}(R_{r,i,\zeta}-ri)\cdots(R_{r,i,\zeta}-(a_i(\zeta)-1)ri). \tag{3.1}$$

This also gives a character formula for $\mathbb{P}T(r,m)$, since it is easy to see that

$$P(y,\mathbb{P}T(r,m),q) = \frac{1}{q-1}P(y,T(r,m),q). \tag{3.2}$$

There is an alternative description of the polynomials $R_{r,i,\theta}$. Define

$$R_i^{(r)} := \sum_{\substack{d|i\\ \gcd(d,r)=1}} \mu(d)(q^{i/d}-1) \in \mathbb{C}[q].$$

Lemma 3.2. *If $t(\theta)$ denotes the order of θ,*

$$R_{r,i,\theta} = \sum_{s|\gcd(r/t(\theta),i)} s\mu(s)R_{i/s}^{(r)}.$$

Proof. Since μ_r is cyclic of order r, we have

$$|\{\zeta\in\mu_r \mid \zeta^d=\theta\}| = \begin{cases} \gcd(d,r), & \text{if } \gcd(d,r)\mid(r/t(\theta)),\\ 0, & \text{otherwise.}\end{cases}$$

Hence

$$R_{r,i,\theta} = \sum_{s|(r/t(\theta))} s \sum_{\substack{d|i\\ \gcd(d,r)=s}} \mu(d)(q^{i/d}-1).$$

The sum over d has no terms unless $s|i$, in which case it equals

$$\sum_{\substack{d|(i/s)\\ \gcd(d,r)=1}} \mu(ds)(q^{i/ds}-1).$$

Since $\gcd(d,r)=1$ implies $\mu(ds)=\mu(d)\mu(s)$, this is $\mu(s)R_{i/s}^{(r)}$. $\square$

This Lemma makes it clear that the $r=2$ case of (3.1) is indeed equivalent to [4, Theorem 5.6].

Now for any $\chi \in \widehat{\mu_n}$, define the generating function

$$\begin{aligned} P(\chi,q) &:= 1+\sum_{m\geq 1}\mathrm{ch}_{S_m}(P^{S_m}(\chi^{-1},T^{(n)}(1,m),q)) \\ &= 1+\sum_{m\geq 1}\frac{1}{m!}\sum_{w\in S_m}P(w,\chi^{-1},T^{(n)}(1,m),q)\,p_w \ \in \mathbb{A}(1)[q]. \end{aligned}$$

We want a formula for this similar to Theorem 3.1. Define

$$P_i^{(r)} := \prod_{s|\,\mathrm{gcd}(r,i)} (1-(-p_i)^{r/s})^{s\mu(s)R^{(r)}_{i/s}/ri} \ \in \mathbb{A}(1)[q].$$

Theorem 3.3. *If $\chi \in \widehat{\mu_n}$ has order e, $P(\chi,q)=\prod_{i\geq 1}P_i^{(e)}$.*

Proof. Using Proposition 2.2, we see that $P(\chi,q)$ equals

$$1+\sum_{m\geq 1}\frac{1}{n^m m!}\sum_{\substack{w\in S_m\\ (\zeta_i)\in\mu_n^m}}\chi(\zeta_1\cdots\zeta_m)P(w(\zeta_1,\cdots,\zeta_m),T(n,m),q)\,p_w,$$

which is precisely the result of applying to $P(n,q)$ the specialization $p_i(\theta)\to\chi(\theta)p_i$. So by Theorem 3.1,

$$\begin{aligned} P(\chi,q) &= \exp\sum_{\substack{i\geq 1\\ \theta\in\mu_n}}\frac{R_{n,i,\theta}}{ni}\log(1+\chi(\theta)p_i) \\ &= \exp\sum_{\substack{i\geq 1\\ \theta\in\mu_n\\ m\geq 1}}\frac{-R_{n,i,\theta}}{nmi}\chi(\theta)^m(-p_i)^m \\ &= \exp\sum_{\substack{i\geq 1\\ d|i\\ \zeta\in\mu_n\\ m\geq 1}}\frac{-\mu(d)}{nmi}\chi(\zeta)^{dm}(q^{i/d}-1)(-p_i)^m \\ &= \exp\sum_{\substack{i\geq 1\\ d|i\\ m\geq 1,\, e|dm}}\frac{-\mu(d)}{mi}(q^{i/d}-1)(-p_i)^m, \end{aligned}$$

since $\sum_{\zeta\in\mu_n}\chi(\zeta)^{dm}=n$ if χ^{dm} is trivial, and vanishes otherwise. Thus we need only show that

$$P_i^{(e)} = \exp\sum_{\substack{d|i\\ m\geq 1,\, e|dm}}\frac{-\mu(d)}{mi}(q^{i/d}-1)(-p_i)^m. \qquad (3.3)$$

The condition $e|dm$ is equivalent to $(e/\gcd(d,e))|m$. Writing f for $\gcd(d,e)$, the right-hand side becomes

$$\begin{aligned}&\exp \sum_{\substack{f|e\\ d|i,\, \gcd(d,e)=f\\ m\geq 1,\, (e/f)|m}} \frac{-\mu(d)}{mi}(q^{i/d}-1)(-p_i)^m\\ &\quad = \exp \sum_{f|\gcd(e,i)} \frac{f}{ei}\log(1-(-p_i)^{e/f}) \sum_{\substack{d|i\\ \gcd(d,e)=f}} \mu(d)(q^{i/d}-1).\end{aligned}$$

By the same argument as in the proof of Lemma 3.2, the sum over d equals $\mu(f)R_{i/f}^{(e)}$ as required. □

Note that $P(\chi, q)$ depends only on e, not on n or χ, and that, as predicted in Remark 2.1, its nonzero homogeneous components all have degree divisible by e.

There is no formula as neat as (3.1) for the individual polynomials $P(w, \chi^{-1}, T^{(n)}(1,m), q)$. But if $w \in S_m$ has a_i cycles of length i, we know that

$$P(w, \chi^{-1}, T^{(n)}(1,m), q) = \prod_{i\geq 1} a_i!\, i^{a_i}(\text{coefficient of } p_i^{a_i} \text{ in } P_i^{(e)}). \tag{3.4}$$

Note that for the right-hand side to be nonzero, a_i must be divisible by $(e/\gcd(e,i))$ for all i. In the special case that $\gcd(e,i)=1$, we have

$$P_i^{(e)} = (1-(-p_i)^e)^{R_i^{(e)}/ei},$$

and the coefficient of $p_i^{a_i}$, where a_i is divisible by e, is

$$(-1)^{a_i - a_i/e}\frac{R_i^{(e)}(R_i^{(e)}-ei)(R_i^{(e)}-2ei)\cdots(R_i^{(e)}-(a_i-e)i)}{(ei)^{a_i/e}(a_i/e)!}. \tag{3.5}$$

Some further special cases of note: the $e=1$ case of Theorem 3.3 says that

$$P(\mathrm{triv}, q) = \prod_{i\geq 1}(1+p_i)^{R_i^{(1)}/i} = P(1,q),$$

reflecting the fact that the quotient of $T^{(n)}(1,m)$ by μ_n is $T(1,m)$. Slightly more interesting is the $e=2$ case. We have

$$P_i^{(2)} = \begin{cases} (1-p_i^2)^{R_i^{(2)}/2i}, & \text{if } i \text{ is odd},\\ (1-p_i^2)^{R_i^{(2)}/2i}(1+p_i)^{-R_{i/2}^{(2)}/i}, & \text{if } i \text{ is even}.\end{cases}$$

Hence if i is even, the coefficient of $p_i^{a_i}$ is

$$\sum_{j=0}^{\lfloor a_i/2\rfloor} (-1)^j \binom{R_i^{(2)}/2i}{j} \binom{-R_{i/2}^{(2)}/i}{a_i-2j}.$$

Returning to Problem 1.1, equations (2.1) and (3.4) tell us that if $w \in S_n$ has a_i cycles of length i,

$$P(w,\chi,ST(1,n),q) = \frac{1}{q-1}\prod_{i\geq 1} a_i!\, i^{a_i}(\text{coefficient of } p_i^{a_i} \text{ in } P_i^{(e)}). \tag{3.6}$$

Since there are $\phi(e)$ characters $\chi \in \widehat{\mu_n}$ of order e, we deduce that

$$P(w,ST(1,n),q) = \frac{1}{q-1}\sum_{e|n}\phi(e)\prod_{i\geq 1} a_i!\, i^{a_i}(\text{coefficient of } p_i^{a_i} \text{ in } P_i^{(e)}). \tag{3.7}$$

Remark 3.4. If q is specialized to a prime power congruent to 1 mod n, the right-hand side of (3.7) equals the formula given in [2, Theorem 5.8] for the number of $\mathbb{F}_q$-points of the regular set of a maximal torus of $SL_n(\mathbb{F}_q)$ obtained from a maximally split one by twisting with w. (To see this, use the expression (3.3) for $P_i^{(e)}$; the coefficient of $p_i^{a_i}$ is called $R^i_{a_i,n}(q)$ in [2].) We have effectively reproved that result, since

$$P(w,ST(1,n),q) = |ST(1,n)(\overline{\mathbb{F}_q})^{wF}|$$

by Grothendieck's Frobenius trace formula and a suitable comparison theorem of complex and ℓ-adic cohomology. Actually, everything we have done with equivariant weight polynomials $P(\gamma,X,q)$ would have worked equally well with the numbers of fixed points $|X(\overline{\mathbb{F}_q})^{\gamma F}|$ of twisted Frobenius maps.

4. Induction

We now aim to interpret the generating function $P(\chi,q)$ in terms of induced characters. Recall that $W(r,m)$ is embedded in S_{rm} as the centralizer of the product of m disjoint r-cycles. For any $\theta \in \mu_r$, let $t(\theta)$ denote the order of θ.

Lemma 4.1. *For any $W(r,m)$-module M,*

$$\mathrm{ch}_{S_{rm}}([\mathrm{Ind}_{W(r,m)}^{S_{rm}}(M)]) = \mathrm{ch}_{W(r,m)}([M])|_{p_i(\theta)\to p_{it(\theta)}^{r/t(\theta)}}.$$

Proof. This is a direct consequence of Frobenius' formula for induced characters, once one observes that a cycle of length i and type θ in $W(r,m)$ beomes the product of $r/t(\theta)$ disjoint $it(\theta)$-cycles when regarded as an element of S_{rm}. □

Lemma 4.2. *For any $W(r,m)$-module M,*

$$\mathrm{ch}_{S_{rm}}([\mathrm{Ind}_{W(r,m)}^{S_{rm}}(\alpha_{r,m}\otimes M)]) \\ = (-1)^{rm-m}\mathrm{ch}_{W(r,m)}([M])|_{p_i(\theta)\to -\theta(-p_{it(\theta)})^{r/t(\theta)}}.$$

Proof. Let $y \in W(r,i)$ be a cycle of length i and type θ. Since $\alpha_{r,i}$ is the product of $\det : W(r,i) \to \mathbb{C}^\times$ and the sign character on S_{ri},

$$\alpha_{r,i}(y) = (-1)^{i-1+(it(\theta)-1)r/t(\theta)}\theta = (-1)^{i-1+ri-r/t(\theta)}\theta.$$

Also, if $y \in W(r,m)$,

$$p_y|_{p_i(\theta)\to(-1)^{ri+i}p_i(\theta)} = (-1)^{rm-m}p_y.$$

Hence

$$\mathrm{ch}_{W(r,m)}([\alpha_{r,m}\otimes M]) = (-1)^{rm-m}\mathrm{ch}_{W(r,m)}([M])|_{p_i(\theta)\to-\theta(-1)^{r/t(\theta)}p_i(\theta)},$$

and the result follows by applying the previous Lemma. □

Now define an element $P'(r,q) \in \mathbb{A}(1)[q]$ by

$$\begin{aligned} P'(r,q) &:= P(r,q)|_{p_i(\theta)\to-\theta(-p_{it(\theta)})^{r/t(\theta)}} \\ &= 1 + \sum_{m\geq 1}\mathrm{ch}_{W(r,m)}(P^{W(r,m)}(T(r,m),q))|_{p_i(\theta)\to-\theta(-p_{it(\theta)})^{r/t(\theta)}} \\ &= 1 + \sum_{m\geq 1}(-1)^{rm-m}\mathrm{ch}_{S_{rm}}(\mathrm{Ind}_{W(r,m)}^{S_{rm}}(\alpha_{r,m}\otimes P^{W(r,m)}(T(r,m),q))). \end{aligned}$$

Proposition 4.3. $P'(r,q) = \prod_{i\geq 1} P_i^{(r)}$.

Proof. By Theorem 3.1 and Lemma 3.2, we have

$$\begin{aligned} P'(r,q) &= \prod_{\substack{i\geq 1\\ \theta\in\mu_r}}(1-\theta(-p_{it(\theta)})^{r/t(\theta)})^{R_{r,i,\theta}/ri} \\ &= \prod_{\substack{i\geq 1\\ \theta\in\mu_r\\ s|\gcd(r/t(\theta),i)}}(1-\theta(-p_{it(\theta)})^{r/t(\theta)})^{s\mu(s)R^{(r)}_{i/s}/ri}. \end{aligned}$$

Applying to this the Möbius inversion formula for cyclotomic polynomials, namely that

$$\prod_{\substack{\theta\in\mu_r\\ t(\theta)=t}}(1-\theta X) = \prod_{u|t}(1-X^{t/u})^{\mu(u)},$$

we obtain

$$P'(r,q) = \prod_{\substack{i\geq 1\\ t|r\\ s|\gcd(r/t,i)\\ u|t}} (1-(-p_{it})^{r/u})^{s\mu(s)\mu(u)R^{(r)}_{i/s}/ri}.$$

Write this as $\prod_{i\geq 1} Q_i^{(r)}$, where $Q_i^{(r)}$ is the product of all factors involving the variable p_i. Thus

$$\begin{aligned} Q_i^{(r)} &= \exp \sum_{\substack{t|\gcd(r,i)\\ s|\gcd(r/t,i/t)\\ u|t}} \frac{st\mu(s)\mu(u)}{ri} R^{(r)}_{i/st}\log(1-(-p_i)^{r/u}) \\ &= \exp \sum_{\substack{v|\gcd(r,i)\\ u|v\\ s|(v/u)}} \frac{v\mu(s)\mu(u)}{ri} R^{(r)}_{i/v}\log(1-(-p_i)^{r/u}), \end{aligned}$$

where we have set $v = st$. Since $\sum_{s|(v/u)}\mu(s)$ is nonzero if and only if $u = v$, we find that $Q_i^{(r)} = P_i^{(r)}$ as required. □

Corollary 4.4. *If $\chi \in \widehat{\mu_n}$ has order e, $P(\chi,q) = P'(e,q)$.*

Proof. Combine Theorem 3.3 and Proposition 4.3. □

Corollary 4.5. *If $\chi \in \widehat{\mu_n}$ has order e, and $e|m$, we have the following equality in $R(S_m)[q]$:*

$$\begin{aligned} &P^{S_m}(\chi^{-1}, T^{(n)}(1,m), q) \\ &\qquad = (-1)^{m-m/e}\mathrm{Ind}^{S_m}_{W(e,m/e)}(\alpha_{e,m/e}\otimes P^{W(e,m/e)}(T(e,m/e),q)). \end{aligned}$$

Proof. Under the isomorphism ch_{S_m}, the left-hand side corresponds to the degree-m term of $P(\chi,q)$, and the right-hand side corresponds to the degree-m term of $P'(e,q)$. □

Taking coefficients of q^{i-m} on both sides and multiplying by $(-1)^i$, we get an isomorphism of S_m-modules:

$$(4.1)\quad H^i_c(T^{(n)}(1,m))_{\chi^{-1}} \cong \mathrm{Ind}^{S_m}_{W(e,m/e)}(\alpha_{e,m/e}\otimes H_c^{i-m+m/e}(T(e,m/e)).$$

By Poincaré duality, this is equivalent to

$$(4.2)\quad H^i(T^{(n)}(1,m))_{\chi} \cong \mathrm{Ind}^{S_m}_{W(e,m/e)}(\alpha^{-1}_{e,m/e}\otimes H^{i-m+m/e}(T(e,m/e)).$$

An equivalent way to state this isomorphism is:

$$(4.3)\quad H^i(T(1,m),\mathcal{L}) \cong \mathrm{Ind}^{S_m}_{W(e,m/e)}(\alpha^{-1}_{e,m/e}\otimes H^{i-m+m/e}(T(e,m/e)),$$

where $\mathcal{L}$ is any of the rank-one local systems on $T(1,m)$ obtained from the covers $T^{(n)}(1,m)$, and e is its order in the group of rank-one local systems.

Finally, we prove Theorem 1.2. Equation (2.1) and Corollary 4.5 together imply

$$\begin{aligned} &P^{S_n}(\chi, ST(1,n), q) \\ &\quad = \frac{(-1)^{n-n/e}}{q-1} \mathrm{Ind}_{W(e,n/e)}^{S_n}(\alpha_{e,n/e} \otimes P^{W(e,n/e)}(T(e,n/e),q)) \\ &\quad = (-1)^{n-n/e} \mathrm{Ind}_{W(e,n/e)}^{S_n}(\alpha_{e,n/e} \otimes P^{W(e,n/e)}(\mathbb{P}T(e,n/e),q)). \end{aligned}$$

Taking coefficients of q^{i-n+1} on both sides and multiplying by $(-1)^i$, we get an isomorphism of S_n-modules:

$$H_c^i(ST(1,n))_\chi \cong \mathrm{Ind}_{W(e,n/e)}^{S_n}(\alpha_{e,n/e} \otimes H_c^{i-n+n/e}(\mathbb{P}T(e,n/e))). \tag{4.4}$$

Since the right-hand side depends only on n and the order of χ, this remains true if χ is replaced by χ^{-1}. Then Theorem 1.2 follows by Poincaré duality.

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