Singularities of Plane Algebraic Curves

Jonathan A. Hillman

School of Mathematics and Statistics, The University of Sydney,
Sydney, NSW 2006, AUSTRALIA

e-mail: jonh@maths.usyd.edu.au

ABSTRACT

We give an exposition of some of the basic results on singularities of plane algebraic curves, in terms of polynomials and formal power series.


Key words and phrases: isolated singularity, Nullstellensatz, plane curve.

A nonconstant polynomial \( f \in \mathbb{C}[X,Y] \) defines a plane curve \( V(f) = \{(x,y) \in \mathbb{C}^2 \mid f(x,y) = 0\} \). If \( f \) is square-free there are only finitely many points \( P \in V(f) \) such that \( f_X(P) = f_Y(P) = 0 \), where \( f_X \) and \( f_Y \) are the partial derivatives of \( f \). This article is an account of the basic properties of such singularities of plane curves, in terms of elementary commutative algebra. Although all the results described here are well-known, some of the arguments may be new.

In §1 we prove the weak Nullstellensatz, and show that an irreducible plane curve \( V(f) \subset \mathbb{C}^2 \) is smooth if and only if its coordinate ring \( \mathbb{C}[X,Y]/(f) \) is integrally closed. In §2 we show that the formal power series ring \( \mathbb{C}[[X,Y]] \) is a 2-dimensional factorial domain and establish the Weierstraß Division and Preparation Theorems. We also show that every square-free formal power series is equivalent to a polynomial after a formal change of coordinates. In §3 we pass to \( A = \mathbb{C}[[X,Y]]/(f) \), the completion of the coordinate ring of \( V(f) \) at 0, where \( f \) is a square-free distinguished polynomial in \( Y \), and describe the method of Puiseux expansions. The next two sections consider numerical invariants of singularities. In §4 we show that the Milnor number \( \mu(f) = \dim_{\mathbb{C}} \mathbb{C}[[X,Y]]/(f_X,f_Y) \) is finite if and only if \( f \) is square-free, and we compute this for \( f \) a weighted homogeneous polynomial. In §5 we show that the codimension \( \delta_A \) of \( A \) in its integral closure \( \bar{A} \) equals the codimension
of the conductor $C$ of $\overline{A}$ into $A$ in $A$, and that $\delta_A = (\mu(f) + 1 - r)/2$, where $r$ is the number of irreducible factors of $f$. These invariants are used in §6 to outline how plane curve singularities may be resolved by iterated quadratic transforms, and in §7 we make some remarks on the links associated to such singularities. The final sections are devoted to calculating the monodromy of the local Gauss-Manin connection in some special cases.

The only algebraic result that we use which is not in the text [AM] is the fact that the integral closure of a complete discrete valuation ring in a finite extension of its fields of fractions is again a complete discrete valuation ring, for which we refer to [Se']. (The issues considered in §6 and §7 are treated in much greater detail in the books [BK], [EN] and [Mi]).

§1. Algebraic curves

If $f$ is a nonconstant polynomial the irreducible components of $V(f)$ correspond to the irreducible factors of $f$. The points of $V(f)$ correspond to the maximal ideals of the coordinate ring $O_{V(f)} = \mathbb{C}[X,Y]/(f)$, by the Nullstellensatz. We shall give a simple ad hoc argument for the 2-variable case, which extends readily to a proof of the (weak) Nullstellensatz for polynomial rings in many variables over algebraically closed fields. (The arguments for Theorems 1 and 4 are based on ones given in [Hi84]).

**Theorem 1.** Let $M$ be a maximal ideal in $\mathbb{C}[X,Y]$. Then $M = (X-\alpha, Y-\beta)$ for some $\alpha, \beta \in \mathbb{C}$.

**Proof.** If $M \cap \mathbb{C}[X] = 0$ then $\mathbb{C}[X]$ embeds in the field $K = \mathbb{C}[[X,Y]]/M$, and so the projection extends to a homomorphism from $\mathbb{C}(X)[Y]$ to $K$. Hence the kernel $M_0 = M\mathbb{C}(X)$ is a proper maximal ideal in the PID $\mathbb{C}(X)[Y]$, and so is generated by $f/d$ say, where $f \in M$ has positive degree in $Y$ and $d \in \mathbb{C}[X] - \{0\}$. Then $f$ divides $dM$ in $\mathbb{C}[X,Y]$ and so $f$ and $M$ generate the same (maximal) ideal in $\mathbb{C}[X,Y][1/d]$. Let $h(X)$ be the coefficient of the leading term of $f$, and let $\gamma \in \mathbb{C}$ be a point at which neither $d$ nor $h$ is 0. Since $X - \gamma$ is a nonzero element of the field $K$ and $K = \mathbb{C}[X,Y][1/d]/(f)$ there are $g, k \in \mathbb{C}[X,Y][1/d]$ such that $(X - \gamma)g - 1 = fk$. Let $\rho : \mathbb{C}[X,Y][1/d] \to \mathbb{C}[Y]$ be the epimorphism with kernel $(X - \gamma)$. Then $\rho(f)$ is a unit in $\mathbb{C}[Y]$, and so $\rho(h) = 0$, contrary to hypothesis. Therefore $M \cap \mathbb{C}[X]$ is a nonzero prime
ideal, and so is generated by $X - \alpha$, for some $\alpha \in \mathbb{C}$. Similarly $M$ contains $Y - \beta$ for some $\beta \in \mathbb{C}$, and so $M = (X - \alpha, Y - \beta)$. □

This theorem can be generalized as follows. Let $R$ be a domain in which the intersection of the maximal ideals is 0, and let $M$ be a maximal ideal in $R[X]$. Then $M \cap R$ is a maximal ideal in $R$, and the intersection of all the maximal ideals of $R[X]$ is 0.

In the above argument we replace $\gamma$ by a maximal ideal $n$ of $R$ which does not contain $dh$, $X - \gamma$ by a nonzero element of $n$ and $\rho$ by the projection of $R[1/d, Y]$ onto $(R/n)[Y]$. (Some hypothesis on $R$ is necessary, for if $R = \mathbb{Z}_p$ is the ring of rationals with denominator prime to $p$ and $f = 1 - pX$ then $R \cap f = 0$ while $R[X]/(f) = \mathbb{Q}$, so $(f)$ is maximal in $R[X]$). This result may be applied inductively to show that if $K$ is an algebraically closed field the maximal ideals of $K[X_1, \ldots, X_n]$ are all of the form $(X_1 - \kappa_1, \ldots, X_n - \kappa_n)$.

(This is the “Weak Nullstellensatz”).

**Theorem 2.** Let $f, g \in \mathbb{C}[X, Y]$ have no common factors. Then

$$|V(f) \cap V(g)| \leq \dim_\mathbb{C}\mathbb{C}[X, Y]/(f, g) < \infty.$$  

**Proof.** The images of $f$ and $g$ in $\mathbb{C}(X)[Y]$ have no common factor there, since $\mathbb{C}(X, Y)$ is a UFD. Hence there are $m, n \in \mathbb{C}(X)[Y]$ such that $mf + ng = 1$, by the Euclidean algorithm. Clearing denominators, there are polynomials $M, N \in \mathbb{C}[X, Y]$ and a monic polynomial $h \in \mathbb{C}[X]$ such that $Mf + Ng = h$. Hence the first coordinate of any point of $V(f) \cap V(g)$ is a root of $h$, and for each such root $\alpha$ the number of points in $V(f) \cap V(g)$ with first coordinate $\alpha$ is bounded by $\dim_\mathbb{C}\mathbb{C}[X, Y]/(X - \alpha, f, g)$. Since $X - \alpha$ does not divide both $f$ and $g$ this dimension is finite, and so $V(f) \cap V(g)$ is finite.

More precisely, let $h = \Pi_{i=1}^r (X - \alpha_i)^{m(i)}$ be the factorization of $h$. Let $A_i = \mathbb{C}[X]/((X - \alpha_i)^{m(i)})$ and let $f_i, g_i$ be the images of $f, g$ in $A_i[Y] = \mathbb{C}[X, Y]/((X - \alpha_i)^{m(i)})$, for all $i \leq r$. Then $\dim_\mathbb{C}\mathbb{C}[X, Y]/(X - \alpha_i, f, g) \leq \dim_\mathbb{C}A_i[Y]/(f_i, g_i) < \infty$, for all $i \leq r$. As $\mathbb{C}[X, Y]/(f, g) \cong \oplus_{i=1}^r A_i[Y]/(f_i, g_i)$ we have $|V(f) \cap V(g)| \leq \dim_\mathbb{C}\mathbb{C}[X, Y]/(f, g) < \infty$. □

Since $\mathbb{C}[X, Y]/(f, g)$ is finite dimensional and hence Artinian, it is the direct sum of its localizations at maximal ideals, by Theorem 8.7 of [AM]. Thus $\mathbb{C}[X, Y]/(f, g) \cong \oplus_{P \in \mathbb{C}[Y]} \mathbb{C}[X, Y]_P/(f, g)$, where $\mathbb{C}[X, Y]_P$ is the ring of ra-
tional functions $r/s \in \mathbb{C}(X,Y)$ with $s(P) \neq 0$, and $\mathbb{C}[X,Y]_{(P)}/(f,g) \neq 0$ if and only if $P \in V(f) \cap V(g)$.

If $f$ and $g$ have total degree $m$ and $n$ (respectively) then Bezout’s Theorem gives $mn$ as the intersection number of the projective completions of $V(f)$ and $V(g)$ in the projective plane $\mathbb{CP}^2$, with multiplicities for intersections at singularities and points where the curves are tangent. (The multiplicity of the intersection at $P$ is $\dim_{\mathbb{C}} \mathbb{C}[X,Y]_{(P)}/(f,g)$. See page 112 of [Fu]). The intersection points are all in the affine plane $\mathbb{C}^2$ if and only if the homogeneous parts of highest degree $f_m$ and $g_n$ have no common factors.

A plane curve is smooth or nonsingular if it has a well defined tangent at every point, i.e., if and only if the partial derivatives $f_X$ and $f_Y$ do not vanish simultaneously at any point of $V(f)$. Let $\text{Sing}(V(f)) = \{(x,y) \in V(f) \mid f_X(x,y) = f_Y(x,y) = 0\}$ be the set of singular points of $V(f)$. Then $V(f)$ is smooth if $\text{Sing}(V(f))$ is empty.

**Corollary.** If $f$ is a square-free polynomial then $\text{Sing}(V(f))$ is finite.

**Proof.** If $f$ is square-free then $f$ has no common factors with either of its partial derivatives, and so $\text{Sing}(V(f)) = V(f) \cap V(f_X) \cap V(f_Y)$ is finite. \(\square\)

The next theorem shows that smoothness is an intrinsic property of the ring $O_{V(f)}$, and does not depend on the embedding of $V(f)$ in $\mathbb{C}^2$. We shall use the following special case of Nakayama’s Lemma in the proof.

**Lemma 3.** Let $R$ be a local ring with maximal ideal $M = (r,s)$. If $I$ is an ideal in $R$ such that $M = I + M^2$ then $I = M$.

**Proof.** Since $M = M^2 + I$ we may find $m,n,p,q \in M$ and $i,j \in I$ such that $r = i + mr + ns$ and $s = j + pr + qs$. Since the determinant $(1-m)(1-q) - pn$ is not in $M$ it is invertible in $R$, and so we may solve this pair of equations for $r$ and $s$ in terms of $i$ and $j$. Hence $M \leq I$ and so $M = I$. \(\square\)

**Theorem 4.** Let $f \in \mathbb{C}[X,Y]$ be irreducible. Then the following are equivalent:

1. $O_{V(f)} = \mathbb{C}[X,Y]/(f)$ is integrally closed;
2. $f \notin M^2$, for all maximal ideals $M < \mathbb{C}[X,Y]$ which contain $(f,f_Y)$;
3. $(f,f_X,f_Y) = \mathbb{C}[X,Y]$;
4. $V(f)$ is nonsingular.
Proof. The nonzero prime ideals of \( O_{V(f)} \) correspond bijectively to the maximal ideals of \( \mathbb{C}[X,Y] \) which contain \( f \), via the canonical epimorphism from \( \mathbb{C}[X,Y] \) to \( O_{V(f)} \), and so \( O_{V(f)} \) is a 1-dimensional noetherian domain. Therefore \( O_{V(f)} \) is integrally closed if and only if each of its localizations is a discrete valuation ring. Let \( A = \mathbb{C}[X] \), and let \( N \) be a maximal ideal of \( O_{V(f)} \), with preimage \( M \) in \( A[Y] = \mathbb{C}[X,Y] \). (Then \( f \in M \)). Let \( Q = M \cap A \), \( B = AQ \) and \( C = A[Y]_M \). Then \( B \) is a discrete valuation ring with maximal ideal \( QB \) and \( C \) is a local ring with maximal ideal \( MC \) generated by \( Q \) and \( g \), for some \( g \) representing an irreducible factor of the image of \( f \) in \( (A/Q)[Y] = (B/QB)[Y] \). Since \( 0 < (f)C < MC \) is a chain of distinct prime ideals, \( MC \) cannot be principal. Therefore \( MC/(MC)^2 \) has dimension 2 as a vector space over \( C/MC \cong \mathbb{C} \), by Nakayama’s Lemma. The maximal ideal of the localization of \( O_{V(f)} \) at \( N \) is \( MC/(f) \) and so is principal if and only if there is some \( t \in C \) such that \( MC = (f,t) \). In this case the images of \( f \) and \( t \) in \( MC/(MC)^2 \) form a basis, so \( f \notin M^2 \). Thus \((1) \Rightarrow (2)\).

If \((f,f_X,f_Y) \leq M = (X-\alpha,Y-\beta) \) then on considering the Taylor expansion of \( f \) around \((\alpha,\beta) \in \mathbb{C}^2 \), we see that \( f \in M^2 \). Hence \((2) \Rightarrow (3)\). Conversely, if \( f \in M^2 \) then \( f_X \) and \( f_Y \) are in \( M \), by an easy application of the Leibniz formula for derivatives of products, so \((3) \Rightarrow (2)\). Moreover if \((2) \) holds and \( M \) is any maximal ideal containing \( f \) then \( f \notin M^2 \). Hence \( MC/(f) + (MC)^2 \) is 1-dimensional and so \( MC/(f) \) is principal, by Nakayama’s Lemma again. Therefore \((2) \Rightarrow (1)\).

The equivalence of \((3) \) and \((4) \) is an immediate consequence of the Nullstellensatz. \( \Box \)

Corollary. \( V(f) \) is smooth if and only if \( O_{V(f)} \) is integrally closed. \( \Box \)

The equivalence \((1) \equiv (2) \) goes through with little change if \( \mathbb{C}[X] \) is replaced by any integrally closed 1-dimensional noetherian domain \( R \). If \((f,f_Y) \leq M \) then \( \varphi = M \cap R \) divides the resultant \( Res(f,f_Y) \) and \( M = (\varphi,g) \), where \( g \) represents an irreducible factor of the image of \( f \) in \( R/\varphi[Y] \). (See §4 below for a definition of resultant). In particular, let \( \Phi_n \in \mathbb{Z}[Y] \) be the \( n^{th} \) cyclotomic polynomial, and \( \zeta_n \) a primitive \( n^{th} \) root of unity. Since \( X^n-1 \) (and hence \( \Phi_n \)) has distinct roots over any field of characteristic prime to \( n \), the only primes dividing \( Res(\Phi_n,\Phi'_n) \) are divisors of \( n \). If \( n = mq \) with \( q = p^r \) and \( (m,p) = 1 \)
then \( \Phi_n(X) = \Phi_m(X^n)/\Phi_m(X^{n/p}) \) so \( \Phi_n \equiv \Phi_m^{\phi(q)} \mod (p) \). Moreover \( \Phi_n(X) \) divides \( \Phi_p(X^{n/p}) \) and so \( \Phi_n(\zeta_m) \) divides \( \Phi_p(1) = p \). Therefore \( \Phi_n \) is not in \( (p,g)^2 \) for any \( g \) representing an irreducible factor of \( \Phi_m \) in \( \mathbb{F}_p[Y] \), and so \( \mathbb{Z}[\zeta_n] = \mathbb{Z}[Y]/(\Phi_n) \) is the full ring of integers in \( \mathbb{Q}(\zeta_n) \). (This argument avoids first computing the discriminant \( \text{disc}(\Phi_n) = \text{Res}(\Phi_n, \Phi_n') \)).

The equivalence (2) \( \equiv \) (3) holds if \( C \) is replaced by any perfect field. (The latter hypothesis is necessary. Let \( K \) be a field of characteristic \( p > 0 \) with an element \( b \) which is not a \( p \)-th power. Then \( f(X,Y) = X^p - b \) is irreducible in \( K[X] \), so \( K[X]/(f) \) is a field and \( K[X,Y]/(f) \) is a PID. However (3) fails, as \( f_X \equiv f_Y \equiv 0 \) and \( V(f) \) is everywhere singular).

In higher dimensions, Zariski showed that an algebraic variety over a perfect field is nonsingular if and only if its local rings are regular. (The notions “regular” and “integrally closed” are equivalent in the 1-dimensional case). See [Em75] for a streamlined account of Zariski’s work.

\section*{2. Power series}

The localization of the polynomial ring \( \mathbb{C}[X,Y] \) at \( O \) is too small to reflect the topology adequately. For instance, \( Y^2 - X^2(X + 1) \) is irreducible in \( \mathbb{C}[X,Y]_{(X,Y)} \) but factors as \( (Y + X(1 + X)^{\frac{1}{2}})(Y - X(1 + X)^{\frac{1}{2}}) \) in \( \mathbb{C}[X,Y] \), the ring of germs of holomorphic functions at the origin \( O \in \mathbb{C}^2 \). (This is the ring of power series which converge on some neighbourhood of \( O \)). From the algebraic point of view it is natural to pass to the completion of this ring with respect to powers of its maximal ideal, which is the formal power series ring \( \mathbb{C}[[X,Y]] \).

The ring \( \mathbb{C}[[X_1, \ldots, X_n]] \) of formal power series in \( n \) variables is a local domain, with maximal ideal \( M = (X_1, \ldots, X_n) \), residue field \( \mathbb{C} \) and field of fractions \( \mathbb{C}((X_1, \ldots, X_n)) \). It is complete and Hausdorff with respect to the \( M \)-adic topology. In particular, \( \mathbb{C}[[X]] \) is a complete discrete valuation ring, with valuation \( v(f) = \max\{n \mid f \in (X)^n\} = \dim_{\mathbb{C}} \mathbb{C}[[X]]/(f) \), for \( f \neq 0 \). (See Chapter 10 of [AM]).

**Theorem 5.** Let \( I \) be an ideal in \( \mathbb{C}[[X,Y]] \). Then the following are equivalent:

1. \( \dim_{\mathbb{C}} \mathbb{C}[[X,Y]]/I < \infty; \)
2. \( I \) contains a power of \( M = (X,Y); \)
(3) I is not contained in any proper principal ideal.

Proof. If \( \dim_\mathbb{C}[X,Y]/I < \infty \) then \( \mathbb{C}[X,Y]/I \) has a finite composition series whose subquotients are isomorphic to \( \mathbb{C}[X,Y]/M \cong \mathbb{C} \), and so \( \mathbb{C}[X,Y]/I \) is annihilated by a power of \( M \). Hence (1) \( \Rightarrow \) (2).

If \( M^n \leq I \) holds then \( \mathbb{C}[X,Y]/I \) is generated as a \( \mathbb{C} \)-vector space by the images of the monomials of total degree less than \( n \). Moreover, if also \( I \leq (f) \) then \( f \) divides \( X^n \) and \( f \) divides \( Y^n \), so \( f \) is a unit in \( \mathbb{C}[X,Y] \). Hence (2) \( \Rightarrow \) (1) and (3).

If (3) holds then \( I \) is not contained in any proper ideal of the localization \( \mathbb{C}((X))[Y] \), and so we may write 1 as a finite sum \( 1 = \Sigma X^{-n_j}r_j \) for some \( r_j \in I \) and \( n_j \geq 0 \). Clearing denominators, we see that \( X^n \in I \) for \( n \geq n_X \).

Similarly, \( Y^n \in I \) for \( n \geq n_Y \). Hence \( M^n \subset I \) for \( n \geq n_X + n_Y - 1 \), and so (3) \( \Rightarrow \) (2). \( \square \)

If \( f \in \mathbb{C}[X,Y] \) the multiplicity of \( f \) is \( \nu(f) = \min \{ n \mid f \in M^n \} \). It is easily seen that multiplicity is additive \( (\nu(fg) = \nu(f) + \nu(g)) \), positive \( (\nu(f) \geq 0) \), and is 0 if and only if \( f \) is a unit. If \( f \neq 0 \) its initial term is the sum of the terms of total degree \( \nu(f) \), i.e., the lowest nonzero homogeneous part of \( f \).

Corollary. If \( \dim_\mathbb{C}[X,Y]/I = d \) then \( M^d \leq I \). \( \square \)

Theorem 6. The ring \( \mathbb{C}[X,Y] \) is a 2-dimensional noetherian local unique factorization domain.

Proof. The above properties of multiplicity imply easily that every element of \( \mathbb{C}[X,Y] \) has a finite factorization into irreducibles. Let \( f \) be an irreducible element which divides a product \( gh \) in \( R \). If \( f = X \) then it is clearly prime, since \( \mathbb{C}[X,Y]/(X) = \mathbb{C}[Y] \) is a domain. Suppose that \( f \neq X \). Then \( f \) remains irreducible in \( \mathbb{C}((X))[Y] \), and divides \( gh \) there. Since \( \mathbb{C}((X))[Y] \) is a PID, \( f \) must divide one of the factors, \( g \) say. On clearing denominators we see that \( f \) divides \( gX^k \) in \( \mathbb{C}[X,Y] \), for some \( k \geq 0 \). It follows easily that \( f \) divides \( g \) in \( \mathbb{C}[X,Y] \), and so irreducibles are prime.

Let \( I \) be a nonzero ideal in \( \mathbb{C}[X,Y] \), and let \( h \) be the highest common factor of the elements of \( I \). (This is well defined up to units, since \( \mathbb{C}[X,Y] \) is factorial). Then \( I = (h)J \), where \( J \) is contained in no proper principal ideal. Hence \( J \) contains \( M^n \), for some \( n \geq 0 \), by Theorem 5, and so is generated by the monomials of degree \( n \) together with a basis for the finite dimensional...
vector space $J/M^n$. Therefore $I$ is finitely generated and so $\mathbb{C}[[X,Y]]$ is noetherian.

In particular, if $I$ is prime and $0 \neq I \neq M$ then $I$ is principal. Thus $\mathbb{C}[[X,Y]]$ is 2-dimensional. □

If $f, g \in R[X]$ are polynomials there are unique polynomials $q$ and $r$ with $r$ of degree $< n$ such that $f = gq + r$, by polynomial long division. This may be extended to formal power series over complete local rings, by the Weierstrass Division Theorem. (The following argument is due to Gersten [Ge83]).

**Theorem 7.** [WDT] Let $A$ be a complete local ring with maximal ideal $M$, such that $\cap_{n \geq 1}M^n = 0$. If $P, D \in A[[Y]]$ and $D \equiv Y^n$ modulo $MA[[Y]]$ there is a unique pair $q \in A[[Y]]$ and $r \in A[Y]$ of degree $< n$ such that $P = Dq + r$.

**Proof.** Define a metric $d$ on $A$ by $d(a, b) = 2^{-n}$ if $a - b \in M^n$ but $a - b \notin M^{n+1}$ and extend this to a metric on $A[[Y]]$ by $d(\Sigma a_iY^i, \Sigma b_iY^i) = \sup_{k \geq 0} d(a_k, b_k)$. Then $A[[Y]]$ is complete with respect to this metric. Define $A$-linear functions $E, F, T$ from $A[[Y]]$ to itself as follows. If $h \in A[[Y]]$ let $F(h)$ be the polynomial given by the terms of $h$ of degree $< n$ in $Y$, and write $h = E(h)Y^n + F(h)$. Let $T(h) = h + E(P - Dh)$. Then it is easily seen that $d(T(h), T(k)) \leq \frac{1}{2}d(h, k)$ and so $T$ is a contraction. Since $A[[Y]]$ is a complete metric space $T$ has an unique fixed point $q$. Then $E(P - Dq) = 0$, so $r = P - Dq$ is a polynomial of degree $< n$. □

The hypothesis $\cap_{n \geq 1}M^n = 0$ is clearly satisfied in $\mathbb{C}[[X,Y]]$, and holds more generally in any noetherian local ring $A$, by Corollary 10.20 of [AM]. See also the 3rd edition of [La] for a short self-contained proof due to Manin.

The Weierstrass Preparation Theorem is a direct consequence. If $A$ is a complete local ring a polynomial $g \in A[Y]$ of degree $n$ is distinguished if $g = Y^n + g_1Y^{n-1} + \cdots + g_n$, where the coefficients $g_i$ are in the maximal ideal of $A$, for $1 \leq i \leq n$.

**Theorem 8.** [WPT] Let $A$ be a complete local ring with maximal ideal $M$, such that $\cap_{n \geq 1}M^n = 0$. If $f \in A[[Y]]$ has nonzero image $\bar{f}$ in $A/M[[Y]]$ then $f = ug$ for some distinguished polynomial $g \in A[Y]$ and unit $u \in A[[Y]]^\times$, which are uniquely determined by $f$. 

Proof. We may write \( \bar{f} = Y^n \bar{v} \), where \( n \) is maximal and \( \bar{v} \) is a unit in \( A/M[[Y]] \). Let \( v \in A[[Y]] \) have image \( \bar{v} \) in \( A/M[[Y]] \). Then \( v \) is a unit in \( A[[Y]] \). Applying the WDT with \( P = Y^n \) and \( D = v^{-1}f \) gives \( q, r \) such that \( Y^n = qv^{-1}f + r \). On reducing modulo \( M \) we see that \( q \) is a unit and \( r \in MA[[Y]] \). Hence \( u = q^{-1}v \) is a unit and \( g = Y^n - r \) is a distinguished polynomial, such that \( f = ug \). The uniqueness follows from the uniqueness in the WDT. \( \square \)

The above theorems hold also for \( C\{X, Y\} \). In particular, if \( P, D \) and \( f \) are holomorphic and \( D(0, Y) = Y^n \) the power series \( q, r, g \) and \( u \) given by the WDT and WPT are holomorphic. These can be deduced from the formal versions by making suitable convergence estimates (cf \([Br10]\)) but it is more natural to use the Cauchy integral formula.

Samuel has shown that square-free formal power series are equivalent to polynomials, up to a formal change of coordinates.

**Theorem 9.** [Sa56] Let \( f \in M = (X, Y) < C[[X, Y]] \) be square-free. Then there is an automorphism \( \alpha \) of \( C[[X, Y]] \) such that \( \alpha(f) \) is a polynomial.

**Proof.** If \( f \notin M^2 \) this is an immediate consequence of the Inverse Function Theorem for formal power series. For if \( f_X(0) \neq 0 \), say, then \( \beta(X) = f \), \( \beta(Y) = Y \) defines an automorphism of \( C[[X, Y]] \), and \( \beta^{-1}(f) = X \).

Suppose that \( f \in M^2 \), and let \( J = (f_X, f_Y) \). Then \( J \) contains some power of \( M \), by Theorem 16, and so \( M^{2k+1} \leq MJ^2 \) for some \( k \geq 1 \). Let \( P \) be a polynomial in the coset \( f + MJ^2 \). (For instance, we may truncate \( F \) after terms of degree at most \( 2k \).) Then \( f = P - A \) where \( A = a_{11}f_X^2 + (a_{12} + a_{21})f_Xf_Y + a_{22}f_Y^2 \), for some \( a_{ij} \in M \). We shall show that there is an automorphism \( \alpha \) of the form \( \alpha(X) = X + u_{11}f_X + u_{12}f_Y, \alpha(Y) = Y + u_{21}f_X + u_{22}f_Y \) with \( u_{ij} \in M \) and such that \( \alpha(f) = P \). We must solve the equation \( f(x + u \nabla f) - f(x) = A \). The left hand side has the form \( u_{11}(1 + G_{11})f_X^2 + u_{12}(1 + G_{12})f_Xf_Y + u_{21}(1 + G_{21})f_Yf_X + u_{22}(1 + G_{22})f_Y^2 \), where the \( G_{ij} \) are power series in the \( u_{ij} \) with coefficients in \( M \) and with constant term 0. Let \( \theta_{ij}(u) = a_{ij}(1 + G_{ij})^{-1} \). Then \( \theta \) defines a contraction mapping on the set of \( 2 \times 2 \) matrices \( M_2(C[[X, Y]]) \), and its unique fixed point gives a solution. \( \square \)

A similar argument applies in \( C\{X_1, \ldots, X_n\} \). If \( f \) and \( g \) are holomorphic series and \( g = \alpha(f) \) for some formal automorphism \( \alpha \) then they are equivalent.
under a holomorphic change of coordinates $\alpha$ [Ar68].

§3. Puiseux series

Let $f \in M = (X, Y) < \mathbb{C}[X, Y]$ be square-free. Then $A = \mathbb{C}[X, Y]/(f)$ is a 1-dimensional noetherian local ring without nilpotent elements. The uniform topology determined by powers of the maximal ideal is Hausdorff (hence metrizable), by Corollary 10.20 of [AM], and $A$ is complete with respect to this uniform structure. After a linear change of coordinates, if necessary, we may assume that $f$ is a distinguished polynomial of degree $n > 0$ in $Y$, by the WPT.

**Theorem 10.** Let $f \in \mathbb{C}[X, Y]$ be an irreducible distinguished polynomial of degree $n > 0$ in $Y$, and let $\hat{A}$ be the integral closure of $A = \mathbb{C}[X, Y]/(f)$ in its field of fractions $K = \mathbb{C}((X))[Y]/(f)$. Then

1. $A = \mathbb{C}[[t]]$ and $K = \mathbb{C}((t))$ for some $t \in K$;
2. $\hat{A}$ is a free $\mathbb{C}[X]$-module of rank $n$;
3. $(X)\mathbb{C}[t] = (t^n)$;
4. $\dim_{\mathbb{C}} A/A < \infty$.

**Proof.** Parts (1), (2) and (3) follow from Proposition II.3 of [Se']. Since $A$ is also clearly free of rank $n$ over $\mathbb{C}[[X]]$ the quotient $\hat{A}/A$ is a finitely generated torsion module and so is finite dimensional over $\mathbb{C}$. \[\square\]

We shall consider the invariant $\delta_A = \dim_{\mathbb{C}} \hat{A}/A$ in more detail in §5 below. Since the prime ideal $(X)$ becomes an $n^{th}$ power in $\mathbb{C}[[t]]$ the extension is totally ramified. In fact we may assume that $t^n = X$, by the following theorem. Let $\hat{Z} = \lim_{n \to \infty} \mathbb{Z}/n\mathbb{Z}$ be the profinite completion of $\mathbb{Z}$.

**Theorem 11.** The algebraic closure of $\mathbb{C}((X))$ is $\overline{\mathbb{C}((X))} = \bigcup_{n \geq 1} \mathbb{C}((X^{1/n}))$, with Galois group $\hat{\mathbb{Z}}$ acting through multiplication of $X^{1/n}$ by $n^{th}$ roots of unity.

**Proof.** Let $K$ be a finite extension of $\mathbb{C}((X))$, of degree $n$. By Theorem 10 the integral closure of $\mathbb{C}[X]$ in $K$ is $R = \mathbb{C}[[t]]$, and $(X) = (t^n)$ in $R$. Therefore $X = t^nu$, where $u$ is a unit. Since $u$ is a power series with nonzero constant term it has an $n^{th}$ root in $R$, which is also a unit. Then $t_1 = u^{1/n}t$ is another uniformizer for $R$, with $X = t_1^n$. Hence $K/\mathbb{C}((X))$ is a Galois
extension, with cyclic Galois group, acting on $t_1$ via multiplication by roots of unity. The theorem follows easily. \qed

This underlies the method of “Puiseux expansions”. If $f \in \mathbb{C}[[X]][Y]$ is an irreducible monic polynomial of degree $n$ in $Y$ and $\zeta$ is a primitive $n^{th}$ root of unity then $f = \Pi_{i=1}^n(Y - h(\zeta^i X))$, for some $h \in \mathbb{C}[[X^\frac{1}{n}]]$. The fractional power series $h$ is well defined up to the Galois action, and shall be called the \textit{Puiseux series} for $f$. (In the holomorphic context, the algebraic closure of the field of germs of meromorphic functions $\mathbb{C}(X)$ is the union $\cup_{n \geq 1} \mathbb{C}\{X^{\frac{1}{n}}\}$. See Exercise 2.8 of Chapter IV of [Se’]).

\textbf{Lemma 12.} Let $f \in \mathbb{C}[[X, Y]]$ be nonzero, and let $f_\nu$ be its initial term. If $f_\nu = g_m h_n$ where $g_m$ and $h_n$ have no common factor then $f = gh$ for some $g$ and $h \in \mathbb{C}[[X, Y]]$ with initial terms $g_m$ and $h_n$, respectively.

\textbf{Proof.} We construct the homogeneous parts of $g = g_m + g_{m+1} + \ldots$ and $h = h_n + h_{n+1} + \ldots$ recursively. It suffices to show that given $k$ homogeneous of degree $m + n + p$ for some $p > 0$ we can solve $g_m h_{n+p} + g_{m+p} h_n = k$ for $g_{m+p}$ and $h_{n+p}$. On making the substitution $Y = XZ$ and dividing by $X^{m+n+p}$ this reduces to an inhomogeneous equation in 1 variable, which may be solved by the Euclidean algorithm. (Note that there are many possible solutions). \qed

Let $x$ and $y$ denote henceforth the images of $X$ and $Y$ in $A$.

\textbf{Theorem 13.} Let $f$ be irreducible. Then $\tilde{A} \cong \mathbb{C}[[t]]$ and $(x, y)\tilde{A} = t^{\nu(f)}\tilde{A}$.

\textbf{Proof.} The initial term of $f$ is a product of linear terms, since it is a homogeneous polynomial in $\mathbb{C}[X, Y]$. Since $f$ is irreducible the linear factors must all be equal, by Lemma 12, and so the initial term has the form $(dX + eY)^n$, where $n = \nu(f)$ and $d$ and $e$ are not both 0. After a linear change of coordinates we may assume that the initial term is $Y^n$. On applying the WPT we may then assume that $f = Y^n + \Sigma a_i X^i Y^{n-i}$, where $a_i \in (X)\mathbb{C}[[X]]$ for $1 \leq i \leq n$. Since $x = t^n$ in $\tilde{A} \cong \mathbb{C}[[t]]$ and $f(y) = 0$ in $\tilde{A}$ we see easily that $y$ must be in $(t)^{n+1}$. Hence $(x, y)\tilde{A} = t^n\tilde{A} = t^{\nu(f)}\tilde{A}$. \qed

In particular, if $f$ is a distinguished polynomial of degree $n$ in $Y$ and $\nu(f) = n$ then $(x, y)\tilde{A} = x\tilde{A}$. 
Let $h(X^{1/n}) = \Sigma h_k X^{k/n}$ be a Puiseux series for $f$, and let $I = \{ k \mid h_k \neq 0 \}$ be the set of indices corresponding to nonzero coefficients in $h$. Then \( \text{hcf} I \) is relatively prime to $n$, since $|K : \mathbb{C}((X))| = n$. Let $\beta(1) = \min \{ k \in I \mid k \not\equiv n \mathbb{Z} \}$ and write $\beta(1)/n = m_1/n_1$ where $(m_1, n_1) = 1$. If $n_1 = n$ then $(m_1, n_1) = (\beta(1), n)$ is the unique characteristic pair for $f$. If $n_1 < n$ then $n_1$ divides $n$, and there are indices $k \in I$ such that $k/n$ cannot be expressed as a fraction with denominator $n_1$. Let $\beta(2)$ be the first such, and write $n_1 \beta(2)/n = m_2/n_2$, where $(m_2, n_2) = 1$. Continuing in this way, after finitely many steps we obtain characteristic pairs $(m_1, n_1), \ldots, (m_g, n_g)$ such that $m_{i-1} < n_i$ for $1 \leq i \leq g$ and $n_1 \ldots n_g = n$. (Conversely, if a finite sequence $(m_1, n_1), \ldots, (m_g, n_g)$ satisfies $m_{i-1} < n_i$ for $1 \leq i \leq g$ then it is the sequence of characteristic pairs associated to a series $h \in \mathbb{C}((X^{1/n}))$, where $n = n_1 \ldots n_g$. The product of the Galois conjugates of $y-h$ gives a series $f \in \mathbb{C}[[X,Y]]$ corresponding to the Puiseux data. See also [Ku89] and pages 56-58 of [EN]).

These pairs may be interpreted in terms of “higher ramification groups”: let $G = \text{Gal}(K/\mathbb{C}((X))) \cong \mathbb{Z}/n\mathbb{Z}$ and $G_i = \{ \sigma \in G \mid \sigma(y) \equiv y \pmod{(t^i)} \}$. Then $\beta(1) = \max \{ m \mid G_m = G \}$, and if $G_{\beta(i)+1} \neq 1$ then $\beta(i+1) = \max \{ m \mid G_m = G_{\beta(i)+1} \}$. Clearly $\beta(1) < \cdots < \beta(g)$ and $G_{\beta(g)+1} = 1$. The denominators $n_i$ are given by $n_i = |G_{\beta(i)} : G_{\beta(i)+1}|$, and so $n_1 \ldots n_g = |G| = n$. (It may be helpful to contemplate a simple example, such as $Y = X^* + X^* + X^*$, with $n = 12$ and characteristic pairs $(1,3), (3,2), (9,2)$).

Suppose now that $f = \Pi_{i=1}^r f_i$ is square-free, with $r$ irreducible factors. On applying Lemma 11 to each such factor we see that the homogeneous part of $f$ of lowest degree is a product of powers of linear terms. (These correspond to the tangent lines to the irreducible components of $V(f)$ at 0). The Puiseux data for $f$ consists of the Puiseux data for the factors together with the “linking numbers” $\ell_{ij} = \dim_{\mathbb{C}} \mathbb{C}[[X,Y]](f_i, f_j)$.

Let $A(i) = \mathbb{C}[[X,Y]]/(f_i)$ and let $K_i$ be the field of fractions of $A(i)$, for $1 \leq i \leq r$. The total ring of quotients of $A$ is the localization $K = A_S$, where $S = A - \cup_{i=1}^r (f_i)$ is the set of non-zerodivisors in $A$, and $A$ embeds in $K$. Moreover $\mathbb{C}[[X]] - \{ 0 \} \subseteq S$, since $f$ is a distinguished polynomial in $Y$. It follows easily that $K = \mathbb{C}((X)) \otimes_{\mathbb{C}[[X]]} A \cong \Pi K_i$. The integral closure of $A$ in $K$ is $\hat{A} \cong \oplus A(i)$. 
Let \( v_i : A \to \mathbb{Z}_{\geq 0} \) be the composite of projection from \( A \) onto \( A(i) \) with the restriction of the canonical valuation on \( A(i) \cong \mathbb{C}[t] \). Then \( v_i(g) < \infty \) for all \( i \) if and only if \( f \) and \( g \) have no common factors. The singularity semigroup \( S(f) = \{ (v_1(g), \ldots, v_r(g)) \mid g \in A, v_i(G) < \infty \forall i \} \) is a subsemigroup of \( \mathbb{Z}^r_{\geq 0} \), which encodes the Puiseux data for \( f \) [Wa72].

§4. The Milnor number

Let \( R \) be an integral domain and let \( f, g \in R[Y] \). The resultant of \( f \) and \( g \) is the product \( \text{Res}(f, g) = (f_m)^n (g_n)^m \Pi (\alpha_i - \beta_j) = \Pi_{i=1}^{i=m} g(\alpha_i) \), where \( f = \Pi_{i=1}^{i=m} (Y - \alpha_i) \) and \( g = g_n \Pi_{j=1}^{j=n} (Y - \beta_j) \) in some field containing \( R \). Then \( \text{Res}(f, g) \) is in \( R \), depends only on the residue class of \( g \) modulo \( f \) and is 0 if and only if \( f \) and \( g \) have a common factor. Moreover \( \text{Res}(f, g) = (-1)^{mn} \text{Res}(g, f) \), and formation of resultants is compatible with change of coefficient domains. The discriminant of \( f \) is \( \text{disc}(f) = (-1)^{\frac{m+1}{2}} \text{Res}(f, f_Y) \).

**Lemma 14.** Let \( R \) be an integral domain and let \( f \in R[Y] \) be a monic polynomial of degree \( m \). If \( g \in R[Y] \) then \( \text{Res}(f, g) \) is the determinant of the endomorphism of \( R[Y]/(f) \cong R^m \) given by multiplication by \( g \).

**Proof.** We may extend coefficients to a field containing \( R \) in which \( f \) splits into linear factors, and the result is then clear. □

If \( R \) is a PID and \( N \) is a finitely generated torsion \( R \)-module let \( \ell_R(N) \) denote its length. (Note that \( \ell_{\mathbb{C}[[X]]}(N) = \dim_{\mathbb{C}} N \) when \( R = \mathbb{C}[[X]] \)).

**Lemma 15.** Let \( R \) be a PID and \( \theta : R^n \to R^n \) a monomorphism. Then \( \ell_R(R^n/\theta(R^n)) = \ell_R(R/\det(\theta)R) \).

**Proof.** Since \( R \) is a PID we may apply elementary row operations, with composition \( E \) say, to reduce the matrix of \( \theta \) to upper triangular form \( E \theta \). Since \( E \) is an automorphism of \( R^n \) it induces an isomorphism \( R^n/\theta(R^n) \cong R^n/E \theta(R^n) \), and \( \det(E) \) is a unit in \( R \). The result now follows by induction on \( n \). □

If \( f, g \in \mathbb{C}[[X,Y]] \) let \( I(f, g) = \dim_{\mathbb{C}} \mathbb{C}[[X,Y]]/(f, g) \). Then \( I(f, g) < \infty \) if and only if \( f \) and \( g \) have no common factor. Let \( M(f) = I(f, f_Y) \).

**Lemma 16.** Let \( f \in \mathbb{C}[[X]][Y] \) be monic (as a polynomial in \( Y \)). Then
If $f$ and $g$ have no common factor $I(f,g)$ is the highest power of $X$ dividing $Res(f,g)$.

(2) if $f = \prod_{i=1}^{\nu} f_i$ is square-free $M(f) = \Sigma M(f_i) + 2\Sigma_{i<j} I(f_i, f_j)$;

Proof. Let $R = \mathbb{C}[[X]]$. Then $\mathbb{C}[[X]][Y]/(f) \cong R^n$, and so (1) follows from Lemma 15, since $I(f,g) = \dim_{\mathbb{C}} \mathbb{C}[[X]]/(Res(f,g))$, which is the highest power of $X$ dividing $Res(f,g)$.

If $f$ and $g$ are monic we have $Res(fg, (fg)_Y) = Res(fg, f_Y g + fg_Y) = Res(f, f_Y g)Res(g, f g_Y) = Res(f, f_Y)Res(g, f g_Y)$. Hence $M(fg) = M(f) + M(g) + 2I(f, g)$, and (2) follows by a finite induction. \(\square\)

In particular, if $f$ is a distinguished polynomial in $Y$ then $M(f)$ is the highest power of $X$ dividing $disc(f)$.

The power series $f$ is nonsingular if $f_X$ and $f_Y$ do not both vanish at $O$. As in the global case, $f$ is nonsingular if and only if it is irreducible and $f \not\in M^2$, i.e., $\nu(f) = 1$. This in turn is so if and only if $A$ is a discrete valuation ring, and hence $A = \hat{A}$. (See Theorem 4). In general, let $J(f) = (f_X, f_Y)$ be the ideal in $\mathbb{C}[[X,Y]]$ generated by the partial derivatives of $f$. The Milnor number of $f$ is $\mu(f) = \dim_{\mathbb{C}} \mathbb{C}[[X,Y]]/J(f) = I(f_X, f_Y)$. Then $f$ is nonsingular if and only if $J(f) = (1)$, i.e., $\mu(f) = 0$.

**Theorem 17.** Let $f \in M = (X,Y) < \mathbb{C}[[X,Y]]$. Then the following are equivalent:

1. $f$ is square-free;
2. $J(f)$ contains a power of $M$;
3. $(f, f_X, f_Y)$ contains a power of $M$;
4. $\mu(f) < \infty$.

Proof. If (2) does not hold then $J(f) \leq (p)$ for some principal prime $(p) < \mathbb{C}[[X,Y]]$, by Theorem 5. If $(p) = (X)$ it is easily seen that $X^2$ divides $f$. Otherwise we may assume that $p$ is a distinguished polynomial of degree $n > 0$ in $Y$, by the WPT. Suppose first that $p = Y - h(X)$ for some $h \in \mathbb{C}[[X]]$ with $h(0) = 0$. Define a retraction $\rho : \mathbb{C}[[X,Y]] \rightarrow \mathbb{C}[[X]]$ with kernel $(p)$ by $\rho(k) = k(X, h(X))$. Then $\rho(f)_X = \rho(f_X + f_Y h_X) = 0$, so $\rho(f) = f(0,0) = 0$. Hence $f = gp$ for some $g \in \mathbb{C}[[X,Y]]$. Then $f_Y = g_Y p + gp_Y = g_Y p + g$, so $f_Y \in (p)$ implies that $g \in (p)$ and so $f \in (p^2)$. In general, $p$ factors into Galois-conjugate linear terms $p_i = Y - h(\zeta^i X^\frac{1}{r})$, where $h \in \mathbb{C}[[X^\frac{1}{r}]]$. The factors $p_i$
are distinct primes in \( \mathbb{C}[[X^\pm]][Y] \), since \( p \) is irreducible in \( R \). Applying the earlier argument to each \( p_i \), we see that \( f \) is divisible by \( p^2 \), and so \((1) \Rightarrow (2)\).

The implications \((2) \Rightarrow (3) \Rightarrow (1) \) and \((2) \equiv (4) \) are clear. \(\square\)

In particular, if \( f \) is square-free then \( f^n \in J(f) \) for \( n \) sufficiently large. In fact \( f^2 \in J(f) \) for any square-free \( f \), by a difficult result of Briançon and Skoda [BS74, LT81].

If \( f \) is holomorphic the singularity of \( f \) at \( O \) is isolated if and only if \( (f, f_X, f_Y) \) contains a power of the maximal ideal in \( \mathbb{C}[X, Y] \). See Proposition 1.2 of [Lo].

We may compute the Milnor number easily in the following important special case. A polynomial \( f \in \mathbb{C}[X, Y] \) is weighted homogeneous of type \((N; a, b)\) if it is equivariant with respect to the actions of the multiplicative group \( \mathbb{C}^\times \) on \( \mathbb{C}^2 \) and \( \mathbb{C} \) given by \( \lambda(x, y) = (\lambda^a x, \lambda^b y) \) and \( \lambda(z) = \lambda^N z \), for all \( x, y, z \in \mathbb{C} \) and \( \lambda \in \mathbb{C}^\times \), where \( a, b \) and \( N \) are positive integers and \( a \) and \( b \) are relatively prime. In other words, \( f(\lambda^a X, \lambda^b Y) = \lambda^N f(X, Y) \), for all \( \lambda \in \mathbb{C} \). It is easy to see that any formal power series satisfying this equation is a polynomial, and is a sum of terms \( c_j X^j Y^k \) where \( aj + bk = N \).

Differentiating each side of this equation with respect to \( \lambda \) at \( \lambda = 1 \) gives \( Nf = aXf_X + bYf_Y \) and so \( f \in J(f) \). Let \( Z = X^{-\frac{1}{a}} Y \). Then we may write \( f(X, Y) = X^{-\frac{1}{a}} F(Z) \), where \( F(Z) \in \mathbb{C}[Z] \). On considering the factorization of \( F(Z) \) we see that the irreducible factors of \( f \) are of the form \( X \) or \( Y^a - cX^b \) for some \( c \in \mathbb{C} \).

Suppose now that \( f \) is monic in \( Y \) and \( g \) is another weighted homogeneous polynomial, of type \((N'; a', b')\), and \( Res(f, g) \neq 0 \). The roots of \( f \) in \( \mathbb{C}(\!(X)\!) \) are of the form \( \alpha_i = \gamma_i X^{\frac{1}{a_i}} \), where \( \gamma_i \in \mathbb{C} \), and so \( g(\alpha_i) = c_i X^{\frac{1}{a_i}} \), for some \( c_i \in \mathbb{C} \) \{-\{0\}\}. Hence \( I(f, g) = \frac{X^m}{ab} \). In particular, if \( f \) is a square free, weighted homogeneous distinguished polynomial in \( Y \) then \( f_X \) and \( f_Y \) are weighted homogeneous of types \((N - a; a, b)\) and \((N - b; a, b)\), respectively, and so \( \mu(f) = \frac{(N - a)(N - b)}{ab} \).

The Tjurina number \( \tau(f) = \dim_{\mathbb{C}} \mathbb{C}[X, Y]/(f, f_X, f_Y) \) is a closely related invariant. Clearly \( \tau(f) \leq \mu(f) \), and it follows easily from Theorem 17 that \( \tau(f) < \infty \) if and only if \( \mu(f) < \infty \). If \( f \) is weighted homogeneous then \( f \in J(f) \) and so \( \tau(f) = \mu(f) \). These conditions are invariant under formal change of coordinates. The converse is also true: \( f \) is weighted homogeneous
up to change of coordinates if and only if $f \in J(f)$ [Sa71]. (Saito credits [Re68] for the two variable case). The polynomial $g = Y^5 + X^2 Y + X^5$ is a simple example of a formal power series which cannot be so obtained, for $\tau(g) = 10 \neq \mu(g) = 11$. (See pages 94-97 of [Di]).

§5. The Conductor

As observed in the paragraph before Theorem 17, a distinguished polynomial $f$ is nonsingular if and only if $A = \mathcal{A}$, and so $\delta_A = \dim_{\mathbb{C}}(\mathcal{A}/A)$ is a numerical measure of the singularity. It is also the codimension in $A$ of $\mathcal{C} = \{ r \in A \mid r \mathcal{A} \leq A \}$, the conductor of $\mathcal{A}$ into $A$. (This is the largest ideal of $A$ which is also an ideal of $\mathcal{A}$). This was first shown geometrically by Apéry [Ap46] and has been extended in various ways (see [Ba82]). Our exposition has its roots in an observation due to Euler.

Let $S = \mathbb{C}[[X]]$ and let $S_0 = \mathbb{C}(X)$ be the field of fractions of $S$. Let $f = \sum_{j \leq n} f_j Y^j \in S[Y]$ be a square-free monic polynomial of degree $n$ in $Y$. Let $A = S[Y]/(f)$ and let $y$ be the image of $Y$ in $A$. Then $K = S_0 \otimes_S A$ is the total ring of quotients of $A$ and we may define an $S_0$-linear homomorphism $\tau : K \to S_0$ by $\tau(sy^j) = 0$ if $0 \leq j < n - 1$ and $\tau(sy^{n-1}) = s$, for all $s \in S_0$.

**Lemma 18.** Let $k \in K$. Then $k \in A$ if and only if $\tau(ky^j) \in S$, for all $0 \leq j < n$.

**Proof.** The condition is clearly necessary, since $\{y^i \mid 0 \leq i \leq n - 1\}$ is a basis for $A$ as an $S$-module. Assume that it holds, and let $k = \sum_{j \leq n} s_j y^j$ where $s_j \in S_0$ for $0 \leq j < n$. Then $s_{n-1} = \tau(k)$, $s_{n-2} = \tau(y(k - s_{n-1}y^{n-1}))$ and in general $s_{n-i} = \tau(y^{i-1}(k - \sum_{j < i} s_j y^j))$, for all $0 < i < n$. A finite induction on $i$ shows that all the coefficients are in $S$, and so $k \in A$. □

**Theorem 19.** $\dim_{\mathbb{C}} A/\mathcal{C} = \dim_{\mathbb{C}} \mathcal{A}/A$.

**Proof.** Let $D(N) = \text{Hom}_S(N, S_0/S)$, for $N$ any $S$-module of finite length. Then $D(C) = D(S/(X)) \cong \mathbb{C}$. Since every $S$-module of finite length has a finite composition series with subquotients $S/(X) \cong \mathbb{C}$, it follows easily that $\dim_{\mathbb{C}} D(N) = \dim_{\mathbb{C}} N$, for any such $N$. Define a pairing $\lambda : A \times \mathcal{A} \to S_0/S$ by $\lambda(a, \alpha) = [\tau(aa)] \in S_0/S$, for all $a \in A$ and $\alpha \in \mathcal{A}$. As $\lambda$ determines monomorphisms from $A/\mathcal{C}$ to $D(\mathcal{A}/A)$ and from $\mathcal{A}/A$ to $D(A/\mathcal{C})$, by the Lemma, the result follows easily. □
The function \( f = Y^2 - X^3 \) gives a simple but nontrivial and instructive example. We have \( A = \mathbb{C}[[t^2, t^3]] < \hat{A} = \mathbb{C}[[t]] \), so \( \delta_A = 1 \), and \( C = t^2 \hat{A} = (x, y)A \). (Note that \( C \) is not principal as an ideal in \( A \)).

When \( f \) is irreducible \( \tau \) is closely related to the trace from \( K \) to \( S_0 \).

**Lemma 20.** Let \( f \in \mathbb{C}[[X]][Y] \) be irreducible. Then \( \tau(a) = \text{tr}_{K/S_0}(a/f_Y(y)) \).

**Proof.** If \( f \) is irreducible then \( K \cong \mathbb{C}(t) \), by Theorems 10 and 11, and we may identify \( \text{tr}_{K/S_0} \) with the \( C((t)) \)-linear homomorphism from \( \mathbb{C}((t)) \) to \( \mathbb{C}((x)) \) determined by \( \text{tr}(1) = n \) and \( \text{tr}(t^i) = 0 \) for \( 1 \leq i < n \).

Let \( \{y_j \mid 1 \leq j \leq n\} \) be the roots of \( f = f(Y) \) in \( \mathbb{C}((t)) \). Expanding \( \frac{1}{f(Y)} \) by the method of partial fractions gives \( \frac{1}{f(Y)} = \sum \frac{1}{f(y_j)Y - y_j} \). On expanding each side as power series in \( Y \) and comparing coefficients, we see that \( \text{tr}(\frac{Y}{r}) = 0 \) if \( 0 \leq i < n - 1 \) and \( \text{tr}(\frac{Y^{n-1}}{r}) = 1 \). \( \square \)

This argument is essentially due to Euler. (See Lemma III.2 of [Se'].)

We shall show next that \( \delta_A \) may be expressed in terms of \( \mu(f) \) and \( r \) (the number of irreducible factors of \( f \)).

**Lemma 21.** Let \( f \in \mathbb{C}[[X,Y]] \) be irreducible, and let \( A = \mathbb{C}[[X,Y]]/(f) \). Let \( \theta : A^n \rightarrow A^n \) be a monomorphism. Then \( \dim_C A^n/\theta(A^n) = \dim_C \hat{A}^n/\hat{\theta}(\hat{A}^n) \).

**Proof.** The monomorphism \( \theta \) extends to an endomorphism of \( \hat{A}^n \), and gives rise to a commutative diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & A^n & \rightarrow & A^n & \rightarrow & A^n/A^n & \rightarrow & 0 \\
\downarrow & & \phi & & \downarrow & & \downarrow & & \\
0 & \rightarrow & A^n & \rightarrow & \hat{A}^n & \rightarrow & \hat{A}^n/A^n & \rightarrow & 0
\end{array}
\]

The “Snake Lemma” (Proposition 2.10 of [AM]) gives an exact sequence

\[
0 \rightarrow \text{Ker}(\theta) \rightarrow A^n/\theta(A^n) \rightarrow \hat{A}^n/\hat{\theta}(\hat{A}^n) \rightarrow \text{Cok}(\theta) \rightarrow 0,
\]

in which the extreme terms have the same dimension, since \( \hat{A}^n/A^n = (\hat{A}/A)^n \) is finite dimensional, by Theorem 10. The result follows easily. \( \square \)

**Lemma 22.** Let \( f = \Pi_{i=1}^r f_i \) be square-free, with \( r \) irreducible factors. Then

(1) \( \delta_A = \Sigma \delta_{A(i)} + \Sigma_{i<j} I(f_i, f_j) \), where \( A(i) = \mathbb{C}[[X,Y]]/(f_i) \) for \( 1 \leq i \leq r; \)
(2) if $f \in \mathbb{C}[X][Y]$ is irreducible and monic in $Y$ then $M(f) = 2\delta_A + n - 1$.

**Proof.** The natural homomorphism from $A$ to $\oplus A(i)$ is injective, with cokernel $\oplus_{i \leq j}(\mathbb{C}[X,Y]/(f_i, f_j))$, while $A = \oplus A(i)$. Part (1) follows immediately.

Assume now that $f$ is in $\mathbb{C}[X][Y]$ and is irreducible. Then $\bar{A}$ is a domain. Let $tr = tr_{K/S_0}$. If $P$ is an $A$-submodule of $\mathbb{C}((t))$ let $P^* = \{ k \in \mathbb{C}((t)) \mid tr(kp) \in A \forall_p \in P \}$. Now $tr(t^{-n}) = \frac{1}{2}$ and $tr(t^i) \in \mathbb{C}[X]$ if $i > -n$, so $\bar{A} = t^{1-n}\bar{A}$. The $n \times n$ matrix with $(i,j)$ entry $tr(\frac{v^{i+j-2}}{y})$ is 0 above the secondary diagonal, and is 1 along that diagonal. Since $\{ y^i \mid 0 \leq i \leq n - 1 \}$ is a basis for $A$ as a $\mathbb{C}[X]$-module it follows that $A^* = f_Y^{-1}A$.

Hence $c \in C \Leftrightarrow c\bar{A} \leq A \Leftrightarrow cf_Y^{-1}\bar{A} \leq A^* \Leftrightarrow tr(cf_Y^{-1}aa) \in A$ for all $a \in \bar{A}$ and $a \in A \Leftrightarrow cf_Y^{-1} \in \bar{A}^* \Leftrightarrow c \in f_Yt^{1-n}\bar{A}$. Since $C = t^{2\delta_A}\bar{A}$, by Theorem 19, and $M(f) = \dim_C \bar{A}/f_Y\bar{A}$ (the $t$-adic valuation of $f_Y$), by Lemma 21, it follows that $M(f) = 2\delta_A + n - 1$. □

The computation of the conductor is based on Proposition III.11 of [Se'].

The next lemma is from [Ri71].

**Lemma 23.** If $f \in \mathbb{C}[[X]][Y]$ is monic (as a polynomial in $Y$) then $I(f, f_Y) = I(mf + Xf_X, f_Y)$ for all $m \in \mathbb{Z}$.

**Proof.** It shall suffice to prove that $I(f, g) = I(mf + Xf_X, g)$ for each irreducible factor $g$ of $f_Y$. The integral closure of $\mathbb{C}[[X]][Y]/(g)$ in its field of fractions is isomorphic to $\mathbb{C}[[u]]$, by Theorem 10; let $v$ be the associated valuation. Then $I(f, g) = v(f)$ and $I(mf + Xf_X, g) = v(mf + Xf_X)$, as in Lemma 21. Now $mf + Xf_X = X^{1-m}d(X^mf)/dX$ and so $v(mf + Xf_X) = (1-m)v(X) + v(d(X^mf)/dX)$. As $dw/du = dw/dX.dX/du$ it follows that $v(dw/du) = v(dw/du) - v(dX/du) = v(u) - 1 - (v(X) - 1) = v(u) - v(X)$. Therefore $v(d(X^mf)/dX) = (m-1)v(X) + v(f)$ and so $I(mf + Xf_X, g) = v(f) = I(f, g)$. □

Suppose that $f = Y^n + \Sigma a_i X^i Y^{n-i}$ is a distinguished polynomial of degree $n = v(f)$ in $Y$. Let $Y = XZ$ and define $f^\sigma$ by $f(X, ZX) = X^nf^\sigma(X, Z)$. Then $f^\sigma = Z^n + \Sigma a_i Z^{n-i}$ is the strict (quadratic) transform of $f$. Moreover if $f$ is irreducible then so is $f^\sigma$.

Quadratic transform is semi-local, in that it replaces the origin by a projective line, and separates the lines through the origin. If $g$ is irreducible and has initial term $(Y - \lambda X)^m$ (i.e., has tangent line $Y - \lambda X = 0$) then $g^\sigma \equiv (Z - \lambda)^n$
mod (X) and \( g^r \) determines the germ of a curve through \((X, Z) = (0, \lambda)\). It follows that if \( f = \Pi_{i=1}^n f_i \) and \( g = \Pi_{j=1}^{n-1} g_j \) then \( I(f, g) = \nu(f)\nu(g) + \Sigma I(f_i^r, g_j^r) \), (with only factors having common tangents contributing to the double sum). Let \( \tilde{I}(f^r, g^r) \) denote the latter sum.

**Theorem 24.** [Ju] Let \( f = \Pi_{i=1}^n f_i \) be square-free, with \( r \) irreducible factors. Then \( \delta_A = (\mu(f) + 1 - r)/2 \).

**Proof.** We may assume that \( f = Y^n + \Sigma a_i X^i Y^{n-i} \) is a distinguished polynomial of degree \( n = \nu(f) > 1 \) in \( Y \), and (hence) that each factor \( f_i \) is also a distinguished polynomial of degree \( n_i \) in \( Y \), for \( 1 \leq i \leq r \). Differentiating the equation \( X^n f^r(X, Z) = f(X, XZ) \) with respect to \( Z \) gives \( X^n(f^r)_Z = X f_Y \) (so \((f_Y)^r = (f^r)_Z = f_Z^2 \), say) and so \( M(f) = I(f, f_Y) = n(n-1) + \tilde{I}(f^r, f_Z^2) \). Differentiation with respect to \( X \) gives \( nX^{n-1} f^r + X^n(f^r)_X = f_Z + Z f_Y \), and so \( \mu(f) = I(f_X, f_Y) = I(f_X + Y f_Y, f_Y) = (n-1)^2 + \tilde{I}(n f^r + X(f^r)_X + (X-1)Z f_Z^2, f_Z^2) \). (Note that since \( f \) is square-free \( f_Z \neq 0 \) and \( \nu(f_X) = n-1 \). This in turn equals \( (n-1)^2 + \tilde{I}(f^r, f_Z^2) \), by Lemma 23. Hence \( \mu(f) = M(f) + 1 - n \). Now \( M(f) = \Sigma M(f_i) + 2 \Sigma_{i<j} I(f_i, f_j) = 2(\Sigma \delta_{A(i)} + \Sigma_{i<j} I(f_i, f_j)) + (\Sigma n_i) - r = 2\delta_A + n - r \), by Lemmas 16 and 22. Therefore \( \mu(f) = 2\delta_A - r + 1 \) and so \( \delta_A = (\mu(f) + 1 - r)/2 \). □

This argument is taken from [Ri71]; the reference to [Ju] (pages 368-370) is from MR46#5334.

The local invariant \( \delta_A \) also enters the Plücker/Riemann-Roch formula \( g = \binom{d-1}{2} + \Sigma \delta_z \) relating the genus \( g \) and degree \( d \) of a projective plane curve as the correction term for a singularity at \( z \). (See Chapter IV of [Se]). Milnor has given a more topological proof of Theorem 24 for \( f \) a polynomial, using this interpretation of \( \delta_A \) and approximating the projective completion of \( V(f) \) by other related curves. (See Theorem 10.5 of [Mi]).

§6. Resolution of singularities

To resolve the singularities of an algebraic variety \( V \) means to give a non-singular variety \( V' \) and a morphism \( p : V' \to V \) which restricts to an isomorphism over the nonsingular points of \( V \). This issue is usually considered in the context of projective varieties.
If $V(f)$ is an irreducible plane curve let $\tilde{O}_{V(f)}$ be the integral closure of the domain $O_{V(f)}$ in its field of fractions. Then $\tilde{O}_{V(f)}$ is finitely generated as an $O_{V(f)}$-module, and hence as a $\mathbb{C}$-algebra. Hence it is the coordinate ring of a smooth algebraic curve $\tilde{V}$ in a (possibly) higher-dimensional affine space $\mathbb{C}^m$. Maximal ideals of $\tilde{O}_{V(f)}$ restrict to maximal ideals of $O_{V(f)}$, so there is a canonical map from $\tilde{V}$ to $V(f)$. However it may not be possible to find such a “smooth model” which is also a plane curve.

For curves the singular set is finite, and the problem may also be considered locally. As observed above, resolution by quadratic transform is semi-local. Thus after a quadratic transform of $f$ centred at $Y = 0$ we may have to apply further quadratic transforms centred at $Z = p_i$, where $p_i$ is the slope of a tangent to $V(f)$ at 0.

The local approach actually gives more information than normalization (passage to the integral closure). Assume that $f$ is a distinguished polynomial of degree $n > 0$ in $Y$ and let $B = \mathbb{C}[[X]][Z]/(f^\sigma) \cong \Lambda[y/x] \leq \tilde{\Lambda}$, where $f^\sigma(X, Z) = X^{-n}f(X, XZ)$. Now $f_Z^\sigma = X^{1-n}f_V$, by the chain rule, so $M(f^\sigma) = M(f) - n(n-1)$ and $\delta_B = \delta_\Lambda - (n(n-1))/2$. Thus after finitely many such quadratic transforms (possibly followed by linear changes of coordinate) we obtain a nonsingular $f^\sigma$. This process of resolution by quadratic transformations is canonical, and if $f$ is irreducible the sequence of multiplicities $\nu(f) \geq \nu(f^\sigma) \cdots \geq \nu(f^\sigma) = 1$ determines $f$ up to topological equivalence. (It is conventional to truncate the sequence before the first “1”). The analogous invariant in the reducible case is somewhat more complex, as one must keep track of how the branches with common tangents interact. (See pages 502ff of [BK]).

Let $\psi_N : X_N \to X_{N-1} \to \cdots \to X_0 = \text{Spec}(\mathbb{C}[[X,Y]])$ be the sequence of quadratic transforms corresponding to the canonical resolution of $f$, let $E_N = \psi_N^{-1}(0)$ and let $V_N = \psi_N^{-1}(V) - E_N$ be the strict preimage of $V(f)$ in $X_N$. The resolution graph $\Gamma_f$ is the weighted graph with one vertex for each irreducible component of $\psi_N^{-1}(V) = E_N \cup V_N$, an edge joining vertices corresponding to components which meet, and weights $i(e) = 1 + \max\{i \mid e \text{ has image 1 point in } X_i\}$ on each vertex $e$ corresponding to a component of $E_N$. (These weights are the negatives of the self-intersection numbers of the components $e$ in $X_N$). An algorithm for determining the resolution graph for
an irreducible \( f \) from its Puiseux data is given in [CS00]. Conversely, each of the multiplicity sequence, the resolution graph and the extended Puiseux data (i.e., including the linking numbers) determines the other sets of invariants (see pages 512ff of [BK]).

§7. Algebraic links

If \( f \) is holomorphic the pair \((S^3, S^3 \cap f^{-1}(0))\) cut out by the sphere of radius \( \epsilon \) centred at an isolated singular point \( P \) determines a fibred link \( L(f) \) whose ambient isotopy type is independent of \( \epsilon \), for \( \epsilon \) small [Mi]. The topology of such algebraic links is well understood, as \( L(f) \) is an “iterated torus link”, with one component for each branch of \( f \) at \( P \). It is determined up to isotopy (among all such links) by the extended Puiseux data for the factors of \( f \), which provide a template for constructing the link iteratively by forming satellites. (This corresponds to the canonical decomposition of the link exterior along essential tori). The branches are the irreducible factors of \( f \) in the ring of germs of holomorphic functions at \( P \), and the \( i^{th} \) component is determined by the Puiseux expansion of the \( i^{th} \) branch. There is no loss of generality in assuming that \( f \) is a polynomial, by the results of Samuel ([Sa56] - reported in Theorem 9 above) and Artin [Ar68].

In principle, topological invariants of such a link can be computed from the Puiseux data (or, equivalently, from the resolution graph \( \Gamma_f \)), and this has largely been done in [EN]. (In particular, if \( f \) and \( g \) are irreducible and relatively prime then \( \text{lk}(L(f), L(g)) = \text{dim}_{\mathbb{C}}[\mathbb{C}[X, Y]]/(f,g) \), and is strictly positive). The link \( L(f) \) is in fact determined among all such links by its component knots and their linking numbers (see Lemma 7.1 of [Za71]), and hence by its multivariable Alexander polynomial \( \Delta_1(L(f)) \) [Ym84]. The invariant \( \delta_A \) is the unknotting number of \( L(f) \); this is a consequence of the Thom conjecture (see the survey article [BW83]), which has been proven by Kronheimer and Mrowka [KM94].

The (reduced) Alexander polynomial of a fibred link is the characteristic polynomial of the monodromy, acting on the cohomology \( H^1(F; \mathbb{C}) \) of the fibre \( F \). Brieskorn has shown that for an algebraic link \( H^1(F; \mathbb{C}) \) may be identified with the kernel of the topological Gauß-Manin connection over a deleted neighbourhood of 0 in \( \mathbb{C} \), and that the Alexander polynomial may (in
principle) be computed in terms of an algebraically defined Gauß-Manin connection on a relative de Rham cohomology module [Br70]. He used the deep coherence theorem of Grauert to show that the latter modules are finitely generated and that the connection is always regular, and his arguments apply also in the many variable cases. In the following sections we shall use elementary calculations to recover Breiskorn’s results for the case when \( f \) is a weighted homogeneous polynomial. (The Seifert form for an algebraic link may also be related to the local Gauß-Manin connection, via integration of \( C^\infty \) forms [Ba85]. However it is not clear how to translate Barlet’s arguments into terms of local commutative algebra).

\[8. \text{The relative de Rham module and the Gauß-Manin connection}\]

Let \( R = \mathbb{C}[X,Y] \) and let \( f = \sum_{k \leq n} a_k(X)Y^k \) be a square-free distinguished polynomial of degree \( n \) in \( Y \). Let \( f^* : S = \mathbb{C}[[s]] \to R \) be the homomorphism sending \( s \) to \( f \), defined by \( f^*(s) = g \circ f \). Then \( R \cong S[[X]][Y]/(f-s) \) and \( A = \mathbb{C}[[X,Y]]/(f) \cong R/(s) \). Let \( \Omega^1 = RdX \oplus RdY \) be the module of 1-differentials of \( R \) over \( \mathbb{C} \) and let \( \Omega^0 = R \) and \( \Omega^2 = \Omega^1 \wedge \Omega^1 = RdX \wedge dY \) be the nonzero exterior powers of \( \Omega^1 \). (Thus \( \Omega^p \) is the completion of the module of germs of holomorphic \( p \)-forms at \( O \) in \( \mathbb{C}^2 \)). The cochain complex \( \Omega^* \) determined by the exterior derivatives \( d : \Omega^p \to \Omega^{p+1} \) gives a resolution \( 0 \to \mathbb{C} \to R \to \Omega^1 \to \Omega^2 \to 0 \).

Let \( \Omega^p_f = \Omega^p / df \wedge \Omega^{p-1} \). The exterior derivative on \( \Omega^p \) induces \( S \)-linear differentials \( df : \Omega^p_f \to \Omega^{p+1}_f \) (via \( f^* \)) and so we obtain a \( S \)-cochain complex \( \Omega^*_f \). Let \( H^i_f = H^1(\Omega^*_f) \), \( H_f = \Omega^1_f/dR = \Omega^1/dR + Rddf \) and \( H''_f = \Omega^2_f/df \wedge dR \).

Then \( H^i_f \), \( H_f \) and \( H''_f \) are \( S \)-modules, \( H^i_f \leq H_f \) and \( H''_f \cong \text{Cok}(\delta) \), where \( \delta : R \to R \) is the \( S \)-linear derivation given by \( \delta(g) = gXf_Y - gYf_X \), for all \( g \in R \). Since \( \delta \) is \( S \)-linear and \( \text{Im}(\delta) \leq J(f) \) it induces a \( \mathbb{C} \)-linear derivation \( \delta_A : A \to A \), with \( \text{Im}(\delta_A) \leq J_A \), where \( J_A \) is the image of \( J(f) \) in \( A \).

Since \( f_X \) and \( f_Y \) are relatively prime wedge product with \( df \) induces a monomorphism \( \kappa : H' \to H'' \), with cokernel \( \Omega^2_f \cong R/(f_X, f_Y) \). Moreover \( d \) induces a \( \mathbb{C} \)-linear bijection \( d' : H' \to H'' \), with \( d'(H^1_f) = \kappa(H'f) \). (For if \( dh = df \wedge dg \) then \( d(\eta + gdf) = 0 \) and so \( \eta + gdf = dr \) for some \( r \in R \), since \( \Omega^* \) is exact above degree 0). This induces an \( S \)-linear isomorphism \( H'/H^1_f \cong \text{Cok}(\kappa) \cong \Omega^2_f \).
A meromorphic connection on a $\mathbb{C}[[s]]$-module $M$ is a $\mathbb{C}$-linear function $\Delta : M \to M[s^{-1}] = \mathbb{C}((s)) \otimes_{\mathbb{C}[[s]]} M$ such that $\Delta(gm) = g\Delta(m) + (\frac{d}{ds} g)m$ for all $g \in \mathbb{C}[[s]]$ and $m \in M$. Such a connection $\Delta$ is regular if $M[s^{-1}]$ has finite dimension and $s\Delta$ maps a lattice in $M[s^{-1}]$ into itself.

The $\mathbb{C}[[s]]$-module $H^1_J$ supports a natural meromorphic connection, defined as follows. If $\eta \in \Omega^1$ represents a class in $H^1_J$ then $d\eta = df \wedge \psi = \kappa([\psi])$, for some 1-form $\psi$. The class of $\psi$ in $H'$ is well-defined, and $d(f^k \psi) = f^k d\psi + kf^{k-1} df \wedge \psi$ is in $df \wedge \Omega^1 = (f_X, f_Y) dX \wedge dY$, for $k$ large, since $f^k \in (f_X, f_Y)$ for $k$ large, by Theorem 17. Hence $f^k \psi \in H^1_J$, and the function defined by $\nabla([\eta]) = s^{-k}[f^k \psi]$ gives a well defined meromorphic connection on $H^1_J$. This is the local Gauß-Manin connection $\nabla$ associated to $f$.

Brieskorn used the coherence theorem of Grauert to show that the $S$-modules $H$, $H'$ and $H''$ are finitely generated and of rank $\mu(f)$, and that the local Gauß-Manin connection of an isolated singularity is always regular [Br70]. Moreover $H''$ is torsion free as an $S$-module [Se70]. Hence so are $H^1_J$ and $H'$. If we identify $H^1_J$ and $H'$ with submodules of $H^1_J[s^{-1}]$ we have $\nabla([\eta]) = \kappa^{-1} d'([\eta])$, and so $\nabla$ maps $H^1_J$ bijectively onto $H'$.

The apparent contradiction of the injectivity of the local Gauß-Manin connection as just defined with the claim that the cohomology of the Milnor fibre may be identified with the kernel of the topological Gauß-Manin connection may be resolved by interpreting the equation $s \nabla(\eta) = 0$ as a linear system of $1^{st}$ order ODEs and extending coefficients from $S$ to a larger ring $\hat{S}$, to include all possible solutions to such a linear system. Let $\hat{S} = \mathbb{C}((s))[\lambda]$, where $\mathbb{C}((s)) = S[s^{-1}, s^{k}; n > 1][\lambda]$ is the algebraic closure of the field of fractions of $S$, and extend the derivation $\frac{d}{ds}$ so that $\frac{d}{ds}(s^k) = ks^{k-1}$ for all rational exponents $k$ and $\frac{d}{ds}(\lambda) = s^{-1}$. (Thus $\lambda$ corresponds to $\log(s)$). Fix a basis $\{v_1, \ldots, v_\mu\}$ for $H^1_J$ over $S$, and write $s \nabla(\epsilon_i) = \Sigma_{j \leq \mu} s_{ij} v_j$. Let $\epsilon_1, \ldots, \epsilon_\mu \in \hat{S}$. Then

$$s \nabla(\Sigma_{i \leq \mu} \epsilon_i v_i) = \Sigma_{i \leq \mu} ((s \frac{d}{ds} \epsilon_i)v_i + \epsilon_i \Sigma_{j \leq \mu} s_{ij} v_j),$$

and so the equation $s \nabla(\eta) = 0$ corresponds to the linear system

$$s \frac{d}{ds} \epsilon_i + \Sigma_{k \leq \mu} s_{ik} \epsilon_k = 0 \quad (1 \leq i \leq \mu)$$

over $\hat{S}$. The monodromy of this linear system is essentially the cohomological monodromy (with complex coefficients) of the Milnor fibration [Br70].
§9. The weighted homogeneous case

In this section we shall establish the results of Brieskorn and Sebastiani for the special case of weighted homogeneous polynomials.

Theorem 25. Let $f$ be a square-free distinguished polynomial of degree $n$ in $Y$ which is weighted homogeneous of type $(N; a, b)$. Then $H'' \cong S^{\mu(f)}$, where

$$\mu(f) = \frac{(N-a)(N-b)}{ab}.$$ 

Proof. Since $aXf_X + bYf_Y = Nf$ we see that $\delta(Y^j) = -jY^{j-1}f_X$ if $0 < j < n$ and $\delta(X^iY^j) = \frac{ai+bj}{a} X^{i-1}Y^j f_Y - \frac{N}{a} X^{-1}Y^{j-1} f$ if $0 < i < \infty$. On passing to $A = R/(s)$ we get $x^i y^j f_Y = \delta_A(\frac{ai+bj}{a} x^{i-1} y^j)$ and $x^i y^j f_X = \delta_A(\frac{N}{a} x^{-1} y^{j-1})$ for all $i, j \geq 0$. Hence $\text{Im}(\delta_A) = J_A$. Since $f \in J(f)$ we have $R/J(f) \cong A/J_A = A/\text{Im}(\delta_A) \cong R/(sR + \text{Im}(\delta))$. Moreover $sR + \text{Im}(\delta) \leq J(f)$ and so $sR + \text{Im}(\delta) = J(f)$. Let $U = \{ u_i \mid 1 \leq i \leq \mu(f) \} \subset R$ represent a basis for $R/J(f)$ as a $\mathbb{C}$-vector space. We may choose functions $h, k : R \to R$ so that if $g \in R$ then $g = \delta(h(g)) + sk(g) + \Sigma c_i(g)u_i$, for some coefficients $c_i(g) \in \mathbb{C}$. Applying a similar expansion to $k(g)$ and iterating, we conclude that $g = \sigma(g) + \delta(h(g))$, where $\sigma(g) = \Sigma_{m \geq 0} c_i(k^m(g))s^m$ and $\delta(h(g)) = \Sigma_{0 \leq m} s^m h(k^m(g))$, and so $R = SU + \text{Im}(\delta)$. Hence $H'' \cong \text{Cok}(\delta)$ is generated by $U$ as an $S$-module.

The ring $R$ is a free $S[[X]]$-module with basis $\{1, \ldots, Y^{n-1}\}$, and so every element $g \in R$ is uniquely expressible as a sum $g = \Sigma_{0 \leq i} \Sigma_{j=0}^{n-1} g_{i,j} X^i Y^j$, with coefficients in $S$. Since $\delta(1) = 0$ it follows that $\delta(g) = \Sigma \Sigma' g_{i,j} \delta(X^i Y^j)$, where $\Sigma'$ denotes summation over indices $(i,j)$ with $0 \leq i < \infty$, $0 \leq q < n$ and $0 < i + q$. Thus if $H''$ is not free of rank $\mu(f)$ as an $S$-module there is a nontrivial linear relation $\Sigma_{k=1}^{\mu(f)} e_k u_k = \Sigma' g_{i,j} \delta(X^i Y^j)$ in $R$, with coefficients $\{ e_k \}$ and $\{ g_{i,j} \}$ in $S$ and not all divisible by $s$. Since $\Sigma_{k=1}^{\mu(f)} e_k(0)u_k = 0$ in $R/J(f)$ we see that $e_k(0) = 0$ for all $k \leq \mu(f)$. Hence $\Sigma' g_{i,j}(0) \delta_A(x^i y^j) = 0$ in $A$. Now $\delta_A(1) = 0$, $\delta_A(y^j) = -jy^{j-1}fx$ if $0 < j < n$ and $\delta_A(x^i y^j) = \frac{ai+bj}{a} x^{i-1} y^j f_Y$ if $0 < i < \infty$, while $axfx + by fy = 0$. Hence $0 = x(\Sigma' g_{i,j}(0) \delta_A(x^i y^j)) = \Sigma_{j=0}^{n-1} g_{0,j}(0)x \delta(y^j) + \Sigma_{i=1}^{\infty} \Sigma_{j=0}^{n-1} g_{i,j}(0)x \delta(x^i y^j) = (\Sigma' g_{i,j}(0) \frac{ai+bj}{a} x^{i-1} y^j) f_Y$.

But $\{1, \ldots, y^{n-1}\}$ is a basis for $A$ as a free $\mathbb{C}[[x]]$-module, and $f_Y$ is a nonzero divisor in $A$. Therefore the coefficients $g_{i,j}(0)$ are all zero. This contradicts our assumption and so $H'' \cong S^{\mu(f)}$. That $\mu(f) = \frac{(N-a)(N-b)}{ab}$ was shown in §4 above. \(\square\)
Since $f \in (f_X, f_Y)$ we have $f \Omega^2 \leq df \wedge \Omega^1$, and so $H''/sH''$ maps onto $\Omega^2_2 \cong R/J(f)$. Since these $C$-vector spaces each have dimension $\mu(f)$ this epimorphism must be an isomorphism. Hence $sH'' = df \wedge \Omega^1 = \kappa(H')$. Since $H'/H_1 \cong \Omega^2_2$ also it follows that $H_1 = sH'$. Moreover $d(f \psi) \in df \wedge \Omega^1$ for all $\psi \in \Omega^1$ and so $\nabla$ is clearly regular.

Suppose that $f = g f_X + h f_Y$ and let $\alpha = -hdX + gdY$ and $\omega = dX \wedge dY$. Then $\kappa(\alpha) = [df \wedge \alpha] = [f \omega]$. Let $U \subset R$ represent an $S$-basis for $H''$. Then $\{u \alpha \mid u \in U\}$ is a basis for $H'$ and so $\{u f \alpha \mid u \in U\}$ is a basis for $H_1$. It is easily verified that $d(u f \alpha) = (u + (gu)X + (hu)Y)f \omega$ for all $u \in R$. (Note that we may always assume that $U$ is a set of monomials $\{X^i Y^j\}$).

If $f = Y^n + X^m$ we may take $\alpha = \frac{1}{m}(-YdX + XdY)$. Let $\eta_{i,j} = [f X^i Y^j \alpha]$. Then $\{\eta_{i,j} \mid 0 \leq i < m - 1, 0 \leq j < n - 1\}$ is a basis for $H_1$. We see easily that $\nabla(\eta_{i,j}) = [\lambda(i,j)X^i Y^j \alpha] = s^{-1}\lambda(i,j)\eta_{i,j}$, where $\lambda(i,j) = 1 + \frac{i+1}{m} + \frac{j+1}{n}$. The corresponding system of ODEs $\frac{d}{ds}(e_{i,j}) + \lambda(i,j)e_{i,j} = 0$ has basic solutions $e_{i,j} = s^{-\lambda(i,j)}$. As $s$ moves once around $0 \in \mathbb{C}$ these solutions change by $\zeta^{-m(j+1)-n(i+1)}$, where $\zeta = \exp(2\pi i / mn)$. In particular, if $(m, n) = 1$ then as $i, j$ vary in the ranges $0 \leq i < m - 1$ and $0 \leq j < n - 1$ these factors $\zeta^{-m(j+1)-n(i+1)}$ run through the $(mn)^{th}$ roots of unity which are neither $m^{th}$ nor $n^{th}$ roots of unity. Hence the characteristic polynomial of the monodromy is $(t^{mn} - 1)(t - 1)/(t^m - 1)(t^n - 1)$, which is the Alexander polynomial of the $(m, n)$-torus knot.

If $f$ is weighted homogeneous of type $(N; a, b)$ we may take $g = \frac{a}{N} X$ and $h = \frac{b}{N} Y$, to get $d\alpha = \frac{a}{N} dX + \frac{b}{N} dY$. In particular, $d(X^i Y^j f \alpha) = (1 + \frac{a(i+1)}{N} + \frac{b(j+1)}{N})X^i Y^j f \omega$. The monodromy is again diagonalizable and its eigenvalues are roots of unity, and so it has finite order. However it is not immediately obvious how to choose monomials representing a $C$-basis for $R/J(f)$.

Since finite generation of $\text{Cok}(\delta)$ is invariant under change of coordinates and a power series form $f$ is weighted homogeneous up to change of coordinates if and only if $f \in J(f)$ [Re68, Sa71], the above conclusions apply whenever $f$ is a square-free distinguished polynomial in $R$ such that $f \in J(f)$. Can a similar approach be extended to establish finiteness for all square-free distinguished polynomials? For the argument of the first paragraph of Theorem 25 it would suffice to show that $\text{Cok}(\delta_A)$ has finite dimension. (An ad hoc argument handles polynomials of the form $f = Y^n + a_1 Y + a_0$ with $n \geq 3$ and $a_0$ and
$a_1$ nonzero, and with $(n-1)d(0) < nd(1)$, where $d(i)$ is the highest power of $X$ dividing $a_i$, for $i = 0$ or 1.

If $f \notin (f_X, f_Y)$ then establishing regularity of $\nabla$ may be more difficult. For if $A \in \mathbb{R}$ is such that $A_X = Xf_X$ then $d(AdY) = df \wedge (-XdY)$, so $\omega = AdY$ represents a class in $H^1$ with $\nabla(\omega) = -XdY$, but $d(fXdY) = f dX \wedge Y + Xdf \wedge dY$ is not in $df \wedge \Omega^1$.

In general, the eigenvalues of the monodromy are always roots of unity, and the largest Jordan block size is 2. (See [Sc80] for a derivation of this from the Briançon-Skoda result that $f^2 \in J(f)$). The monodromy has finite order if $f$ is irreducible [Le72]. On the other hand the monodromy of $(Y^3 + X^2)(Y^2 + X^3)$ has infinite order and so is not diagonalizable [A'C73].

§10. An hermitean pairing

Barlet defines a finitely generated $\mathbb{C}[[s, \bar{s}]]$-module $\mathcal{M}$ with an involution extending the involution $s \leftrightarrow \bar{s}$ on the coefficient ring and with a meromorphic connection $\partial_s$ such that $\partial_s(s.m) = s.\partial_s m$, for all $m \in M$. He then uses integration of representative compactly supported $\mathcal{C}^1$ forms to define a pairing $\hat{H}$ on $H'$ with values in $\mathcal{M}$ which is hermitean and horizontal, in the following senses:

(i) it is $S$-linear in the first variable;
(ii) $\hat{H}(\alpha, \beta) = \widehat{H}(\beta, \alpha)$, for all $\alpha, \beta \in H'$;
(iii) (horizontal) $s^k\partial_s \hat{H}(\alpha, \beta) = \hat{H}(s^k \nabla \alpha, \beta)$, for all $\alpha, \beta \in H'$ and $k >> 0$.

He shows that $(H', \nabla, \hat{H})$ is essentially equivalent to the intersection form on $H_1(F; \mathbb{C})$ together with the monodromy automorphism $h_C$, after localization away from the 1-eigenspace of the monodromy [Ba85]. The 1-eigenspace is trivial for any knot, and has dimension $r - 1$ if $f$ has $r$ irreducible factors [Du75]. Barlet’s argument applies in all dimensions and does not require a resolution of the singularity in its application. On the other hand, it is not clear how to translate it into terms of local commutative algebra.

Acknowledgment. I would like to acknowledge the support of Grey College and the Department of Mathematical Sciences at the University of Durham (through a Grey College Mathematics Fellowship) while revising §5 and writing the final three sections of this paper.
References


[Em75] Emerson, J.D. Simple points on an affine algebraic variety, Amer. Math. Monthly 82 (1975), 132-147.


