THE DIRICHLET BOUNDARY VALUE PROBLEM FOR REAL SOLUTIONS OF THE FIRST PAINLEVÉ EQUATION ON SEGMENTS IN NON-POSITIVE SEMI-AXIS

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Abstract. We develop a qualitative theory for real solutions of the equation \( y'' = 6y^2 - x \). In this work a restriction \( x \leq 0 \) is assumed. An important ingredient of our theory is the introduction of several new transcendental functions of one, two, and three variables that describe different properties of the solutions. In particular, the results obtained allow us to completely analyse the Dirichlet boundary value problem \( y(a) = y_0 \), \( y(b) = y_0 \) for \( a < b \leq 0 \).

1. Introduction

1.1. In 1898, in the framework of the theory of functions of one complex variable and the analytic theory of ODEs, P. Painlevé [15] introduced the equation

\[
y'' = 6y^2 - x,
\]

now called the first Painlevé equation. The way it was introduced suggested the following two main directions of research:

(1) The proof of the transcendency of the solutions of Equation (1.1), i.e., that they are actually “new” functions that cannot be expressed (within a certain class of operations) in terms of already known functions;

(2) The study of solutions as the functions of complex variable \( x \). This concerned mainly the singularity structure of its solutions, that is: (i) the proof of the Painlevé property, i.e., all solutions are meromorphic functions in \( \mathbb{C} \) possessing second order poles of strength unity; (ii) the distribution of the poles; and (iii) the structure of the essential singularity at the infinity, i.e., asymptotic behaviour of the solutions in a neighbourhood of \( x = \infty \).

Both directions were substantially worked out by Painlevé. The singularity structure and asymptotic behaviour in a neighbourhood of infinity is due to Boutroux [3]. Contemporary researchers actually have to make considerable efforts in clarifying the many statements made by Painlevé and Boutroux and putting them on a solid mathematical foundation. We do not review here the subsequent modern studies, this requires a special focus and considerable space, whilst they do not concern directly this work. To complete the general picture of the “complex theory”, the connection formulae, which allow to connect asymptotics as \( x \to \infty \) for...
1.2. Real solutions of Equation (1.1) appear in applications. One of the sources is similarity reductions of integrable PDEs. In particular, there are such reductions for the famous Korteweg de Vries, Boussinesq, and Kadomtsev-Petviashvili equations (see [1]). To study related similarity regimes a knowledge of the behaviour of real solutions for finite values of $x$ is important.

There is another, probably even more important source of applications of equation (1.1) viz. in asymptotic analysis of nonlinear PDEs, ODEs, and difference equations. This concerns both integrable and non-integrable situations. In this analysis Equation (1.1) serves as a model equation in the description of asymptotics in transition layers and caustic-type domains [4, 12], so that particular solutions of Equation (1.1) at some finite point $x$ appears in the corresponding asymptotic formulae. The point $x$ in such formulae serves as a parameter describing the domain of validity of these asymptotics. To further use these asymptotics, say, in the techniques related with the matching of asymptotic expansions, the knowledge of behaviour of real solutions of Equation (1.1) for finite $x$ is important.

1.3. Despite the growing importance of the real solutions for applications, there are only a few papers about them in the literature. Some information concerning qualitative behaviour of real solutions of Equation (1.1) in the finite domain can be found in the following works [2, 5, 6, 7, 8, 14, 18]. Among them the main emphasis in [6, 18] is made on asymptotic analysis of solutions for $x < 0$ and the only paper [2] concerns the solutions for $x > 0$.

The major problem involved here is that the standard methods of qualitative analysis for ODEs applicable to Equation (1.1), based on the maximum principle [17] or equivalent statements, work only for solutions whose graphs in $(x, y)$-plane are located in the upper half plane $y > 0$, i.e., positive solutions. At the same time integral curves corresponding to the most number of interesting solutions cross the $x$-axis and than return back to the upper half-plane. Therefore, whilst the analysis in the upper half plane is almost trivial, still little is known about the global behaviour of the integral curves of Equation (1.1). Another interesting problem, how to connect asymptotic description of the solutions, which is now well known for all solutions complex and real, with their Cauchy initial values, even on the qualitative level, still remains unsolved.

We address these problems in our series of works on the qualitative studies of Equation (1.1). The first work [8] is devoted to the most famous Boutroux tritronquée solution of Equation (1.1), which very often appears in different applications. This is the second work, where we develop the theory of the integral curves for $x \leq 0$ and our third work [9] provides a qualitative description of the integral curves for $x \geq 0$ and, in particular, of the Cauchy initial value problem for this semi-axis. The works [5, 6, 7, 8, 14, 18] concerns real solutions on the non-negative semi-axis. We give their overview in our [9]. The only work devoted to the qualitative studies of solutions of Equation (1.1) on the non-positive semi-axis, we were able to find, is [2]. The results of the last work are reviewed in Subsection 1.5 below, along with a description of our work.
1.4. Before explaining our results it is important to stress that the word \textit{solution} is used throughout this paper to denote a real-valued solution of Equation (1.1) which is defined in the \textit{interval of existence}, i.e., the largest connected open domain in $\mathbb{R}$ where the solution has no poles. Moreover, solutions that have different intervals of existence are referred to as different solutions, despite the fact that some of them could be analytic continuations of each other in the complex plane $x$, i.e., represent the same meromorphic solution from the standpoint of analytic theory of ODEs. In few places where we mean solutions in the latter sense we call them \textit{complex solutions}. Note that this convention is different from the one in our previous work [8], where by real solutions we meant those complex solutions whose imaginary part vanishes on the real axis. Real solutions in the last sense have an infinite number of poles on the real axis.

1.5. In our previous work we remarked (cf. [8], Remark 2) that the intervals of existence of real solutions, whose closure contains $x = 0$, are uniformly bounded on the left. In § 2 we prove the uniform lower boundedness of intervals of existence of solutions that are regular at any point $x_0 \leq 0$, i.e., for any point $x_0$ there exists a minimal finite interval $(X(x_0), x_0)$, such that all solutions regular at $x_0$ have a pole in this interval. Further, for brevity, we call this property \textit{the property of uniform boundedness}. This is a specific property of the real solutions; it is not valid for the restriction of general complex solutions on the real axis. To see it, we recall the special solutions, which Boutroux called “intégrales tronquées” (see [3, 8]). The restrictions on the real axis have intervals of existence which contain the whole negative semi-axis. This property is not valid for complex solutions even if we exclude the intégrales tronquées as there are complex solutions that approximate them on the intervals of arbitrary length.

The property of uniform boundedness has not been observed before and does not follow from the results known about the real solutions in the literature. It is known that asymptotics as $x \to -\infty$ of all real solutions, if we understand them in the sense of analytic continuation, are given by a Boutroux type asymptotic formula in terms of the Weierstrass $\wp$-function with fixed period parallelogram. One can deduce from this that all such solutions have an infinite number of poles on the real axis and find their asymptotic distribution [3, 10, 11]. However, the property of uniform boundedness cannot be derived from the asymptotic results since, there is no universal value $X$ such that for all $x < X$ the correction term to the asymptotic formula would have a uniformly (with respect to all real solutions) small estimate. If we suppose that such $X$ exists, we immediately arrive at a contradiction by taking for any $x_0 < X$ initial data, such that the corresponding solution does not have the universal behaviour prescribed by the Boutroux asymptotic formulae in the neighbourhood of $x_0$, though, of course, for sufficiently large $x$ this solution fits the Boutroux asymptotic regime. The existence of such data is easy to observe: if we assume the uniform estimate of the asymptotic correction term, we can estimate the initial data of the Weierstrass $\wp$-function in terms of the initial data for the Equation (1.1). The latter initial data can be taken arbitrary, whilst the data for the $\wp$-function are related via the Weierstrass differential equation of the first order with fixed invariants.

It is worth mentioning that Bartashevich [2] was very close to formulation of the property of uniform boundedness. He observed a partial uniformity with respect
to the initial slope of the solutions. More precisely, the property he noticed can be formulated as follows. Let \((a, b)\) with \(a < 0\) be the interval of existence of solution \(y(x)\) of Equation (1.1) with initial data: \(y(x_0) = y_0\) and \(y'(x_0) = y_1\) at \(x_0 < 0\). Then \(X(x_0, y_0) \equiv \inf_{y_1} \{a(x_0, y_0, y_1)\}\) is finite. However, the proof given in [2] is incomplete, as it addresses only the simplest case, in our notation, \(y(x) > y_0 > 0\) for \(x < x_0\) or, equivalently, \(y_1 \leq 0\). Moreover, his estimate is divergent as \(y_0 \rightarrow +0\).

To prove and further study the property of uniform boundedness we introduce in § 2 three special functions: \(X(x_0), X_-(x_0),\) and \(X_{\min}(x_0)\). We not only prove the lower boundedness of these functions but also find the first nontrivial term of their asymptotic expansion as \(x_0 \rightarrow -\infty\). This term for the function \(X(x_0)\) looks very similar to the leading term of asymptotics of the difference of two neighbouring poles (the pole spacing) for real solutions that is governed by the Boutroux asymptotic regime. This leading term of the pole spacing is universal for all real solutions. However, a more precise comparison (to appear elsewhere) shows that asymptotics of the pole spacing has a smaller coefficient than that for \(X(x_0) - x_0\), i.e., \(2C\) in Equation (2.20). So that as \(x_0 \rightarrow -\infty\) the asymptotic spacing of the poles is not the maximal possible on the real axis.

1.6. The central technical idea of the paper is to reparameterize solutions, \(y(x)\), which are originally considered as the functions of initial data, \(y_0 = y(x_0)\) and \(y_1 = y'(x_0)\), i.e., the functions \(y(x) = y(x; x_0, y_0, y_1)\) (or \(y(x) = y(x; x_0, c)\), where \(c\) is the so called pole parameter, if \(x_0\) is a pole), in terms, as we call it, the level parameterization: \(y(x) = y(x; x_0, y_0, y_l)\) (or, respectively, \(y(x) = y(x; x_0, y_l)\)), where \(y_l\) is the minimum value of the corresponding solution. Most of the results in Section 3 aim to establish a possibility of this reparameterization and different properties of the functions \(y_l = y_l(x_0, y_0, y_l)\) and \(c = c(x_0, y_l)\) defining it. In particular, we found that \(c = c(x_0, y_l)\) is a strictly monotonically increasing function of the second argument. This property can be viewed as a “visualization” of the parameter \(c\) which is, in analytical definition, hidden being the fourth coefficient of the Laurent expansion at \(x_0\).

Our theory is essentially real-valued. Even where we prove the smoothness of our reparameterization, we substantially exploit specific properties of real solutions. Unexpectedly, one pure analytic fact plays an important role in our study of the geometry of the integral curves. This is existence of an analytic mapping of the parameters \((x_0, c_0)\) defining Laurent expansion of the general complex solution at \(x_0\) to the parameters \((x_1, c_1)\) of Laurent expansion of the same solution at \(x_1\). This mapping was actually discussed by Boutroux in his landmark work [3], where he made deep comments about this mapping lying in the heart of the Painlevé method.\(^1\) In particular, he indicated that an attempt to understand the structure of its singularities can viewed as the motivation for his studies. Boutroux, actually, did not give a definition of this mapping, possibly, considering it as self-evident. We give an accurate definition of the mapping in Proposition 3.4 of Section 3. It shows that existence of such mapping relies upon a specific dependence of the Laurent expansion on parameters \(x_0\) and \(c_0\), so that for equations of a non-Painlevé type

\(^1\)In Boutroux’s own words (p.261 of [3]): “C’est là un fait gros de conséquences: en approfondissant l’étude de la fonction \(X_1(X_0, C)\) et des fonctions connexes, il semble que nous touchions au cœur des nouveaux êtres analytiques introduits dans la Science par M. Painlevé.” Here the notation \(X_1(X_0, C)\) coincides with our \(x_1(x_0, c_0)\).
there could be some situations, where this mapping does not exist, or has some unusual properties.

1.7. One of the basic technical tools that make it possible to study the function $X(x_0)$ and $X_-(x_0)$ as well as to establish many properties of the integral curves is Lemma 4.3 and its two limiting cases Lemmas 4.1 and 4.2. These Lemmas concern the behaviour of the integral curves to the left of their minima. We provide slightly different proofs for the last two lemmas in Section 4, both due to their importance and having in mind possible generalizations. In Sections 4 and 5 we explore the opportunities suggested by these Lemmas to study the functions $X(x_0)$ and, correspondingly, $X_-(x_0)$. In Section 5 we actually study the function inverse to $X_-(x_0)$ since we considering there solutions as being initially defined at the left bound of their interval of existence.

1.8. Another very important technical tool, the Moore–Nehari Lemma 6.1, is invoked in Section 6. This Lemma concerns the number of intersections of the integral curves. It is convenient for us to call two solutions having a common pole, solutions intersecting at the point at infinity. In particular, we prove that the Moore–Nehari lemma can be generalized to include the points of intersection at infinity. We call this generalization of Moore–Nehari lemma Projective Lemma 6.1.

The latter Lemma allows us to prove the important statement that for any $x_0$ the solution which has the maximal interval of existence $(X(x_0), x_0)$ is unique. Another interesting property that we establish in Section 6 is that the integral curves can be viewed as the lines of a model of geometry where: any two lines can intersect only at two, one, or zero points. Moreover, the usual duality principle of projective geometry, with the interchange of lines and points, is valid: any two points can belong only to two, one, or zero lines.

The latter result can be reformulated in terms of the Dirichlet boundary value problem; we pursue this in Section 7. To make such a reformulation we introduce and study an auxiliary function, $Z = Z(x_0, y_0, y^0)$. This function define the largest segment $[Z, x_0]$ where the boundary value problem for Equation (1.1), $y(x_0) = y_0$ and $y(Z) = y^0$ has a solution. We prove that the problem with the same boundary values on the segments $[z, x_0]$ have: two solutions if $z \in (Z, x_0)$, one solution if $z = Z$, and no solutions if $z < Z$.

The function $Z(x_0, y_0, y^0)$ for finite values $y_0$ and $y^0$ can be calculated numerically with MAPLE 8 code by using the dsolve procedure for boundary value problems. However, it is important to know an actual number of the solutions as this procedure, in case of existence, gives always only one solution.

1.9. We provide a numerical illustration of some results obtained in § 2–5 in the last Section 8. This illustration requires rather precise calculations as some data for different solutions are very close.

In this Section we also formulate four conjectures. The first one concerns an interesting property of approximate symmetry for a function $f(x_0, y_0, y_l)$ describing initial slope of solutions at $x_0$ in terms of their initial, $y_0$, and the minimum, $y_l$, values. The other three are about uniqueness and behaviour of special solutions corresponding to the functions $X_{\text{min}}(x_0)$ and $\Xi_{\text{min}}(x_0)$. 
1.10. In our next work [9] we, in particular, consider the continuation of the functions \(X_{\min}(x_0), X_-(x_0),\) and \(X(x_0)\) to the whole real axis. Some interesting further questions, apart of the conjectures mentioned above, concerns further terms in the asymptotic expansions of these functions derived in Section 2. For example, the leading terms of asymptotics for the functions \(X_{\min}(x_0)\) and \(X_-(x_0)\) coincide, however, the next terms should be different. Another interesting further question might be about analytic continuation of these functions to the complex plane. The natural question about the Neumann and mixed Neumann–Dirichlet boundary value problems on the non-positive semi-axis is closely related with the uniqueness mentioned in the last sentence of Subsection 1.9.

An interesting development would be a “continuation” of the theory developed here to a wider class of second order ODEs.

2. Uniform Bound on Intervals of Existence

**Proposition 2.1.** The interval of existence of any solution of Equation (1.1) is bounded below.

**Proof.** Consider solutions whose intervals of existence have non-empty intersection with the negative semi-axis. Change variables to \(t = -x > 0, \ u(t) = y(x),\) then Equation (1.1) becomes

\[
(2.1) \quad u''(t) = 6u(t)^2 + t.
\]

Assume there is no pole for \(t > 0.\) Since \(u''(t) > t,\) there exists a point \(t_0\) such that for \(t > t_0, \ u(t)\) is monotonically increasing to \(+\infty.\) We assume that \(t_0\) is chosen such that \(u_0 = u(t_0) > 0, \ u_1 = u'(t_0) > 0.\)

We have \(u''(t) > 6u(t)^2.\) Integration gives \(u'(t)^2/2 > 2u(t)^3 + N(u_0, u_1),\) where \(N(u_0, u_1) = u_1^2/2 - 2u_0^3.\) Choose now \(t_1 \geq t_0\) such that for all \(t \geq t_1, \ 3u(t)^3/2 + N(u_0, u_1) > 0.\) Hence \(u'(t)^2 > u(t)^3\) for \(t > t_1.\)

Integration gives

\[-\frac{1}{\sqrt{u(t)}} + \frac{1}{\sqrt{u(t_1)}} > \frac{t - t_1}{2},\]

which is a contradiction. \(\Box\)

**Remark 2.1.** The proof of Proposition 2.1 shows that every solution with a positive derivative at some point \(x_0 \leq 0\) has a unique minimum, \(x_{\min},\) achieved to the left of \(x_0.\)

**Definition 2.1.** Let \(x_0 \leq 0,\) define

\[X_{\min}(x_0) \equiv \inf_{y_0 \in \mathbb{R}, y_1 \geq 0} x_{\min},\]

where \(x_{\min} = x_{\min}(x_0) < x_0\) is the minimum of the solution \(y(x)\) with the initial values: \(y(x_0) = y_0\) and \(y'(x_0) = y_1 > 0.\)

**Lemma 2.1.** The function \(X_{\min}(x_0)\) is finite for all \(x_0 \leq 0.\)

**Proof.** Without loss of generality we can assume that \(x_0 < 0: \) the boundedness of \(X_{\min}(0)\) follows from the boundedness of \(X_{\min}(x_0)\) for \(x_0 \neq 0.\) Actually, for any \(\varepsilon > 0,\) either a solution with initial data given at \(x_0 = 0\) blows up in the semi-segment \([-\varepsilon, 0),\) or its minimum is lower-bounded by \(X_{\min}(-\varepsilon) \leq X_{\min}(0) < 0.\) In fact, as \(\varepsilon \to 0,\) by continuity, \(X_{\min}(-\varepsilon) \to X_{\min}(0),\) therefore \(X_{\min}(0) = \sup \{X_{\min}(-\varepsilon)\}.\)
We again use notation (2.1) for Equation (1.1). Denote \( t_0 = -x_0 > 0, \) \( t_{\min} = -x_{\min}, \) \( u(t_0) = y_0 \equiv u_0, \) and \( u'(t_0) = -y_1 \equiv u_1 < 0. \) Consider now Equation (2.1) in the segment \([t_0, t_{\min}]\). Multiplying it by \( u'(t) \), integrating then from \( t_0 \) to \( t \), and solving for \( u'(t) \), we obtain,

\[
-u'(t) = \sqrt{4u^3 + 2N(u_0, u_1) + 2 \int_{t_0}^t tu'(t)dt},
\]

where \( N(u_0, u_1) = u_1^2/2 - 2u_0^3 \) and the positive branch of the square root is assumed. Putting \( t = t_{\min} \) in Equation (2.2) we get,

\[
0 = 4u_{\min}^3 + 2N(u_0, u_1) + 2 \int_{t_0}^{t_{\min}} tu'(t)dt.
\]

Subtracting now the right-hand side of Equation (2.3) from the expression under the square root in Equation (2.2), dividing both sides of the latter equation by this square root, and integrating from \( t_0 \) to \( t \) we arrive at

\[
t - t_0 = \int_{u(t_0)}^{u(t)} \frac{du}{\sqrt{4(u^3 - v_{\min}^3) + 2 \int_{v_{\min}}^{u} t(\tilde{u})d\tilde{u}}},
\]

Note that \( u(t) \) is monotonic on \([t_0, t_{\min}]\), therefore its inverse, \( t(u) \), is properly defined. Now, substituting \( t = t_{\min} \) into Equation (2.4), noting that \( t(\tilde{u}) \geq t_0 \), and changing variables to \( u = \sqrt{t_0}v \), we get the following estimate

\[
t_{\min} - t_0 < \frac{I(v_0, v_{\min})}{t_0^{1/4}},
\]

\[
I(v_0, v_{\min}) \equiv \int_{v_{\min}}^{v_0} \frac{dv}{\sqrt{4(v^3 - v_{\min}^3) + 2(v - v_{\min})}},
\]

where, \( v_{\min} = u_{\min}/\sqrt{t_0} \) and \( v_0 = u_0/\sqrt{t_0}. \)

The function \( I(v_0, v_{\min}) \) is bounded for all values of \( v_0 \in \mathbb{R} \) and \( v_{\min} < v_0. \) Actually, \( I(v_0, v_{\min}) < I(+\infty, v_{\min}) \). The function \( I(+\infty, v_{\min}) \) is a continuous function of \( v_{\min} \in \mathbb{R} \) which vanishes as \( v_{\min} \to \pm \infty: \)

\[
|v_{\min}| > \epsilon > 0, \quad I(v_0, v_{\min}) < \frac{I_\nu}{2\sqrt{|v_{\min}|}}, \quad \nu = \text{sign} \{v_{\min}\}1,
\]

where

\[
I_\nu := \int_\nu^{+\infty} \frac{dw}{\sqrt{w^4 - w^3}} = 2\int_0^{+\infty} \frac{dv}{\sqrt{(v^4 + 3v^2 + 3v^2)}}.
\]

Thus, \( \sup_{v_{\min} < v_0 \in \mathbb{R}} I(v_0, v_{\min}) \) exists and

\[
\sup_{v_{\min} < v_0 \in \mathbb{R}} I(v_0, v_{\min}) = \max_{v_{\min} \in \mathbb{R}} I(+\infty, v_{\min}),
\]

where the maximum is achieved at some finite value of \( v_{\min}. \) Under the change of variables, \( \sqrt{v - v_{\min}} = w \) one finds that

\[
I(+\infty, v_{\min}) = \int_0^{+\infty} \frac{dw}{\sqrt{w^4 + 3v_{\min}w^2 + 3v_{\min}^2 + 1/2}},
\]
so that it is clear that the maximum is achieved at a negative value of $v_{\min}$. Now, introducing the constant 

\begin{equation}
C = \max_{v_{\min} < 0} \int_0^{+\infty} \frac{dw}{\sqrt{w^4 + 3v_{\min}w^2 + 3v_{\min}^2 + 1/2}}
\end{equation}

we get (cf. (2.5)) the estimate 

\begin{equation}
x_0 - \frac{C}{|x_0|^{1/4}} < X_{\min}(x_0) < x_0 < 0.
\end{equation}

\begin{remark}
We can further specify a location of the maximum, $v_{\max}^\prime$, in Equation (2.8). Indeed, making the change of variables: $w = \sqrt{-v_{\min}u}$, $2x = 1/v_{\min}^2$, one rewrites Equation (2.8): $C = \max_{x > 0} (2x)^{1/4} \int_0^{+\infty} \frac{du}{\sqrt{u^4 - 3u^2 + 3 + x}}$. Now, differentiating the function under the max-sign with respect to $x$ and combining all the resulting terms under one integral, one observes that the integrand is non-negative for $x \leq 3/4$. Thus, all extrema of the integral in Equation (2.8) are located in the interval, $v_{\min} \in (-\sqrt{2/3}, 0) \subset (-0.8165, 0)$. Numerical calculations (with MAPLE 8) show that there is only one extremum, the global maximum, which is achieved at 

$v_{\min}^\max = -0.2260038763530209\ldots$, $x_{\max} = \frac{1}{2(v_{\min}^\max)^2} = 9.78899773742578347\ldots$.

This many (18) decimal digits allows us to calculate 36 decimal digits of $C$: 

\begin{equation} C = 2.32470720434237566413065947435242998\ldots. \end{equation}

Our attention to the precision of calculation of $C$ is clarified by Corollary 2.1.

\begin{remark}
For small $-1 \leq x_0 < 0$ the estimate for $X_{\min}(x_0)$ in Equation (2.9) becomes too rough. The reason lies in the rough approximation we made of the integral inside the square root in Equation (2.4). For these values of $x_0$, we can improve the estimate: $-1 - C < X_{\min}(x_0) < x_0$. The improved estimate is also valid for $x_0 = 0$. This follows by similar arguments as in the first paragraph of the proof of Lemma 2.1. The numerical value of $C$ (2.10) allows us to make a slight improvement of the above small-$x_0$-estimate. Actually, the function $x_0 - C/|x_0|^{1/4}$ has the minimum $\hat{x}_0 = -(C/4)^{1/4} = -0.64780846\ldots \in [-1, 0]$. Thus for $x_0 \leq \hat{x}_0$ Inequality (2.9) still delivers the best available estimate, while for $x \in [\hat{x}_0, 0]$ the better estimate is $-5(C/4)^{1/5} < X_{\min}(x_0) < x_0$. However, this is really a minor improvement as the numerical values $5(C/4)^{1/5} = 3.23904230\ldots$ and $1 + C = 3.32470720\ldots$ are very close.

\begin{corollary}
For $x_0 \leq 0$, denote by $\eta(x_0)$ the unique positive root of the equation $\eta^5 - |x_0|\eta = C$. Then, 

\begin{equation} X_{\min}(x_0) < -\eta(x_0)^4 = x_0 - \frac{C}{\eta(x_0)}. \end{equation}

For $x_0 < 0$, 

\begin{equation} x_0 - \frac{C}{|x_0|^{1/4}} < X_{\min}(x_0) < x_0 - \frac{C}{||x_0| + C/|x_0|^{1/4}|^{1/4}}. \end{equation}

\end{corollary}
In particular, as \( x_0 \to -\infty \),
\[
X_{\min}(x_0) = x_0 - \frac{C}{|x_0|^{1/4}} + O\left(\frac{1}{|x_0|^{3/2}}\right),
\]
where \( 0 < O\left(\frac{1}{|x_0|^{3/2}}\right) < \frac{C^2}{4|x_0|^{3/2}} \) for \( |x_0| > C^{4/5} \).

**Proof.** We turn back to the notation of Lemma 2.1 and consider Equation (2.4). However, now we estimate the function \( t(\hat{u}) \) from above \( t(\hat{u}) < t_{\min} \) and arrive at the following estimate,
\[
t_{\min} - t_0 > \frac{l(v_0, u_{\min})}{t_{\min}^{1/4}},
\]
where now, \( u_{\min} = v_{\min}/\sqrt{t_{\min}} \) and \( v_0 = u_0/\sqrt{t_{\min}} \). The key point now is that there exists a solution of Equation (1.1) which corresponds to the value \( v_{\min} = v_{\max} \) (see the last paragraph of this proof). For this solution, Inequality (2.13) reads \((t_{\min} - t_0)^{1/4} > C \). Now, noting that \( \sup t_{\min} = |X_{\min}(x_0)| \) is finite according to Lemma 2.1, we obtain \(|X_{\min}(x_0)| - |x_0|^2|X_{\min}(x_0)|^{1/4} > C \). So that \( |X_{\min}(x_0)|^{1/4} > \eta(x_0) \) and we arrive at estimate (2.11). The next simpler inequality is obtained by substitution of the first Inequality (2.9) instead of \(|X_{\min}(x_0)|^{1/4} \) in the derivation above.

We now prove the existence of the solution of Equation (1.1) corresponding to the value \( v_{\min} = v_{\max} \) for all \( x_0 \). This proof is based on the results established in § 3 and can be omitted at the first reading as it is not used anywhere in the subsequent sections. This solution must have a pole at \( x_0 \), and such minimum, \( x_{\min} \), that \( y_{\min} = v_{\min} / \sqrt{|x_{\min}|} \), where \( y_{\min} \) is the corresponding minimum value. From Theorem 3.1 it follows that for any \( x_0 \leq 0 \) and \( y_{\min} \in \mathbb{R} \), there is a unique solution of Equation (1.1). The corresponding minimum \( x_{\min} = x_{\min}(x_0, y_{\min}) < x_0 \leq 0 \) for all values of its arguments is a continuous function (see the proof of Corollary 3.3). Moreover, it is lower bounded according to Lemma 2.1. Thus, the function \( y_{\min} / \sqrt{|x_{\min}(x_0, y_{\min})|} \) is a continuous function of \( y_{\min} \in \mathbb{R} \), which maps \( \mathbb{R} \) onto \( \mathbb{R} \) and therefore achieves at some point the value \( v_{\min} \).

**Remark 2.4.** The lower bound estimates of \( X_{\min}(x_0) \) in terms of \( \eta(x_0) \) is much sharper for small values of \( x_0 \) than the upper bounds, see § 8 for a comparison with the numerical results.

**Definition 2.2.** Let \((a, b)\) with \( a < 0 \) be the interval of existence of the solution, \( y(x) \), of Equation (1.1) with initial data: \( y(x_0) = y_0 \) and \( y'(x_0) = y_1 \) at \( x_0 \leq 0 \). Define functions:
\[
X(x_0) = \inf_{y_0, y_1 \in \mathbb{R}} \{a(x_0, y_0, y_1)\}, \quad X_-(x_0) = \inf_{y_0 \in \mathbb{R}, y_1 \leq 0} \{a(x_0, y_0, y_1)\}
\]
which map \((-\infty, 0]\) into \([-\infty, 0)\).

**Lemma 2.2.** Let \( x_0 \leq 0 \) and \( C \) and \( \eta(x_0) \) be the same as in Corollary 2.1. Then
\[
X_-(x_0) < -\eta(x_0)^4 = x_0 - \frac{C}{\eta(x_0)},
\]
For any \( x_0 \leq 0 \) the function \( X_-(x_0) \) is finite and for \( x_0 < 0 \),
\[
x_0 - \frac{C}{|x_0|^{1/4}} < X_-(x_0) < x_0 - \frac{C}{|x_0| + C/|x_0|^{1/4}}.
\]
In particular, as \( x_0 \to -\infty \),

\[
X_-(x_0) = x_0 - \frac{C}{|x_0|^{1/4}} + \mathcal{O} \left( \frac{1}{|x_0|^{3/2}} \right),
\]

where \( 0 < \mathcal{O} \left( \frac{1}{|x_0|^{3/2}} \right) < \frac{C^2}{4|x_0|^{3/2}} \) for \( |x_0| > C^{4/5} \).

**Proof.** The proof is analogous to that of Lemma 2.1 and Corollary 2.1: we exclude from consideration the point \( x_0 = 0 \), as done in the first paragraph of the proof of Lemma 2.1, and switch to notation (2.1). As \( u'(t) > 0 \) now, instead of Equation (2.2), we get:

\[
u'(t) = \sqrt{4u^3 + 2N(u_0, u_1) + 2\int_0^t tw'(t)dt} > \sqrt{4(u^3 - u_0^3) + 2t_0(u - u_0)}.
\]

This results in the following upper bound for the position of the nearest (to the right of \( t_0 \)) pole, \( t_p \), of the solution \( u(t) \),

\[
t_p - t_0 < \frac{I(+\infty, v_0)}{t_0^{1/4}},
\]

where \( I(+\infty, v_0) \) is defined by Equation (2.6) with the change \( v_0 \to +\infty \) and \( v_{\min} \to v_0 \). The left Inequality (2.15) follows from Inequality (2.16). The right Inequality (2.15) is a consequence of Inequality (2.14). The latter follows from the estimates analogous to the ones in the first paragraph of the proof of Corollary 2.1 (with \( t_{\min} \to t_p \)). Thus we arrive at the estimate analogous to (2.13), \( t_p^{1/4}(t_p - t_0) > J \), where \( J \) is an integral like \( I(+\infty, v_0) \), but with an extra term \( +u_1^2/t_p^{5/2} \) under the square root. This term disappears under the passage to the supremum.

The proof of existence of the solution with \( y_0 = v_{\max} \max \sqrt{t_p} \) is even easier here than in Corollary 2.1 as it does not require references to any specific results: the pole \( t_p = |x_p| > |x_0| \geq 0 \) is a continuous function of the initial data, \( y_0 \) and \( y_1 = -u_1 \) at \( x_0 \), \( t_p = t_p(x_0, y_0, y_1) \). Moreover, as established above it is upper bounded. Thus, \( y_0/|x_p(x_0, y_0, 0)| \) is a continuous function of \( y_0 \) which maps \( \mathbb{R} \) onto \( \mathbb{R} \). \( \square \)

**Remark 2.5.** Concerning the behaviour of \( X_-(x_0) \) for small \( x_0 \), one can make the same comments and estimates as in Remark 2.3. It is interesting that for two different functions, \( X_{\min}(x_0) \) and \( X_-(x_0) \), we have exactly the same bounds and asymptotics; see § 8 for a numerical comparison.

**Theorem 2.1.** Let \( x_0 \leq 0 \) and the constant \( C \) and function \( \eta(\cdot) \) be the same as in Corollary 2.1. Then

\[
X_-(X_{\min}(x_0)) \leq X(x_0) < x_0 - \frac{2^{4/5}C}{\eta(x_0/2^{4/5})},
\]

in particular, \( X(x_0) \) is finite. Moreover, for \( x_0 < 0 \)

\[
x_0 - \frac{C}{|x_0|^{1/4}} - \frac{C}{\eta(x_0)} < X_{\min}(x_0) - \frac{C}{|X_{\min}(x_0)|^{1/4}} < X(x_0)
\]

\[
< x_0 - \frac{2^{4/5}C}{|X_{\min}(x_0/2^{4/5})|^{1/4}} < x_0 - \frac{2C}{|x_0| + 2C/|x_0|^{1/4}}.
\]
in particular, as \( x_0 \to -\infty \),

\[
X(x_0) = x_0 - \frac{2C}{|x_0|^{1/4}} + \mathcal{O}\left(\frac{1}{|x_0|^{3/2}}\right),
\]

where \( 0 < \mathcal{O}\left(\frac{1}{|x_0|^{3/2}}\right) < \frac{C^2}{|x_0|^{1/2}} \) for \( |x_0| > (2C)^{4/5} \).

**Proof.** The first Inequality (2.17) is an evident consequence of Definitions 2.1 and 2.2. To prove the second Inequality (2.17), we have to combine corresponding estimates for the upper bounds from Corollary 2.1,

\[
t_{\text{min}} - t_0 > I(+\infty, v_{\text{min}}) t_{\text{min}}^{-1/4},
\]

with \( v_{\text{min}} = u_{\text{min}}/\sqrt{t_{\text{min}}} \), and Lemma 2.2,

\[
p - t_{\text{min}} > I(+\infty, v_{\text{min}}) t_{\text{min}}^{-1/4},
\]

where \( v_{\text{min}} = u_{\text{min}}/\sqrt{t_{\text{min}}} \) and we have assumed that \( u_1 = 0 \). Since \( p > t_{\text{min}} \), we can never get for the same solution \( u_{\text{min}} = v_{\text{min}} = v_{\text{max}} \). Therefore, we make the first estimate above a bit more rough, i.e., in the original integral (2.4) we put \( p \) instead of \( t(\tilde{u}) \). This formally results in changing \( t_{\text{min}}^{-1/4} \) to \( t_{\text{min}}^{-1/4} \) and \( v_{\text{min}} \) to \( v_{\text{min}} \) in the first estimate. Then summing the above estimates, after the same procedure of passing to the supremum as described in Lemma 2.2, we arrive at the inequality,

\[
|X(x_0)|^{1/4} (|X(x_0)| - |x_0|) > 2C.
\]

The first Inequality (2.18) follows from Corollary 2.1. The second Inequality (2.18) is the consequence of the first Inequality (2.17) and Lemma 2.2. The first Inequality (2.19) follows from the second Inequality (2.17) and Corollary 2.1, finally, the last Inequality (2.19) also results from Corollary 2.1. \( \square \)

**Remark 2.6.** The reader can find numerical values of \( X(0) \) and \( X(-1) \) in § 8.

### 3. The Functions \( X_{\text{min}}(x_0, y_1) \) and \( X_{\text{min}}(x_0) \)

**Proposition 3.1.** Given \( y_0 > y_1 \), there exists only one solution \( y(x) \) such that \( y(x_0) = y_0 \) and \( y_{\text{min}} = y_1 \), where \( y_{\text{min}} \) is the minimum of \( y(x) \) achieved to the left of \( x_0 \leq 0 \).

**Proof.** It is convenient to use the notation \( t = -x \), \( u(t) = y(x) \), with \( t_0 = -x_0 > 0 \), \( u(t_0) = y_0 \equiv u_0 \), \( u'(t_0) = -y'(x_0) \equiv u_1 \) and \( y_1 = u_1 \). See Equation (2.1).

Consider a straight line \( L := \{(t, u) : u = u_1\} \). The graph of the solution with initial data \( u(t_0) = u_0 \), \( u'(t_0) = 0 \) does not intersect \( L \). On the other hand, the solutions with initial data \( u(t_0) = u_0 \), \( u'(t_0) = -\mu < 0 \) all intersect \( L \) for large enough \( \mu \). This follows from the inequality

\[
-4u_{\text{min}}^3 + 2u_{\text{min}}(u_0 - u_{\text{min}}) > 2N(u_0, \mu),
\]

which can be obtained from Equation (2.3), and the fact that \( 2N = \mu^2 - 4u_0^3 \to +\infty \) as \( \mu \to +\infty \). By continuity, it follows that there exists a solution tangent to \( L \).

This solution is unique. Actually, suppose there are two such solutions \( u_1(t) \), \( u_2(t) \) with \( u_1(t_0) = u_2(t_0) = u_0 \), \( u_1'(t_0) < u_2'(t_0) < 0 \) and the same minimum \( u_1_{\text{min}} = u_2_{\text{min}} = u_1 \). Since their derivatives are non-zero before their respective minima, \( u_1(t), u_2(t) \) are monotonic and we can define the inverse functions \( t_1(u), \)
In some neighbourhood of \((t_0, u_0)\), \(t_1(u) < t_2(u)\). Therefore, \(u_1''(t_1(u)) < u_2''(t_2(u))\). Integrating the last inequality, we get

\[
0 < u_1''(t_0) - u_2''(t_0) < u_1''(t_1(u)) - u_2''(t_2(u)).
\]

From this inequality follows that the graphs of \(t_1(u), t_2(u)\) cannot cross. Therefore, Inequality (3.2) holds on the whole interval \([u_1, u_0]\). Substituting \(u = u_1\) into (3.2), we get a contradiction, since the right side is zero while the left side is positive.

**Lemma 3.1.** Let \(y_1(x)\) and \(y_2(x)\) be solutions of Equation (1.1) whose graphs intersect at \(x_0\):

\[
y_1(x_0) = y_2(x_0) = y_0, \quad y_1''(x_0) > y_2''(x_0) \geq 0.
\]

Denote by \(b_1\) and \(b_2\) the right bounds of the interval of existence of the solutions \(y_1(x)\) and \(y_2(x)\), respectively. Then \(b_1 > b_2\) and \(y_2(x) > y_1(x)\) for \(x \in (x_0, b_2)\).

**Proof.** In some right neighbourhood of \(x_0\), \(y_2(x) > y_1'(x) > 0\). Therefore, the functions \(x_1(y)\) and \(x_2(y)\) inverse to the solutions \(y_1(x)\) and \(y_2(x)\), respectively, are defined in some right neighbourhood of \(y_0\). Moreover, there exists \(\hat{y} > y_0\) such that for \(y \in [y_0, \hat{y}]\)

\[
x_1'(y) > x_2'(y).
\]

Thus, \(0 > x_1(y) > x_2(y)\) for \(y \in (y_0, \hat{y})\). Denote

\[
\hat{y}_0 = \sup\{y : \text{Condition (3.3) holds}\}.
\]

Suppose, \(\hat{y}_0\) is finite. Then \(x_1'(\hat{y}_0) = x_2'(\hat{y}_0)\) and \(y_2''(x_2(y)) \geq y_1''(x_1(y))\) for \(y \in [y_0, \hat{y}_0]\). Integrating the last inequality from \(y_0\) to \(y \leq \hat{y}_0\) we get

\[
y_2''(x_2(y)) - y_1''(x_1(y)) \geq y_2''(x_2(y_0)) - y_1''(x_1(y_0)) > 0,
\]

implying \(y_2'(x_2(y)) > y_1'(x_1(y)) > 0\). Therefore we deduce that Inequality (3.3) actually holds for any \(y \leq \hat{y}_0\). But for \(y = \hat{y}_0\) we get a contradiction. Consequently, Inequality (3.3) holds for all \(y \in [y_0, +\infty)\). Integrating it now from \(y_0 + \epsilon\) to \(y > y_0 + \epsilon\), where \(\epsilon > 0\) is some fixed number, we have

\[
x_1(y) - x_2(y) > x_1(y_0 + \epsilon) - x_2(y_0 + \epsilon) \equiv \infty > 0.
\]

This inequality means that there are no intersections of the graphs for any \(y > y_0\). Finally, taking the limit \(y \to +\infty\) in this inequality we get \(b_1 - b_2 \geq \infty > 0\).

**Proposition 3.2.** Given \(\hat{y}_0 > y_0 > y_1\), consider solutions \(y(x), \hat{y}(x)\) such that \(y(x_0) = y_0, \hat{y}(x_0) = \hat{y}_0\) and \(y_{\min} = \hat{y}_{\min} = y_1\). Then \(x_{\min} = \hat{x}_{\min}\) and \(\hat{y}'(x_0) > y'(x_0) > 0\). Moreover, \(\hat{y}(x) > y(x)\) for \(x \in [x_{\min}, x_0]\).

**Proof.** For \(\mu \in \mathbb{R}\), consider the solutions \(\hat{y}_\mu(x)\) with the same initial value \(\hat{y}_0\) and \(\mu = \hat{y}_\mu'(x_0)\). From the proof of the previous proposition, it is clear that for sufficiently large \(\mu > M > 0\), these solutions cross the graph of \(y(x)\), while for all \(\mu \leq 0\) they do not.

 Continuously increasing \(\mu\) from 0 to \(M\), the first intersection of the graphs \(\hat{y}_\mu(x)\) with \(y(x)\) cannot occur for \(x \in [x_{\min}, x_0]\). Otherwise, there would exist \(\mu_1\), such that the graphs of \(\hat{y}_{\mu_1}(x)\) and \(y(x)\) are tangent at some point in this interval. This contradicts the uniqueness of solution of the Cauchy problem at that point.

 The first “intersection” occurs for some \(\mu = \mu_2\) at \(x = x_p\), the pole of both solutions \(y(x)\) and \(\hat{y}_{\mu_2}(x)\). Actually, the solutions asymptote to the vertical line
\( x = x_p \) without intersection. By increasing \( \mu \) from \( \mu_2 \), we observe that a finite intersection point \( A \) with coordinates \((A_x, A_y)\) appears. This point \( A = A(\mu) \) moves continuously on the graphs of both \( y(x) \) and \( \tilde{y}_\mu(x) \) as \( \mu \) increases. For \( \mu > 0 \), the graph of \( \tilde{y}_\mu(x) \) has a minimum \( \{x_{\min}(\mu), y_{\min}(\mu)\} \). For large \( \mu \), \( A_x \in (x_{\min}, x_0) \). Thus, for some value of \( \mu = \mu_{\min} \), the point \( A \) coincides with the minimum point of the graph of \( y(x) \). For this value of \( \mu = \mu_{\min} \), the minimum point of the graph of \( \tilde{y}_\mu(x) \) is lower and to the left of the minimum point of the graph of \( y(x) \). Consequently, for some \( \mu < \mu_{\min} \), \( A(\mu) \) crossed the minimum point of the graph of \( \tilde{y}_\mu(x) \). Such a crossing may occur more than once. We choose the largest such value of \( \mu \) and denote it by \( \tilde{\mu}_{\min} \). Therefore, for all \( \mu \in (\tilde{\mu}_{\min}, \mu_{\min}) \), \( \tilde{x}_{\min}(\mu) < A_x < x_{\min} \) and, for some \( \mu \) in this interval, the minimum value of \( \tilde{y}_\mu(x) \) equals \( y_1 \). This proves the existence of the desired solution \( \tilde{y}(x) \) with \( \tilde{x}_{\min} < x_{\min} \). It is unique by Proposition 3.1.

Denote by \( \tilde{x} \) the point where the straight line \( y = y_0 \) first crosses the graph of \( \tilde{y}(x) \). Arguments similar to that of the previous proposition show that \( \tilde{y}'(\tilde{x}) > y'(x_0); \) otherwise, \( y_{\min} > y_1 \). It is clear that \( \tilde{y}'(\tilde{x}) > y'(\tilde{x}) \).

\[ \square \]

**Remark 3.1.** There is another proof of Proposition 3.2 in the spirit of Lemma 3.1.

**Corollary 3.1.** Consider the solution with initial value \( y(x_0) = y_0 \) and minimum value \( y_1 \) achieved to the left of \( x_0 \). The initial slope \( y_1 = f(x_0; y_0, y_1) \) at \( x_0 \) is a continuous function defined for \( x_0 \leq 0 \) and \( y_0 \geq y_1 \in \mathbb{R} \) with \( f(x_0; y_0, y_0) = 0 \). Moreover, \( f(x_0; y_0, y_1) \) is a smooth function of all its variables for \( x \leq 0 \) and \( y_1 < y_0 \) with \( \partial_y f(x_0; y_0, y_0) < 0 \).

For fixed \( x_0 \) and \( y_1 \) it is a monotonically increasing function with the asymptotic behaviour

\[ f(x_0; y_0, y_1) \xrightarrow[\infty]{\text{as} y_0} 2y_0 \sqrt{y_0} + \mathcal{O} \left( \frac{1}{\sqrt{y_0}} \right). \]

For fixed \( x_0 \) and \( y_0 \), it is monotonically decreasing with the asymptotics

\[ f(x_0; y_0, y_1) \xrightarrow{-\infty}{\text{as} y_1} -2y_1 \sqrt{-y_1} + \mathcal{O} \left( \frac{1}{\sqrt{-y_1}} \right). \]

For fixed \( y_0 \) and \( y_1 \), \( f(x_0; y_0, y_1) \) is a monotonically decreasing function of \( x_0 \) with the asymptotics

\[ f(x_0; y_0, y_1) \xrightarrow{x_0 \to -\infty} \sqrt{2|x_0|(y_0 - y_1)} \left( 1 + \frac{y_0^2 + y_1^2 + y_0 y_1}{|x_0|^5/4} \right) + \mathcal{O} \left( \frac{1}{|x_0|^{5/4}} \right). \]

**Proof.** The continuity, actually even the smoothness of the inverse function \( y_1 = g(x_0; y_0, y_1) \), is evident as \( y_1 = y(x_{\min}; x_0, y_0, y_1) \), where \( y(x; x_0, y_0, y_1) \) is a solution of Equation 1.1 with initial data \( y_0 \) and \( y_1 \) given at \( x_0 \), and \( x_{\min} = x_{\min}(x_0; y_0, y_1) \) is a smooth function by the implicit function theorem as the solution of the equation \( y'(x_{\min}; x_0, y_0, y_1) = 0 \), with \( y''(x_{\min}; x_0, y_0, y_1) = 6y_1^2 - x_{\min} > 0 \). Continuity of \( f(x_0; y_0, y_1) \) follows from continuity of \( g(x_0; y_0, y_1) \) by standard arguments, we recall them below in the particular case of continuity on \( y_1 \). Consider a convergent sequence \( \{y_{in}\} \) of values of \( y_1 \) and let \( y_{\infty} = \lim y_{in} \). Define \( y_{in} = f(x_0; y_0, y_{in}) \), then the sequence \( y_{in} \) is bounded, because unbounded slopes, \( y_1 \to -\infty \), correspond to unbounded “levels”, \( y \to -\infty \). If we suppose that \( \lim y_{in} \) does not exist, then there exist two subsequences, \( y_{in_k} \) and \( y_{in_m} \), \( k, m = 1, 2, \ldots \), convergent to \( y_{10} \) and...
\(y_{11}\), respectively, where \(y_{10} \neq y_{11}\). However, using the continuity of \(g\) we obtain:
\[
g(y_{10}) = g(\lim y_{1n_0}) = \lim y_{1n_k} = y_{10} \quad g(y_{11}) = g(\lim y_{1n_m}) = \lim y_{1n_m} = y_{10}.
\]
This gives a contradiction, since it implies that \(y_{10} = y_{11}\). Hereafter we suppress the dependence of \(f\) and \(g\) on \(x_0\) and \(y_0\). Now, again by continuity of \(g\), we find:
\[
\lim f(y_{1n}) = \lim y_{1n} = f(g(\lim y_{1n})) = f(\lim g(y_{1n})) = f(\lim y_{1n}) = f(y_0).
\]
To prove smoothness of the function \(f\), we rewrite Equation (2.3) in original variables, \(y\) and \(x\), and differentiate the latter with respect to \(y_1\) recalling that \(y_t = g(x_0; y_0, y_1)\):
\[
0 = y_t'(6y_t^2 - x_m) + y_1 - \int_{y_0}^{y_1} x'(u; x_0, y_0, y_1)du,
\]
where the prime is the derivative with respect to \(y_1\) and the minimum \(x_m = x(y_t; x_0, y_0, y_1)\). A key observation now is that \(x(u; x_0, y_0, y_1)\) is a monotonically increasing function of \(y_t\), for fixed values of the other variables (cf. the last paragraph of the proof of Proposition 3.1), so that the integrand, \(x'\), is nonnegative. Therefore, the integral in Equation (3.4) is convergent, as \(y_t'\) exists, and (strictly) negative since we assume that \(y_t < y_0\). At the same time \(y_1 > 0\) under the same condition, \(y_t < y_0\), and the factor multiplying \(y_t'\) is positive as \(-x_m > -x_0 > 0\). Thus Equation (3.4), together with continuity of \(f\), implies that \(y_t'\) is continuous and strictly negative for \(x_0 \leq 0\) and \(y_t < y_0\). So, the derivative of the function inverse to \(y_t\), i.e., the function \(f\), has the same properties. The proof of smoothness with respect to \(x_0\) and \(y_0\) is analogous with the help of the observation that \(\partial_{x_0} x \geq 0\) and \(\partial_{y_0} x \leq 0\), so that the corresponding integrals exist and are strictly positive or, respectively, negative.

Rewriting Equation (2.3) in terms of \(y(x)\), we get the estimate
\[
2|x_{\text{min}}|(y_0 - y) - 4y_t^2 > y_t^2 - 4y_0^2 > 2|x_0|(y_0 - y) - 4y_t^2
\]
The first two asymptotics follow immediately, whilst the proof of the last one requires a reference to estimate (2.9). \(\square\)

Remark 3.2. It follows from Corollary 3.1 that instead of a function, say, \(h(x_0, y_0, y_t)\) smoothly depending on initial data \(x_0, y_0\), and \(y_t\), we can always consider a function \(h(x_0, y_0, y_t) = h(x_0, y_0, f(x_0; y_0, y_t))\), which depends smoothly on \(x_0\), \(y_0\), and \(y_t\). Where it does not cause confusion, we admit an abuse of notation and denote both functions, \(h\) and \(\hat{h}\), by the same symbol, \(h\).

Remark 3.3. Corollary 3.1 shows that the functions \(f(x_0; y_0, y_t)\) and \(f(x_0; -y_0, -y_t)\) have similar monotonicity properties. Moreover, the difference \(\Delta(x_0; y_0, y_t) = f(x_0; y_0, y_t) - f(x_0; -y_0, -y_t) = o(1)\) when one of the variables \(|x_0|\), \(|y_0|\) or \(|y_t|\) is large. This estimate also holds when two or all three of the variables are large, as can be deduced from the corresponding asymptotics of \(f(x_0; y_0, y_t)\). It is straightforward to obtain these asymptotics from Inequality (3.5) and estimate (2.9). They, clearly, inherit from (3.5) an invariance under the change of variables \(y_0, y_t \to -y_0, -y_t\). Therefore the function \(\Delta(x_0; y_0, y_t)\) is bounded in its domain of definition, \(D := \{x_0 \leq 0, y_t \leq y_0\} \subset \mathbb{R}^3\). Note also that \(\Delta(x_0; y_0, \pm y_0) = 0\). Some other numerical values of \(\Delta\) are given in § 8.
One can define also the function $f^+(x_0; y_0, y_l)$ as initial slope of the solution with the initial value $y(x_0) = y_0$ and minimum value $y_l \leq y_0$ achieved to the right of $x_0$. It is clear that $f^+ < 0 \mid f^+(x_0; y_0, y_l) \mid < f(x_0; y_0, y_l)$. Actually, the function $|f^+|$ has properties similar to those of $f$. Moreover, using $f^+$ one can make a continuous prolongation of $f$ from $D$ to $\mathbb{R} \times \mathbb{R}^2$, by putting $f(x_0; y_0, y_l) := f^+(x_0; y_0, y_l)$ for $y_l \geq y_0$. It would be interesting to know whether this continuation is smooth as $x'$ in Equation (3.4) is unbounded as $y_l \to y_0$, and thus a finite limit of $y_l'$ is possible.

**Definition 3.1.** Given $x_0 \leq 0$, $y_0 > y_l$, consider the solution $y(x) = y(x; x_0, y_0, y_l)$ with initial value $y(x_0) = y_0$ and minimum value $y_l$ at $x_{min} = x_{min}(x_0, y_0, y_l) < x_0$.

Define

$$X_{min}(x_0, y_l) = \inf_{y_0 > y_l} \{x_{min}(x_0, y_0, y_l)\}$$

**Proposition 3.3.** $X_{min}(x_0, y_l)$ is finite. The solution $y_m(x)$ with initial data

$$y_m(X_{min}(x_0, y_l)) = y_l, \quad y'_m(X_{min}(x_0, y_l)) = 0$$

has a pole at $x_0$.

**Proof.** Definitions 2.1 and 3.1 imply

$$X_{min}(x_0) \leq X_{min}(x_0, y_l) < x_0.$$  

The solution $y_m(x)$ cannot be regular at $x_0$ as it is a limit of solutions with initial value $y_0 \to +\infty$, $y_0' \to +\infty$ at $x_0$. (See Proposition 3.2 and Corollary 3.1.) Therefore, it has a pole $x_p \leq x_0$. Suppose $x_p < x_0$.

For any given $\varepsilon > 0$ and a sequence $x_n \nearrow x_p$, one finds $y_{0n} \nearrow +\infty$ such that for all values of $y_0 \geq y_{0n}$, we have

$$|y_m(x) - y(x; x_0, y_0, y_l)| \leq \varepsilon, \quad |y'_m(x) - y'(x; x_0, y_0, y_l)| \leq \varepsilon,$$

for $x \in [X_{min}(x_0, y_l), x_n]$. In particular, this means that

$$y(x_n; x_0, y_{0n}, y_l) \underset{y_0 \to +\infty}{\to} L_n \to +\infty, \quad y'(x_n; x_0, y_{0n}, y_l) \underset{y_0 \to +\infty}{\to} M_n \to +\infty.$$  

Arguments analogous to that of Lemma 2.2 show that any solution with initial values $y(x_n) = L_n, y'(x_n) \geq 0$ has a pole $x_{pn} > x_n$, whose location can be estimated by

$$\frac{1}{2\sqrt{L_n}} \int_1^{+\infty} \frac{dv}{\sqrt{v^2 - 1}} > x_{pn} - x_n.$$  

Therefore, for all sufficiently large $L_n$, $x_{min} < x_{pn} < x_0$. This contradicts the fact that $y(x; x_0, y_0, y_l)$ is finite on $[x_{min}, x_0]$.

**Lemma 3.2.** Given $y_l \in \mathbb{R}$ there is only one solution that has minimum value $y_l$ with a pole at $x_0$ and an interval of existence to the left of $x_0$.

**Proof.** Existence of the solution $y(x; x_0, y_l)$ follows from Proposition 3.3. Any solution with a pole at $x_0$ has the following convergent Laurent expansion:

$$(3.6) \quad y(x) = \frac{1}{(x - x_0)^2} + \frac{x_0}{10(x - x_0)^2} + \frac{1}{6}(x - x_0)^3 + c(x - x_0)^4 + \ldots$$
where \( c \in \mathbb{R} \) is a parameter characterising the solution. Suppose \( y_1(x), y_2(x) \) are solutions corresponding to \( c_1 \) and \( c_2 \) with \( c_1 > c_2 \). Consider the difference

\[
y_1 - y_2 = \frac{(c_1 - c_2)(x - x_0)^4 + \mathcal{O}((x - x_0)^5)}{x - x_0},
\]

(3.7)

\[
y_1' - y_2' = 4(c_1 - c_2)(x - x_0)^3 + \mathcal{O}((x - x_0)^4).
\]

(3.8)

From this it follows that

\[
y_1(x) > y_2(x), \quad 0 < y_1'(x) < y_2'(x)
\]

(3.9)

in some (small) left neighbourhood of \( x_0 \). Consider a horizontal line which crosses the graphs of \( y_1(x), y_2(x) \) at points \( x_1, x_2 \) in the neighbourhood of \( x_0 \) where Inequalities (3.9) hold. We have

\[
y_1'(x_1) < y_2'(x_2).
\]

Repeating almost exactly the proof of Proposition 3.1, we see that the minimum values of the solutions satisfy \( y_{1\text{min}} > y_{2\text{min}} \).

**Theorem 3.1.** Any solution with an interval of existence that is contained in the non-positive semi-axis can be uniquely characterised by the position of its right pole \( x_0 \) and minimum value \( y_1 \). Moreover, it achieves its minimum value at \( X_{\text{min}}(x_0, y_1) \).

**Proof.** The proof follows from the above propositions and Lemma 3.2.

**Remark 3.4.** From Theorem 3.1 we see that there is a natural one-to-one correspondence between solutions of Equation (1.1) and the functions \( X_{\text{min}}(x_0, y_1) \). The solution \( y(x; x_0, y_1) \) that corresponds to the function \( X_{\text{min}}(x_0, y_1) \) has the following extremal property: its minimum \( X_{\text{min}}(x_0, y_1) < x_{\text{min}} \) where \( x_{\text{min}} \) is the minimum of any solution regular at \( x_0 \) with minimum value \( y_1 \). Moreover, the properties of the solutions can be formulated as monotonicity properties of the functions \( X_{\text{min}}(x_0, y_1) \).

**Corollary 3.2.** For any \( y_1 \), if \( x_1 < x_2 < 0 \) then

\[
X_{\text{min}}(x_1, y_1) < X_{\text{min}}(x_2, y_1)
\]

**Proof.** If \( X_{\text{min}}(x_1, y_1) = X_{\text{min}}(x_2, y_1) \) then the two corresponding solutions (see Remark 3.4) would coincide. Suppose we have \( X_{\text{min}}(x_1, y_1) > X_{\text{min}}(x_2, y_1) \). Then there is a solution regular at \( x_1 \), with \( x_{\text{min}} < X_{\text{min}}(x_1, y_1) \). This contradicts the definition of \( X_{\text{min}}(x_0, y_1) \).

**Proposition 3.4.** For any pair of numbers \( (x_0, c_0) \in \mathbb{C}^2 \) consider a Laurent expansion (3.6) (with \( c \to c_0 \)) as a germ defining a complex solution \( y(x) \) by analytic continuation on \( x \). Denote such a complex solution by \( y(x; x_0, c_0) \). It is a meromorphic function of \( x \) which, for any choice of \( (x_0, c_0) \), has an infinite discrete set of second order poles, \( \mathcal{P} = \mathcal{P}(x_0, c_0) \). For any \( x \in \mathbb{C} \setminus \mathcal{P} \) the function \( y(x; x_0, c_0) \) is an analytic (locally holomorphic) function of the parameters \( x_0 \) and \( c_0 \).

Fix any pair \( (\hat{x}_0, \hat{c}_0) \in \mathbb{C}^2 \). If \( \hat{x}_1 \) is any other pole of \( y(x; \hat{x}_0, \hat{c}_0) \) and \( \hat{c}_1 \) is the corresponding pole parameter of expansion (3.6), then there exist a unique pair of functions \( x_1 = x_1(\hat{x}_0, \hat{c}_0), c_1 = c_1(\hat{x}_0, \hat{c}_0) \) which is (1) holomorphic at \( (\hat{x}_0, \hat{c}_0) \); (2) \( \hat{x}_1 = x_1(\hat{x}_0, \hat{c}_0), \hat{c}_1 = c_1(\hat{x}_0, \hat{c}_0) \); and (3) \( x_0, x_1 \) and \( c_0, c_1 \) are respectively the poles and corresponding pole parameters of the complex solution \( y(x; x_0, c_0) \).
Proof. Let \( r_k, k = 0, 1 \) be the radii of convergence of the Laurent expansions at \( x_k \) and \( \epsilon > 0 \) be chosen sufficiently small, \( \epsilon < r_k \). Then in the annuli \( R_k := \{ x : \epsilon < |x - x_k| < r_k \} \), the complex solutions \( y(x; x_k, c_k) \) are (locally) holomorphic functions of variables \( x_k \) and \( c_k \) since the coefficients of the Laurent expansions (3.6) are polynomials in \( x_k \) and \( c_k \). Take any points \( \tilde{x}_k \in R_k \) and consider the analytic functions of \( x_k \) and \( c_k \): \( y_k = y(\tilde{x}_k; x_k, c_k) \) and \( y_{1k} = y'(\tilde{x}_k; x_k, c_k) \), as the initial data for the complex solution \( y(x; x_k, c_k) \), where now \( x \) is an arbitrary point in \( \mathbb{C} \setminus \mathcal{P} \). For each \( k \), \( y(x; x_k, c_k) \) is an analytic function of \( x_k \) and \( c_k \) as a composition of analytic functions \( x_k, c_k \mapsto y_k, y_{1k} \) and \( y_k, y_{1k} \mapsto y(x; x_k, c_k) \).

Consider now the system of equations which guarantees that the complex solutions \( y(x; x_k, c_k) \) have the same “initial data” at \( x = \tilde{x}_1 \):

\[
y(\tilde{x}_1; x_0, c_0) = y(\tilde{x}_1; x_1, c_1), \quad y'(\tilde{x}_1; x_0, c_0) = y'(\tilde{x}_1; x_1, c_1),
\]

where the left (respectively, right) sides of the equations are considered as holomorphic functions of \( x_0 \) and \( c_0 \) (respectively, \( x_1 \) and \( c_1 \)). We claim that the unique solvability of this system in neighbourhoods of \((\tilde{x}_k, \tilde{c}_k)\) and the existence of the corresponding derivatives follow from the implicit function theorem. In fact, consider the Jacobian,

\[
J(\tilde{x}_1) = \begin{pmatrix}
\partial_{x_1} y(\tilde{x}_1; x_1, c_1) & \partial_{c_1} y(\tilde{x}_1; x_1, c_1) \\
\partial_{x_1} y'(\tilde{x}_1; x_1, c_1) & \partial_{c_1} y'(\tilde{x}_1; x_1, c_1)
\end{pmatrix}.
\]

This Jacobian is independent of \( \tilde{x}_1 \) as it coincides with the Wronskian of two complex solutions \( \partial_{x_1} y(\tilde{x}_1; x_1, c_1) \) and \( \partial_{c_1} y(\tilde{x}_1; x_1, c_1) \) of the linearization of Equation (1.1), \( Y''(\tilde{x}_1) = 12y(\tilde{x}_1; x_1, c_1)Y(\tilde{x}_1) \). Direct calculation with the help of Laurent expansion (3.6) gives \( J(\tilde{x}_1) = 14 \). Finally, we remark that the functions \( x_1(x_0, c_0) \) and \( c_1(x_0, c_0) \) do not actually depend on the “connection” point \( \tilde{x}_1 \). In fact, if these functions solve System (3.10) at some point \( \tilde{x}_1 \), then they solve it at any other point in the annulus \( R_1 \), as this means that \( y(x; x_0, c_0) \) and \( y(x; x_1(x_0, c_0), c_1(x_0, c_0)) \) have the same initial data at \( \tilde{x}_1 \) and thus they coincide for all \( x \) due to the uniqueness of solution of the Cauchy initial value problem. Suppose now that there is another pair of functions \( x_1 = x_1(x_0, c_0) \) and \( c_1 = c_1(x_0, c_0) \) with the properties (1)–(3). Then clearly, these functions solve System (3.10) and therefore coincide with \( x_1(x_0, c_0) \) and \( c_1(x_0, c_0) \), respectively.

\[\square\]

Remark 3.5. To establish most of the qualitative properties of the solutions we need continuity rather than analyticity of the corresponding functions. However, in §§6 and 7 we actually use complex analyticity of the function \( x_1 = x_1(x_0, c_0) \) with respect to \( c_0 \). Below we use the adjective smooth to indicate that corresponding real function has one continuous derivative; in fact, most of our functions are real analytic.

Remark 3.6. A solution with a pole at \( x_0 \) and interval of existence to the left is uniquely characterised on the one hand by the parameter \( c \) in Laurent expansion (3.6) and on the other hand, by its minimum \( y_1 \) achieved at \( x_{\min} = x_{\min}(x_0, c) \). Therefore, for any given \( x_0 \leq 0 \) there exists a bijection \( y_1(x_0, c) : \mathbb{R} \to \mathbb{R} \).

Corollary 3.3. The function \( y_1(x_0, c) \) is a smooth function of both variables. For any \( x_0 \leq 0 \), \( y_1(x_0, c) \) is a monotonically growing function of \( c \in \mathbb{R} \). Moreover, \( y_1(x_0, c) \to \pm \infty \) as \( c \to \pm \infty \), respectively. The inverse function \( c(x_0, y_1) \) has the same properties. The function \( x_{\min} = x_{\min}(x_0, c) \) is a smooth function of both variables.
Proof. To denote more explicitly the dependence of the solution on the parameters $x_0$ and $c$, we use the notation $y(x) = y(x; x_0, c)$. As follows from Proposition 3.4 both $y(x; x_0, c)$ and $y'_1(x; x_0, c)$, where the subscript 1 means that the derivative is taken with respect to the first variable, are smooth functions of $x_0$ and $c$.

Now $x_{\text{min}}(x_0, c)$ is the unique solution of the equation $y'_1(x; x_0, c) = 0$. By the implicit function theorem, both derivatives $x'_{\text{min},k}(x_0, c)$, $k = 1, 2$, exist and are given by

$$x'_{\text{min},k}(x_0, c) = -\frac{y''_{1(k+1)}(x_{\text{min}}; x_0, c)}{y''_{11}(x_{\text{min}}; x_0, c)},$$

where the denominator on the right is always positive because $x_{\text{min}} < x_0 \leq 0$.

Clearly, $y_t(x_0, c) = y(x_{\text{min}}(x_0, c); x_0, c)$ is also a smooth function of $x_0$ and $c$. Monotonicity of $y_t(x_0, c)$ with respect to $c$ is proved in fact in Lemma 3.2.

Let $c \searrow -\infty$. Then $y_t(x_0, c)$ monotonically decreases to a limit $\hat{y}(x_0)$. Suppose $\hat{y}(x_0) > -\infty$. In this case, $c(x_0, y_t)$ would not be defined in the interval $(-\infty, \hat{y}(x_0))$.

One now proves continuity of $c(x_0, y_t)$ by almost a repetition of the analogous proof for continuity of the function $y_t = f(x_0, y_0, y_1)$ in Corollary 3.1. Since the function $x_{\text{min}} = x_{\text{min}}(x_0, c)$ is smooth, as proved above, this continuity of $c(x_0, y_t)$ implies continuity of $x_{\text{min}}(x_0, y_t)$.

To prove smoothness of $c(x_0, y_t)$, one starts with the initial value problem at the minimum of the solution $y(x; x_0, c)$, $y(x_{\text{min}}; x_0, c) = y_t$, $y'_1(x_{\text{min}}, x_0, c) = 0$. The solution of this initial value problem for Equation (1.1) is a smooth function of the parameters $x_{\text{min}}$ and $y_t$. So we have another parameterization of the same solution,

$$y(x; x_0, c) = y(x; x_{\text{min}}, y_t)$$

with a slight abuse of notation. From this equation one deduces that the functions $x_0 = x_0(x_{\text{min}}, y_t)$ and $c = c(x_{\text{min}}, y_t)$ are smooth functions of their arguments. To prove this, define the functions $F_k(x_0, x_{\text{min}}, y_t) = \frac{1}{2\pi i} \oint \frac{y(x; x_{\text{min}}, y_t)}{(x-x_0)^{k+2}} dx$, $k = 1, 2$, where integration is taken anti-clockwise along a circle centred at $x_0$. Then, from Laurent expansion (3.6) one finds the following representation for the function $c = F_2(x_0, x_{\text{min}}, y_t)$, where $x_0 = x_0(x_{\text{min}}, y_t)$ solves equation $F_1(x_0, x_{\text{min}}, y_t) = x_0/10$. Using again Laurent expansion (3.6) we prove that $\partial_{x_0} F_1 = 3/6 = 1/2$, thus the implicit function theorem implies the smoothness of $x_0(x_{\text{min}}, y_t)$.

Now we differentiate Equation (3.11) with respect to $x$ to get analogous equation for the $x$-derivatives and consider both equations as a system to determine the functions $x_{\text{min}}(x_0, y_t)$ and $c(x_0, y_t)$. To employ the implicit function theorem we have to check that the corresponding Jacobian, $J = y'_3(x; x_0, c)y''_{12}(x; x_{\text{min}}, y_t) - y'_2(x; x_{\text{min}}, y_t)y'_{13}(x; x_0, c)$, is nonzero for all $x_0 \leq 0$ and real $y_t$. It is straightforward to see that $J$ is independent of $x$ (see Proposition 3.4). Considering it for $x$ in a proper neighbourhood of $x_0$, we find that $J = -14\partial_{x_0} x_0(x_{\text{min}}, y_t)$. Our goal now is to prove that $\partial_{x_{\text{min}}} x_0(x_{\text{min}}, y_t) > 0$. The fact that $\partial_{x_{\text{min}}} x_0(x_{\text{min}}, y_t) \geq 0$ is easy to deduce from Proposition 3.1: it is a monotonically growing function. Let us rewrite Equation (2.4) by turning back to the the original $x$, $y$ variables, putting now that $t_0 = -x_0$ is the pole, so that $u_0 = +\infty$,

$$x_0 - x_{\text{min}} = \int_{y_t}^{+\infty} \frac{dy}{\kappa}, \quad \kappa = \sqrt{4(y^3 - y_t^3) - 2 \int_{y_1}^{y_t} x(\tilde{y}; x_{\text{min}}, y_t) d\tilde{y}}.$$
Differentiating this equation with respect to \( x_{\min} \) we obtain
\[
(3.13) \quad \partial_{x_{\min}} x_0 - 1 = \int_{y_l}^{+\infty} \frac{dy}{K^3} \int_{y_l}^{y} \partial_{x_{\min}} x(y; x_{\min}, y_l) dy.
\]
We note that \( \partial_{x_{\min}} x(\tilde{y}; x_{\min}, y_l) \) exists for \( \tilde{y} > y_l \) and from Proposition 3.1 follows that \( \partial_{x_{\min}} x(\tilde{y}; x_{\min}, y_l) \geq 0 \). The integral (3.13) is improper, however it is straightforward to see that it converges, obviously at infinity and at \( y = y_l \) in virtue of the following limit: \( \lim_{y \to y_l} \partial_{x_{\min}} x(\tilde{y}; x_{\min}, y_l) = 1 \). Thus, \( \partial_{x_{\min}} x_0(x_{\min}, y_l) > 1 \) as the right-hand side of Equation (3.13) is positive.

**Remark 3.7.** From the above corollary it follows that the function \( X_{\min}(x_0, y_l) = x_{\min}(x_0, c(x_0, y_l)) \) is a smooth function of both variables.

**Proposition 3.5.** For any \( y_l \) and \( x_1 < x_2 < 0 \),
\[
(3.14) \quad 0 < X_{\min}(x_2, y_l) - X_{\min}(x_1, y_l) < x_2 - x_1.
\]

**Proof.** Let \( y_1(x), y_2(x) \) be the solutions with poles at \( x_1, x_2 \) and the same minimum value \( y_l \). The inverse functions \( x_1(y) \) with range \([x_{1,\min}, x_1]\), and \( x_2(y) \) with range \([x_{2,\min}, x_2]\) exist and are differentiable on the interval \((y_l, +\infty)\). Of course, by this definition \( x_{i,\min} = x_i(y_l) \) for \( i = 1, 2 \). Moreover, for \( y \in (y_l, +\infty) \), we have
\[
y_1'(x(y_l)) > y_2'(x(y_l)) \quad \Rightarrow \quad y_1'(x_1(y_l)) > y_2'(x_2(y_l)) > 0
\]
where we have used the fact that \( y_i'(x_i(y_l)) = 0 \). This implies
\[
x_1'(y_l) < x_2'(y_l).
\]
Integration from \( y_l \) to \(+\infty\) yields the right Inequality (3.14). The left inequality follows from Corollary 3.2. \( \square \)

**Proposition 3.6.** For \( |y_l| \geq \epsilon > 0 \)
\[
0 < x_0 - X_{\min}(x_0, y_l) < \frac{I_{\nu}}{2\sqrt{|y_l|}},
\]
where \( \nu = \text{sign}\{y_l\}1 \) and \( I_{\nu} \) is defined by Equation (2.7).

**Proof.** Neglecting in Equation (2.4) the integral term under the square root, sending the upper limit of the remaining integral to \(+\infty\) and \( t \to t_{\min} \), making the change of variables, \( u = |y_l|w \) \((u_{\min} = y_l)\), and using Proposition 3.2 and Definition 3.1 we arrive at the stated result. \( \square \)

**Proposition 3.7.** For any \( x_1 < x_2 < 0 \),
\[
(3.15) \quad 0 < X_{\min}(x_2) - X_{\min}(x_1) < x_2 - x_1.
\]

**Proof.** From Remark 3.7, we have for any \( y_l \) and \( x_0 < 0 \), that \( X_{\min}(x_0, y_l) \) is a continuous function of \( y_l \). Moreover, as follows from Proposition 3.6,
\[
0 < x_0 - X_{\min}(x_0, y_l) \to 0, \quad \text{as} \ y_l \to \pm\infty.
\]
Therefore, \( \exists \hat{y}_l(x_0) \leq 0 \) such that
\[
X_{\min}(x_0) = X_{\min}(x_0, \hat{y}_l(x_0)).
\]
The left Inequality (3.15) follows from
\[
X_{\min}(x_2) = X_{\min}(x_2, \hat{y}_l(x_2)) > X_{\min}(x_1, \hat{y}_l(x_2)) \geq X_{\min}(x_1)
\]
and the right one can be proved analogously:

\[ x_1 - X_{\min}(x_1) = x_1 - X_{\min}(x_1, \hat{y}(x_1)) \leq x_2 - X_{\min}(x_2, \hat{y}(x_1)) \leq x_2 - X_{\min}(x_2) \]

where we have again used Proposition 3.5. \( \square \)

**Remark 3.8.** It is easy to prove that \( X_{\min}(x_0, y_1) \) is monotonically decreasing as \( y_1 \searrow +0 \). We expect that this monotonic decrease continues until some negative value of \( y_1 = \hat{y}(x_1) \). After that value, \( X_{\min}(x_0, y_1) \) monotonically grows with \( y_1 \searrow -\infty \). Therefore, we conjecture that the solution with minimum at \( X_{\min}(x_0) \) is unique. This uniqueness was not assumed in the proof of the above proposition.

**Proposition 3.8.** For any \( y_0 > y_1 \) and four points \( x_k, k = 1, 2, 3, 4 \), such that \( x_1 < x_3 < x_1 \leq 0, x_4 < x_2 < x_1, \) and \( 0 < x_1 - x_2 \leq x_3 - x_4 \), consider four solutions \( y_k(x; x_k, y_0, y_1) \). Denote their minima \( x_{\min}(x_k, y_0, y_1) \). Then,

\[ x_{\min}(x_1; y_0, y_1) - x_{\min}(x_2; y_0, y_1) \leq x_{\min}(x_3; y_0, y_1) - x_{\min}(x_4; y_0, y_1) \]

and

\[ X_{\min}(x_1, y_1) - X_{\min}(x_2, y_1) \leq X_{\min}(x_3, y_1) - X_{\min}(x_4, y_1). \]

**Proof.** Restrict our solutions \( y_k(x; x_k, y_0, y_1) \) on the segments \( [x_{\min}(x_k; y_0, y_1), x_k] \), then their inverse functions, \( x_k(y; x_k, y_0, y_1) \), are properly defined on the segment \( y \in [y_1, y_0] \). For brevity, we denote them \( x_k(y) \). In particular we have \( x_k = x_k(y_0) \).

It is convenient to introduce the following notation:

\[ \Delta_{km}(y) := x_k(y) - x_m(y), \quad w_k(y) := 4(y^3 - y_1^3) - 2 \int_{y_1}^{y} x_k(u)du. \]

Using them we can rewrite Equation (2.4) as follows,

\[ x_k(y) - x_k(y_0) = \int_{y_1}^{y_0} \frac{du}{w_k(u)}, \]

and obtain equation for the differences,

\[ \Delta_{km}(y_0) - \Delta_{km}(y) = 2 \int_{y_1}^{y_0} \frac{\int_{y_1}^{y} \Delta_{km}(u)du}{\sqrt{w_k(u)w_m(u)}(\sqrt{w_k(u)} + \sqrt{w_m(u)})}. \]

Assume that

\[ \int_{y_1}^{y} \Delta_{12}(\tilde{u})d\tilde{u} > \int_{y_1}^{y_0} \Delta_{34}(\tilde{u})d\tilde{u} \quad \text{for} \quad y = y_0. \]

Then by continuity this inequality holds for all \( y : y^* < y \leq y_0 \), for some \( y^* \geq y_1 \). We further assume that \( y^* \) denotes the infimum of such numbers \( y^* \). Then Equation (3.18) implies for this values of \( y \), \( \Delta_{12}(y_0) - \Delta_{12}(y) > \Delta_{34}(y_0) - \Delta_{34}(y) \), since by our conditions \( w_1(u) < w_3(u) \) and \( w_2(u) < w_4(u) \) for all \( u \in [y_1, y_0] \). Our assumption for points \( x_k \) reads \( \Delta_{12}(y_0) \leq \Delta_{34}(y_0) \), therefore we obtain \( \Delta_{12}(y) < \Delta_{34}(y) \) for \( y : y^* < y \leq y_0 \). If \( y^* = y_1 \), then we integrate the last inequality for the differences from \( y_1 \) to \( y_0 \) and arrive at a contradiction with Inequality (3.19). Thus \( y^* > y_1 \). In this case we have \( \int_{y_1}^{y} \Delta_{12}(\tilde{u})d\tilde{u} = \int_{y_1}^{y^*} \Delta_{12}(\tilde{u})d\tilde{u} + \int_{y^*}^{y_0} \Delta_{12}(\tilde{u})d\tilde{u} < \int_{y_1}^{y^*} \Delta_{34}(\tilde{u})d\tilde{u} \). Now summing up the last inequality and equation we again arrive at a contradiction with Inequality (3.19).
Thus conditions of our proposition implies that

\[(3.20) \quad \int_{y_1}^{y_0} \Delta_{12}(\tilde{u})d\tilde{u} \leq \int_{y_1}^{y_0} \Delta_{34}(\tilde{u})d\tilde{u}.\]

Now we have two logical possibilities:
1. \(\Delta_{12}(y) \leq \Delta_{34}(y)\) for \(y^* \leq y \leq y_0\); and
2. \(\Delta_{12}(y) \geq \Delta_{34}(y)\) for \(y^* \leq y \leq y_0\);

where again \(y^* < y_0\) denotes the infimum of such values of \(y\) that the corresponding condition holds. Consider the second case. If we suppose that \(y^* = y_1\), then we integrate the second condition from \(y_1\) to \(y_0\) to arrive at

\[\int_{y_1}^{y_0} \Delta_{12}(\tilde{u})d\tilde{u} \geq \int_{y_1}^{y_0} \Delta_{34}(\tilde{u})d\tilde{u}.\]

The latter inequality does not contradict Inequality (3.20) only if \(\Delta_{12}(y) = \Delta_{34}(y)\) for all \(y \in [y_0, y_1]\). Otherwise \(y^* > y_1\). In both situations of case (2) we proved that there exists \(y^* < y_0\) and \(\Delta_{12}(y^*) = \Delta_{34}(y^*)\). Now note that if we combine this result with the statement of case (1) we arrive at the existence of a point \(y^* : y_1 \leq y^* < y_0\) where

\[(3.21) \quad \Delta_{12}(y^*) \leq \Delta_{34}(y^*).\]

Moreover, as the curves \(x_k(y)\) for \(y \in [y, y_0]\) do not intersect. We observe, that the assumptions of our proposition holds at \(y^*\).

Now we consider the set \(Y^* = \{y^* : y^* \in [y_1, y_0], \text{ Inequality (3.21) holds}\}\). Clearly, the set \(Y^*\) is non-empty and closed, moreover by previous construction if a point \(y \in Y^*\) and \(y < y_1\), then there is a point \(y^* < y\) that belongs to \(Y^*\). This means that \(\inf Y^* = y_1\).

Finally, Inequality (3.17) follows from Inequality (3.16) by taking a limit \(y_0 \to +\infty\).

**Corollary 3.4.** In the notation of Proposition 3.8 and Corollary 3.1 the following inequality is valid

\[(3.22) \quad f(x_2; y_0, y) \frac{\partial}{\partial y_0} x_{\min}(x_2; y_0, y_1) \leq f(x_1; y_0, y) \frac{\partial}{\partial y_0} x_{\min}(x_1; y_0, y_1),\]

where \(x_2 < x_1 \leq 0\).

**Proof.** Consider four solutions: \(y_1(x) = y(x; x_1, y_0, y_1), y_2(x) = y(x; x_1, y_1, y_1), y_3(x) = y(x; x_2, y_0, y_1), \) and \(y_4(x) = y(x; x_2, y_1, y_1),\) where \(\tilde{y}_1 > y_1 > y_0 > y_1\). Consider the inverse functions, \(x_k(y),\) as in the proof of Proposition 3.8. We have \(x_1 = x_1(y_0), x_2 = x_3(y_0).\) Now we impose an additional condition on the parameter \(\tilde{y}_1; x_2(y_0) - x_1(y_0) = x_3(y_0) - x_4(y_0)\) and write Equation (3.16) for the case under consideration

\[x_{\min}(x_1; y_0, y_1) - x_{\min}(x_1; y_1, y_1) \leq x_{\min}(x_2; y_0, y_1) - x_{\min}(x_2; \tilde{y}_1, y_1).\]

Divide now both parts of the last inequality by \(\tilde{y}_1 - y_0,\) and make a passage to a limit \(\tilde{y}_1 \to y_0,\) taking into account that \(\lim(\tilde{y}_1 - y_0)/(x_3(y_0) - x_4(y_0)) = f(x_2; y_0, y_1)\) and \(\lim(y_1 - y_0)/(x_1(y_0) - x_2(y_0)) = f(x_1; y_0, y_1).\)

**Remark 3.9.** Actually, we expect that there is a strict monotonicity in Equations (3.16), (3.17), and (3.22).
4. The Functions $\mathcal{X}(x_0, y_0)$ and $X(x_0)$

**Lemma 4.1.** Let $y_1(x)$ and $y_2(x)$ be solutions that intersect at $x_0$, i.e., $y_1(x_0) = y_2(x_0)$, such that $y_1'(x_0) < y_2'(x_0) \leq 0$. Denote by $a_k$ the left end of the interval of existence of $y_k(x)$ ($k = 1, 2$). Then $a_2 < a_1$ and

$$y_1(x) > y_2(x), \quad y_1'(x) < y_2'(x) < 0, \quad \text{for } x \in (a_1, x_0).$$

**Proof.** The conditions on the solutions can be reformulated as $u_1(t_0) = u_2(t_0)$, $u_1'(t_0) > u_2'(t_0) \leq 0$ in our usual notation $t = -x$, $u(t) = y(x)$. Denote $\hat{a}_k = -a_k$.

In some open right neighbourhood of $t_0$ we have

$$u_1(t) > u_2(t) \quad \text{and} \quad u_1'(t) > u_2'(t). \quad (4.1)$$

Denote

$$\hat{a} = \sup \{t \mid \text{Condition (4.1) holds} \}.$$

Suppose $\hat{a} < \min\{a_1, a_2\}$. We have $u_1'(\hat{a}) = u_2'(\hat{a})$ and $u_1(\hat{a}) > u_2(\hat{a})$. Moreover,

$$u_k^2(\hat{a}) - 4u_k^3(\hat{a}) = u_k^2(t_0) - 4u_k^3(t_0) + 2 \int_{t_0}^{\hat{a}} tu_k(t) \, dt \quad (4.2)$$

for $k = 1, 2$. Subtracting Equation (4.2) for $k = 2$ from that for $k = 1$, we get

$$4(u_2^3(\hat{a}) - u_2^3(\hat{a})) = u_1^2(t_0) - u_2^2(t_0) + 2 \int_{t_0}^{\hat{a}} t(u_1(t) - u_2(t)) \, dt$$

This is a contradiction as the right-hand side of the equation is positive whilst the left-hand side is negative. Therefore, $\hat{a} = \min\{a_1, a_2\}$. Since $u_1(t) > u_2(t)$, we have $\hat{a}_1 \leq \hat{a}_2$. Suppose that $\hat{a}_1 = \hat{a}_2 = \hat{a}$. Since $u_1'(t) > u_2'(t)$, for $t \leq \hat{a}$, we have $u_1(t) - u_2(t)$ grows as $t \to \hat{a}$. However, according to Equation (3.7), $u_1(t) - u_2(t) \to 0$ as $t \to \hat{a}$. \hfill \Box

**Lemma 4.2.** Let $y_1(x)$ and $y_2(x)$ be solutions with the same minimum value $y_1$. More precisely, $y_1(x_1) = y_2(x_2) = y_1$, $y_1'(x_1) = y_2'(x_2) = 0$, for $x_1 < x_2$. Then $a_2 > a_1$ and

$$y_1(x) < y_2(x), \quad y_1'(x) < y_2'(x) < 0, \quad \text{for } x \in (a_2, x_1).$$

**Proof.** In terms of the variables $t = -x$, $u(t) = y(x)$, the conditions read: $t_1 > t_2 > 0$, $u_1(t_1) = u_2(t_2) = y_1 =: u$, $u_1'(t_1) = u_2'(t_2) = 0$. In some open right neighbourhood of $t_1$ we have

$$u_2(t) > u_1(t) \quad \text{and} \quad u_2'(t) > u_1'(t). \quad (4.3)$$

Denote

$$\hat{a} = \sup \{t \mid \text{Condition (4.3) holds} \}.$$

Suppose $\hat{a} < \min\{a_1, a_2\}$. Then $u_1'(\hat{a}) = u_2'(\hat{a})$ and $u_2(\hat{a}) > u_1(\hat{a})$. Integrating Equation (2.1) from $t_k$, we have

$$u_k^2(\hat{a}) - 4u_k^3(\hat{a}) = -4u_1^3 + 2 \int_{t_k}^{\hat{a}} tu_k(t) \, dt$$

for $k = 1, 2$. Subtracting and rearranging the integral, we get

$$u_1^2(\hat{a}) - u_2^2(\hat{a}) = 4(u_1^3(\hat{a}) - u_2^3(\hat{a})) + 2 \int_{t_1}^{\hat{a}} t(u_1'(t) - u_2'(t)) \, dt - 2 \int_{t_2}^{\hat{a}} t u_2(t) \, dt \quad (4.4)$$
The right-hand side of Equation (4.4) is strictly negative. Hence we have $u_1'^2(\hat{a}) < u_2'^2(\hat{a})$ which is a contradiction. The remainder of the proof is the same as the proof of the previous lemma with interchange of the indices 1 and 2. \hfill \Box

**Lemma 4.3.** Let $y_1(x)$ and $y_2(x)$ be solutions satisfying the following conditions: $y_1(x_1) = y_2(x_2) = y_0$, $0 \leq y'_1(x_1) \leq y'_2(x_2)$, for $x_1 < x_2$. Then $a_2 > a_1$ and

$$y_1(x) < y_2(x), \quad y_2'(x) < y_1'(x) < 0, \quad \text{for} \quad x \in (a_2, x_1).$$

**Proof.** This is a generalization of the previous Lemma. The proof is literally the same. The only difference is an additional term, $u_1'^2(t_1) - u_2'^2(t_2) = y_1'^2(x_1) - y_2'^2(x_2) \leq 0$, which does not spoil the proof. \hfill \Box

**Definition 4.1.** Let $x_0 \leq 0$ and $x_{\text{min}}$ be the minimum of the solution $y(x)$ corresponding to initial data $(y_0, y_0')$ at $x_0$. Define

$$\mathcal{X}(x_0, y_0) = \inf_{y_0, y_0'} \left\{ a(x_0, y_0, y_0') \mid y(x_{\text{min}}) = y_0 \right\}.$$

**Proposition 4.1.** For any $x_0 \leq 0$, $y_0 \in \mathbb{R}$, $\mathcal{X}(x_0, y_0)$ is finite. The unique solution with a pole at $x_0$ and $x_{\text{min}} = \mathcal{X}_0(x_0, y_0)$ has a pole at $\mathcal{X}(x_0, y_0)$. Moreover, $\mathcal{X}(x_0, y_0)$ is a smooth function of $(x_0, y_0)$.

**Proof.** Since $X(x_0) \leq \mathcal{X}(x_0, y_0)$, it follows from Theorem 2.1 that $\mathcal{X}(x_0, y_0)$ is bounded for any $x_0 \leq 0$, $y_0 \in \mathbb{R}$. The statement that the solution with minimum at $\mathcal{X}_0(x_0, y_0)$ has a pole at $\mathcal{X}(x_0, y_0)$ follows from Theorem 3.1 and Lemma 4.2. The smoothness of $\mathcal{X}(x_0, y_0)$ is a consequence of Proposition 3.4 and Corollary 3.3. \hfill \Box

**Corollary 4.1.** For $y_l \in \mathbb{R}$, if $x_1 < x_2 < 0$ then

$$\mathcal{X}(x_1, y_l) < \mathcal{X}(x_2, y_l).$$

**Proof.** The proof follows from the monotonicity result for $\mathcal{X}_0(x_0, y_0)$ (Corollary 3.2) and Lemma 4.2. \hfill \Box

**Proposition 4.2.** The function $\mathcal{X}(x_0, y_l)$ satisfies the following inequalities,

(4.5) $x_1 < x_2 \leq 0 : \quad 0 < \mathcal{X}(x_2, y_l) - \mathcal{X}(x_1, y_l) < \mathcal{X}_0(x_2, y_l) - \mathcal{X}_0(x_1, y_l),$

(4.6) $x_0 \leq 0 : \quad \mathcal{X}_0(x_0, y_l) < \frac{1}{2} (x_0 + \mathcal{X}(x_0, y_l)).$

**Proof.** The left inequality (4.5) is proved in Corollary 4.1. The right inequality (4.5) follows from the fact that the graph of the solution starting at $\mathcal{X}_0(x_1, y_l)$ goes to its pole more steeply than the one starting at $\mathcal{X}_0(x_2, y_l)$. Analogous arguments also justify Inequality (4.6); if we denote $y_1(x) = y(x)$ for $x \in (x_0, \mathcal{X}_0(x_0, y_l)]$ and $y_2(x) = y(x)$ for $x \in [\mathcal{X}_0(x_0, y_l), \mathcal{X}(x_0, y_l))$, then $y_2(x)$ is steeper than $y_1(x)$. The formal proof is analogous to that of Proposition 3.5 with corresponding changes of notation. \hfill \Box

**Proposition 4.3.** For $|y_l| \geq \epsilon > 0$

$$0 < x_0 - \mathcal{X}(x_0, y_l) < \frac{I_\nu}{\sqrt{|y_l|}},$$

where $\nu = \text{sign} \{y_l\} 1$ and $I_\nu$ is defined by Equation (2.7).
Proof. Note that $0 < \mathcal{X}_{\min}(x_0, y_l) - \mathcal{X}(x_0, y_l) < x_0 - \mathcal{X}_{\min}(x_0, y_l)$, as the graph of the solution with poles at $x_0$ and $\mathcal{X}(x_0, y_l)$ is steeper to the left of its minimum $x_{\min} = \mathcal{X}_{\min}(x_0, y_l)$ (see Proposition 4.1) than that to the right. The formal proof is very similar to the one of Proposition 3.5. Now, the result follows from the identity, $x_0 - \mathcal{X}(x_0, y_l) < x_0 - \mathcal{X}_{\min}(x_0, y_l) + \mathcal{X}_{\min}(x_0, y_l) - \mathcal{X}(x_0, y_l)$, and Proposition 3.6. □

**Proposition 4.4.** For any $y_l \in \mathbb{R}$ and $x_0 \leq 0$,

\begin{equation}
0 \leq \frac{\partial}{\partial x_0} \mathcal{X}_{\min}(x_0, y_l) < 1, \quad 0 \leq \frac{\partial}{\partial x_0} \mathcal{X}(x_0, y_l) < 1.
\end{equation}

Moreover,

\begin{equation}
\lim_{\kappa_- + \kappa_+ \to +\infty} \frac{\partial}{\partial x_0} \mathcal{X}_{\min}(x_0, y_l) = 1, \quad \lim_{\kappa_- + \kappa_+ \to +\infty} \frac{\partial}{\partial x_0} \mathcal{X}(x_0, y_l) = 1,
\end{equation}

where $\kappa_- = x_0$, $\kappa_+ = |y_l|$.

**Proof.** The fact that the values of the derivatives belong to $[0, 1]$ follows from Propositions 3.5 and 4.2. If we consider $x_{\min}$ in Equation (3.12) as a function of $x_0$ and $y_l$, then it is nothing but $\mathcal{X}_{\min}(x_0, y_l)$. Differentiating now both parts of the equation with respect to $x_0$, we get on the left $1 - \frac{\partial}{\partial x_0} \mathcal{X}_{\min}(x_0, y_l)$ and on the right an integral similar to the one in the right part of Equation (3.13) but with $x_{\min}$ changed to $x_0$. Taking now into account that $0 \leq \frac{\partial}{\partial x_0} x(y; x_0, y_l) \leq 1$, one finds that the integral vanishes as $x_0 \to -\infty$ and/or $|y_l| \to +\infty$. The partial derivative cannot be identically equal to 0 for all $y_l$, as it implies that the partial derivative of $y(x; x_0, y_l)$ (with respect to $x_0$) identically vanishes for all $x$. This contradicts Laurent expansion (3.6). Thus, the right Inequalities (4.7) are strict.

The proof for $\frac{\partial}{\partial x_0} \mathcal{X}(x_0, y_l)$ is similar; instead of one integral in the right-hand side of the equation for $1 - \frac{\partial}{\partial x_0} \mathcal{X}(x_0, y_l)$ we get two similar integrals, and then, the same argument as above shows that both vanish as $x_0 \to -\infty$. □

**Remark 4.1.** In Proposition 5.8 of § 5 we prove that both left Inequalities (4.7) are also strict for all $x_0$ and $y_l \in \mathbb{R}$.

**Proposition 4.5.** For any $x_0 \leq 0$, there exists a solution with interval of existence $(\mathcal{X}(x_0), x_0)$. If some solution has an interval of existence $I \supset (\mathcal{X}(x_0), x_0)$, then $I = (\mathcal{X}(x_0), x_0)$.

**Proof.** By definition, we have $\inf_{y_l} \mathcal{X}(x_0, y_l) \geq \mathcal{X}(x_0)$. On the other hand, for any $\epsilon > 0$, there exists a solution $y(x)$ with initial data $y(x_0) = y_l$ and $y'(x_0) = y_1$ such that its left pole $a(x, y_0, y_1) < \mathcal{X}(x_0) + \epsilon$. Let $y_l$ be its minimum. Then $\mathcal{X}(x_0, y_l) < a(x, y_0, y_1) < \mathcal{X}(x_0) + \epsilon$. Therefore

\begin{equation}
\mathcal{X}(x_0) = \inf_{y_l} \mathcal{X}(x_0, y_l).
\end{equation}

Since $\mathcal{X}(x_0, y_l)$ is differentiable in $y_l$ and by virtue of Proposition 4.3 $\mathcal{X}(x_0, y_l) \to x_0$ as $y_l \to \pm \infty$, it achieves its supremum at some finite value $\hat{y}_l$. Actually $\hat{y}_l \leq 0$. Thus, $X(x_0) = \mathcal{X}(x_0, \hat{y}_l)$, and Proposition 4.1 implies that there is a solution with the interval of existence $(\mathcal{X}(x_0), x_0)$.

Denote by $y_l(x)$ the solution with interval of existence $I$. If $y_l(x)$ has a pole at $x_0$, then it has another pole $p \geq \mathcal{X}(x_0)$, as follows from Equation (4.9) and thus its interval of existence $I \subset (\mathcal{X}(x_0), x_0)$. 

Suppose now that \( y_1(x) \) is finite at \( x_0 \), then by Proposition 3.2 and Definition 3.1, \( \mathcal{X}_{\min}(x_0, y_1) < x_{\min} \), where \( y_1 \) and \( x_{\min} \) are the minimum value and minimum of \( y_1(x) \). Denoting \( a \) the left bound of \( I \), and applying Lemma 4.2 and Definition 4.1, we get that \( \mathcal{X}(x_0, y_1) < a \). This inequality contradicts Equation (4.9) as on the other hand \( a = X(x_0) \); by definition of \( X(x_0) \) we have \( a \geq X(x_0) \), whilst by definition of \( I \), \( a \leq X(x_0) \).

\[ \square \]

Remark 4.2. If \( I \subset (-\infty, 0] \), then the last paragraph of the proof of Proposition 4.5 can be simplified: Denote by \( b \) the right bound of \( I \), then Corollary 4.1 implies \( \mathcal{X}(x_0, y_1) < \mathcal{X}(b, y_1) \equiv a \leq X(x_0) \) and we arrive at a contradiction with Equation (4.9).

Corollary 4.2. \( x_0 \leq 0 \) : \( X(x_0) < X_{\min}(x_0) < \frac{1}{2}(x_0 + X(x_0)) \).

Proof. Immediate consequence of Propositions 4.5 and 4.2.

\[ \square \]

Proposition 4.6. The function \( X(x_0) \) is monotonically increasing with increase of \( x_0 \leq 0 \) and satisfies the Lipschitz condition,

\[ 0 < X(x_2) - X(x_1) < x_2 - x_1. \]

Proof. The proof follows from Proposition 4.2 and Proposition 3.5 by similar arguments as in the proof of Proposition 3.7.

\[ \square \]

Remark 4.3. The solution introduced in Proposition 4.5 is unique and smooth. The proof is given in § 6, see Theorem 6.2 and Corollary 6.1, respectively. Most probably instead of the Lipschitz condition proved in Proposition 4.6, a stronger inequality holds, analogous to the one proved for the level-functions in Proposition 4.2 (see Conjecture 8.5 in § 8).

5. The functions \( \Xi_{\min}(x_0, y_1), \Xi_{\max}(x_0), \Xi(x_0, y_1), \Xi(x_0), \) and \( X_{\max}(x_0) \)

Remark 5.1. In those statements below where no condition is given on \( x_0 \), we assume that \( x_0 \leq \mathcal{X}(0, y_1) \). This condition guarantees that each solution under consideration has an interval of existence which belongs to the negative semi-axis.

Proposition 5.1. Given \( y_0 > y_1 \), there exists only one solution \( y(x) \) such that \( y(x_0) = y_0 \) and \( y_{\min} = y_1 \) where \( y_{\min} \) is the minimum of \( y(x) \) achieved to the right of \( x_0 \).

Proof. The existence of the solution follows by the similar arguments as those in the first two paragraphs of the proof of Proposition 3.1, with \( t_0 \) substituted for \( t_{\min} \) in Equation (3.1). The uniqueness follows from Lemma 4.2.

\[ \square \]

Proposition 5.2. Given \( \tilde{y}_0 > y_0 > y_1 \), consider solutions \( y(x), \tilde{y}(x) \) such that \( y(x_0) = y_0, \tilde{y}(x_0) = \tilde{y}_0 \) and with the same minimum value \( y_1 = y_{\min} = \tilde{y}_{\min} \) achieved to the right of \( x_0 \). Then \( x_0 < x_{\min} < \tilde{x}_{\min} \) and \( \tilde{y}'(x_0) < y'(x_0) < 0 \).

Proof. Follows from Lemma 4.2.

\[ \square \]

Corollary 5.1. Consider the solution with initial value \( y(x_0) = y_0 \) and minimum value \( y_1 \) achieved to the right of \( x_0 \). The initial slope \( y'_0 = f^+(x_0, y_0, y_1) \) (see Remark 3.3) at \( x_0 \) is a function of \( y_0 \) and, for fixed \( y_1 \), has asymptotic behaviour

\[ \frac{y'_0}{y_0} \xrightarrow{y_0 \to +\infty} 4y_0^2 + O(1). \]
Lemma 4.2. 

Proof. Literally the same as the proof of Corollary 3.1 with the corresponding change of $x_0 \leftrightarrow x_{\min}$. \qed

Definition 5.1. Given $x_0, y_0 > y_l$, consider the solution $y(x) = y(x; x_0, y_0, y_l)$ with initial value $y(x_0) = y_0$, minimum value $y_l$ at $x_{\min} = x_{\min}(x_0, y_0, y_l) > x_0$, and the right bound of its interval of existence $b = b(x_0, y_0, y_l)$. Define

$$\Xi_{\min}(x_0, y_l) = \sup_{y_0} x_{\min}(x_0, y_0, y_l), \quad \Xi(x_0, y_l) = \sup_{y_0} b(x_0, y_0, y_l).$$

Remark 5.2. The functions $\Xi_{\min}(x_0, y_l)$ and $\Xi(x_0, y_l)$ are finitely defined for $y_l \in \mathbb{R}$, $-\infty < x_0 \leq \mathcal{X}(0, y_l)$, and satisfy evident inequalities:

$$x_0 < \Xi_{\min}(x_0, y_l) < \Xi(x_0, y_l) < 0.$$

Proposition 5.3. The solution $y_\mu(x)$ with initial data

$$y_\mu(\Xi_{\min}(x_0, y_l)) = y_l, \quad y_\mu'(\Xi_{\min}(x_0, y_l)) = 0$$

has an interval of existence $(x_0, \Xi(x_0, y_l))$. 

Proof. From Proposition 5.2 and Definition 5.1 it follows that the solution $y_\mu(x)$ has a pole $x_p, x_0 \leq x_p < \Xi_{\min}(x_0, y_l)$. The equality $x_p = x_0$ can be confirmed by making an analogous construction to that in the last paragraph of the proof of Proposition 3.3.

According to Proposition 5.2, any solution with finite initial value at $x_0$ has a minimum strictly less than $\Xi_{\min}(x_0, y_l)$. The graph of such a solution crosses the graph of $y_\mu(x)$ at a point to the left of its minimum $(\Xi_{\min}(x_0, y_l), y_l)$ and does not cross to the right of it by virtue of Proposition 3.1. Thus, the right bound of the interval of existence, $b$, of any solution finite at $x_0$ is less than $b_\mu$, the right bound of the interval of existence of $y_\mu(x)$. On the other hand, by similar arguments as above we establish that any sequence of solutions $y_n(x)$ with finite initial data $y_n(x_0) \nearrow +\infty$, has a sequence of minima $x_{\min,n} \nearrow \Xi_{\min}(x_0, y_l)$ and the sequence of the right bounds of their intervals of existence, $b_n \nearrow b_\mu$. \qed

Corollary 5.2. $x_0 = \mathcal{X}(\Xi(x_0, y_l), y_l)$. In other words, $\Xi(x_0, y_l) = \mathcal{X}^{-1}(x_0, y_l)$, where $\mathcal{X}^{-1}(x_0, y_l)$ denotes the section of the fibre $\mathcal{X}^{-1}(x_0)$ by the straight line $y = y_l$. 

Proof. By virtue of Theorem 3.1 the solution with a pole at $\Xi(x_0, y_l)$ and minimum value $y_l$ is unique. Therefore, it coincides with the solution defined in Proposition 5.3. From Proposition 4.1 it follows that this solution has a pole at $\mathcal{X}(\Xi(x_0, y_l), y_l)$ while Proposition 5.3 implies that this pole is $x_0$. \qed

Proposition 5.4. Given $x_0 \leq 0$ and $y_l \in \mathbb{R}$ and $x_0 \leq \mathcal{X}(0, y_l)$, there is only one solution with a pole at $x_0$, an interval of existence to the right of $x_0$, and minimum value is $y_l$. 

Proof. Existence of the solution follows from Proposition 5.3, uniqueness from Lemma 4.2. \qed

Theorem 5.1. Any solution with an interval of existence that is contained in the non-positive semi-axis can be uniquely characterised by the position of the left pole $x_0$ and its minimum value $y_l$. Moreover, it has the following extremal property. Its minimum $\Xi_{\min}(x_0, y_l) > x_{\min}$ where $x_{\min}$ is the minimum of any solution regular at $x_0$ with minimum value $y_l$. 

Proof. The proof follows from Propositions 5.1–5.4.

The above results can be formulated as monotonicity properties of $\Xi_{\min}(x_0, y_l)$.

**Corollary 5.3.** If $x_1 < x_2 < \mathcal{X}(0, y_l)$, where $y_l \in \mathbb{R}$, then

$$\Xi_{\min}(x_1, y_l) < \Xi_{\min}(x_2, y_l)$$

**Proof.** If $\Xi_{\min}(x_1, y_l) = \Xi_{\min}(x_2, y_l)$ then the two corresponding solutions (see Proposition 3.3) would coincide. Suppose we have $\Xi_{\min}(x_1, y_l) > \Xi_{\min}(x_2, y_l)$. Then there is a solution regular at $x_2$, with $x_{\min} > \Xi_{\min}(x_2, y_l)$. This contradicts the definition of $\Xi_{\min}(\cdot, y_l)$. \qed

**Proposition 5.5.** For any $x_0 \leq 0$, $x_1 < x_2 \leq \mathcal{X}(0, y_l)$, where $y_l \in \mathbb{R}$:

1. $\mathcal{X}_{\min}(x_0, y_l) = \Xi_{\min}(\mathcal{X}(x_0, y_l), y_l) < x_0$.
2. $0 < x_2 - x_1 < \Xi_{\min}(x_2, y_l) - \Xi_{\min}(x_1, y_l) < \Xi(x_2, y_l) - \Xi(x_1, y_l)$.
3. $\Xi_{\min}(x_0, y_l) < \frac{1}{2} (x_0 + \Xi(x_0, y_l))$.

**Proof.** Equation (5.1) holds because the solutions corresponding to its left- and right-hand sides coincide. To prove Inequality (5.2) it is enough to notice that the graph of the solution starting at $\Xi_{\min}(x_1, y_l)$ goes to its pole more steeply than the one starting at $\Xi_{\min}(x_2, y_l)$. A similar argument works to prove Inequality (5.3); a graph of the solution to the left of $\Xi_{\min}(x_0, y_l)$ is steeper than that to the right. The formal proof of Inequalities (5.2) and (5.3) is analogous to that of Proposition 3.5 with corresponding changes of notation. \qed

**Remark 5.3.** For $x_0 \leq \mathcal{X}(0, y_l)$ we also have $\Xi_{\min}(x_0, y_l) = \mathcal{X}_{\min}(\Xi(x_0, y_l), y_l) < 0$.

**Remark 4.** A solution with a pole at $x_0$ and interval of existence to the right of it, is uniquely characterised on the one hand by the parameter $c$ in the Laurent expansion (3.6) and on the other hand, by its minimum $y_l$ achieved at $x_{\min} = x_{\min}(x_0, c)$. Therefore, there exists a bijection $y_l(c) : \mathbb{R} \to \mathbb{R}$.

**Corollary 5.4.** For any $c$, $y_l(c)$ is a smooth monotonically growing function of $c$. The inverse function $c(y_l)$ has the same properties. Moreover, $y_l(c) \to \pm \infty$ as $c \to \pm \infty$. For any $x_0 \leq \mathcal{X}(0, y_l)$, $x_{\min}(c) = x_{\min}(x_0, c)$ is a smooth function of $c$.

**Proof.** The proof is analogous to the one for Corollary 3.3. \qed

**Corollary 5.5.** The functions $\Xi_{\min}(x_0, y_l)$ and $\Xi(x_0, y_l)$ are smooth functions of both variables.

**Proof.** Follows from Propositions 5.3, 3.4 and Corollary 5.4. \qed

**Definition 5.2.** For $x_0 \leq X(0)$ define

$$\Xi_{\min}(x_0) = \sup_{y_0, y_l} x_{\min}(x_0; y_0; y_l), \quad \Xi(x_0) = \sup_{y_0, y_l} b(x_0; y_0; y_l)$$

where $x_{\min}(x_0; y_0; y_l)$ and $b(x_0; y_0; y_l)$ are the same as in Definition 5.1.

**Proposition 5.6.** For any $x_0 \leq X(0)$: (1) There exists a solution of Equation (1.1) that has a minimum at $\Xi_{\min}(x_0)$. Any such solution has a pole at $x_0$; (2) There exists a solution with the interval of existence $(x_0, \Xi(x_0))$. \qed
Proof. Taking into account that $\Xi_{\min}(x_0, y_1) - x_0 > 0$ and $\Xi(x_0, y_1) - x_0 > 0$ vanish as $|y_1| \to \infty$ (Proposition 4.3 and Corollary 5.3) and continuity of $\Xi_{\min}(x_0, y_1)$ and $\Xi(x_0, y_1)$ with respect to $y_1$, one proves the following representations for $\Xi_{\min}(x_0)$ and $\Xi(x_0)$:

$$\Xi_{\min}(x_0) = \max_{y_1} \Xi_{\min}(x_0, y_1), \quad \Xi(x_0) = \max_{y_1} \Xi(x_0, y_1)$$

The maxima are achieved at some finite values of $y_1$. \hfill \Box

Remark 5.5. We conjecture that the solution from Proposition 5.6 corresponding to $\Xi_{\min}(x_0)$ is unique; the argument is analogous to the one in Remark 3.8. The solution corresponding to $\Xi(x_0)$ is unique. This follows from Proposition 5.7 and Theorem 6.2.

**Proposition 5.7.** $\Xi_{\min}(X_-(x_0)) = x_0$, $X_-(\Xi_{\min}(x_0)) = x_0$, $\Xi(X(x_0)) = x_0$, $X(\Xi(x_0)) = x_0$, i.e., the functions $\Xi_{\min}(x_0)$ and $\Xi(x_0)$ are inverse to $X_-(x_0)$ and $X(x_0)$ respectively.

Proof. Lemma 4.1 implies that in the definition of $X_-(x_0)$ (see Definition 2.2) the infimum is achieved for $y_1 = 0$. Moreover, for any $x_0 \leq 0$ there exists a solution $y(x)$ with $y'(x_0) = 0$ and a pole at $X_-(x_0)$. Actually, denote by $y_n$ a sequence of initial data at $x_0$ such that the corresponding solutions $y_n(x)$ ($y_n(x_0) = y_n$ and $y_n'(x_0) = 0$) have the left bounds $a_n \to X_-(x_0)$. The sequence $y_n$ is bounded as $0 < x_0 - a_n < x_0 - X(x_0) < I_0/|y_n|$ (see Proposition 4.3). So there is a convergent subsequence $y_{nk} \to \hat{y}_n$. The solution $y(x)$ with initial data $y(x) = \hat{y}_n$ and $y'(x_0) = 0$ has a pole at $X_-(x_0)$.

It follows from the previous paragraph that $\Xi_{\min}(X_-(x_0)) \geq x_0$. Suppose that $\Xi_{\min}(X_-(x_0)) > x_0$. Then, as it follows from Proposition 5.6, there is a solution $y_1(x)$ with the pole at $X_-(x_0)$ whose graph crosses the vertical line $x = x_0$ with a slope $y_1'(x_0) < 0$. According to Lemma 4.1 a solution $y_2(x)$ with the initial data $y_2(x_0) = y_1(x_0)$ and $y_2'(x_0) = 0$ has a left bound of the interval of existence less than $X_-(x_0)$; this contradicts the definition of $X_-(x_0)$. Now, we prove that $X_-(\Xi_{\min}(x_0)) = x_0$. Proposition 5.6 and the definition of the function $X_-(\cdot)$ imply that $X_-(\Xi_{\min}(x_0)) \leq x_0$. If we suppose that $X_-(\Xi_{\min}(x_0)) < x_0$, then this means that there exist a solution finite at $x_0$ with the minimum at $\Xi_{\min}(x_0)$. This contradicts the definition of $\Xi_{\min}(x_0)$ by virtue of Proposition 5.2. The proof that $X(x_0) = \Xi^{-1}(x_0)$ is very similar. \hfill \Box

**Corollary 5.6.** The functions $\Xi_{\min}(x_0)$ and $\Xi(x_0)$ are continuous and satisfy the following inequalities:

$$\Xi_{\min}(X(x_0, y_1)) \geq \Xi_{\min}(x_0, y_1), \quad \Xi_{\min}(X(x_0)) \geq \Xi_{\min}(x_0),$$

$$0 < x_2 - x_1 < \Xi_{\min}(x_2) - \Xi_{\min}(x_1), \quad \Xi_{\min}(x_0) < \frac{1}{2}(x_0 + \Xi(x_0)).$$

Proof. The continuity follows from the fact that both functions, $\Xi(x_0)$ and $\Xi_{\min}(x_0)$ are the inverses of the continuous monotonic functions $X(x_0)$ and $X_-(x_0)$, respectively (concerning $X_-(x_0)$ see Remark 5.6). The inequalities follow from Equation (5.1), Inequalities (5.2) and (5.3), and Equations (5.4). \hfill \Box
Remark 5.6. The function \( X_-(x_0) \) has very similar properties to those of the function \( X_{\min}(x_0) \). To prove this one can develop the technique based on the introduction of the level function, \( X_-(x_0, y_1) \), using Lemmas 4.1-4.3 as the main instrument. Analogously, one can define a function \( \Xi_+(x_0) \) as

\[
\Xi_+(x_0) := \sup_{y_0 \in \mathbb{R}, y_1 \geq 0} \{b(x_0, y_0, y_1)\},
\]

where \( b(x_0, y_0, y_1) > x_0 \) is the right bound of the interval of existence of solution \( y_+(x; x_0, y_0, y_1) \). The function \( \Xi_+(x_0) \) is the inverse to \( X_{\min}(x_0) \) and has properties similar to those of \( \Xi_{\min}(x_0) \). The proof again can be based on the corresponding level function, \( \Xi_+(x_0, y_1) \) with Lemma 3.1 as the main instrument. None of these constructions require any new ideas and we omit them.

Introduction of these functions has not only some notational convenience, as we see below in Proposition 5.8, it allows us to prove an important property of the derivatives of \( \lambda^- \) and \( \lambda \)-functions, that otherwise would be difficult to establish directly by working with the corresponding integral equation.

**Proposition 5.8.** For any \( y_1 \in \mathbb{R} \) and \( x_0 \leq 0 \),

\[
0 < \frac{\partial}{\partial x_0} \lambda_{\min}(x_0, y_0) < 1, \quad 0 < \frac{\partial}{\partial x_0} \lambda(x_0, y_0) < 1.
\]

For any \( y_1 \in \mathbb{R} \) and \( x_0 \leq \lambda(x_0, y_1) \),

\[
1 < \frac{\partial}{\partial x_0} \lambda_{\min}(x_0, y_0) < +\infty, \quad 1 < \frac{\partial}{\partial x_0} \lambda(x_0, y_0) < +\infty.
\]

**Proof.** We have only to prove that the derivatives are strictly positive as the rest is proved in Proposition 4.4. Consider the functions \( \lambda(x_0, y_0) \) and \( \Xi(x_0, y_0) \). According to Corollary 5.2 these functions are mutually inverse. Proposition 4.1 and Corollary 5.5 imply that they are smooth for all values of \( x_0 \) and \( y_1 \). Thus, their derivatives cannot vanish. The proof for \( \lambda_{\min}(x_0, y_0) \) and \( \lambda_{\min}(x_0, y_0) \) is similar with the help of the functions \( \Xi_+(x_0, y_0) \) and \( X_-(x_0, y_0) \), respectively (see Remark 5.6).

**Definition 5.3.**

\[
L_{\Xi_{\min}} := \sup_{y_1 \in \mathbb{R}, x_0 \leq \lambda(0, y_1)} \frac{\partial}{\partial x_0} \Xi_{\min}(x_0, y_0), \quad L_{\Xi} := \sup_{y_1 \in \mathbb{R}, x_0 \leq \lambda(0, y_1)} \frac{\partial}{\partial x_0} \Xi(x_0, y_0).
\]

**Proposition 5.9.** The numbers \( L_{\Xi_{\min}} \) and \( L_{\Xi} \) are finite and satisfy the following inequalities:

\[
1 < L_{\Xi_{\min}} \leq L_{\Xi} < +\infty.
\]

Moreover, for \( x_0 \leq \hat{x}_0 \leq \lambda(0, y_1) \),

\[
0 < \hat{x}_0 - x_0 < \Xi_{\min}(\hat{x}_0, y_0) - \Xi_{\min}(x_0, y_0) < L_{\Xi_{\min}}(\hat{x}_0 - x_0),
\]

\[
0 < \hat{x}_0 - x_0 < \Xi(\hat{x}_0, y_0) - \Xi(x_0, y_0) < L_{\Xi}(\hat{x}_0 - x_0).
\]

**Proof.** Consider \( \frac{\partial}{\partial x_0} \lambda(x_0, y_0) \). It is a bounded continuous function in the whole left half-plane \( x_0 \leq 0 \). It follows from Proposition 4.4 (see Equation (4.8)), that the infimum of this function, which is strictly less than 1, is achieved at some finite point of the half-plane. This infimum is a positive number, as \( \frac{\partial}{\partial x_0} \lambda(x_0, y_0) > 0 \) at any finite point according to Proposition 5.8. Exactly the same conclusion is valid for the domain \( (x_0, y_1) : y_1 \in \mathbb{R}, x_0 \leq \lambda(0, y_1) \). In this domain \( 1 > \inf \frac{\partial}{\partial x_0} \lambda(x_0, y_0) = 1/L_{\Xi} > 0 \). Now from Inequalities (5.2) follows that \( 1 \leq L_{\Xi_{\min}} \leq L_{\Xi} < +\infty \). The
fact that $L_{\text{even}} \neq 1$ is clear from the first Inequality (5.9). The uniform bounds for the derivatives imply the Lipschitz estimates in Inequalities (5.10) and (5.11).

Remark 5.7. In § 6, Corollary 6.1 and Remark 6.2, we prove that the functions $X(x_0)$ and $\Xi(x_0)$ are smooth and in Corollary 6.2 state the properties of their derivatives similar to those for the corresponding level functions proved above.

6. GEOMETRY OF INTEGRAL CURVES

Definition 6.1. The graphs of two solutions are said to intersect at the point at infinity iff these solutions have a common pole, i.e., if their intervals of existence have a common end point.

Theorem 6.1. The graphs of any two (different) solutions of Equation (1.1) have at most two points of intersection, counting the points at infinity, with non-positive abscissas.

Proof. Given two intersecting solutions, we count intersection points starting from the one with abscissa closest to the origin and counting to the left. The first intersection point exists. To prove this, denote by $x_0 \leq 0$ the supremum of abscissas of the set of intersection points. If $x_0$ belongs to the interval of existence of both solutions, then it is also an intersection point by continuity. Suppose that $x_0$ is a pole of one of the solutions, then it should be also a pole of the other. From the Laurent expansion (3.6) it follows that any two solutions with the pole at the same point do not intersect in some neighbourhood of this point. Therefore in the latter case $x_0$ is an isolated intersection point at infinity.

Now, we have to consider several different cases:

(1) The solutions have a common pole: one solution has the interval of existence to the left of the pole and the other to the right. The theorem is obviously true.

(2) The solutions have a common pole at the right end of their intervals of existence. Then they do not intersect until both of them achieve their minimum values (see Lemma 3.1). One of these minimum values is larger than the other and we call the corresponding solution the upper-solution. The finite point of intersection may occur at the minimum value of the upper solution or after it. In the first case we use Lemma 4.3 and in the second Lemma 4.1 to prove that there are no other intersection points.

(3) The first point of intersection is finite and solutions at the intersection point have positive derivatives. The respective location of graphs of the solutions after the first intersection is the same as in the second item.

(4) At the first intersection point one solution has a positive derivative and the other non-positive. Then the second intersection point may occur and at the latter point, both solutions have negative derivative and we use Lemma 4.1 to prove that there are no other intersection points.

(5) At the first intersection point one solution has non-positive derivative and the other a negative one. These solutions have no other intersection points according to Lemma 4.1.

Lemma 6.1. Let $y_k(x)$ for $k = 1, 2, 3$, be solutions of Equation (1.1) whose intervals of existence contain the segment $[\alpha, \beta] \in (-\infty, 0]$. Suppose that: (i)
Proof. If the conditions $y_1(x) > y_3(x)$ and $y_2(x) > y_3(x)$ hold for all $x \in [\alpha, \beta]$, then this statement is a particular case of a general lemma proved by Moore and Nehari (Lemma 2 in [13]). Therefore, we consider only the case when all three solutions intersect at $\alpha$ and/or $\beta$, since for all other possibilities our Lemma follows from the one by Moore and Nehari [13] by applying the latter to possibly a smaller segment $[\alpha', \beta'] \subset [\alpha, \beta]$.

Suppose $y_k(\alpha) = y_0$ for $k = 1, 2, 3$ and solutions $y_1(x)$ and $y_2(x)$ have one more point of intersection $\gamma \in (\alpha, \beta)$. For definiteness, let $y_1(x) > y_2(x)$ for $x \in (\alpha, \gamma)$. In that case for $\epsilon > 0$ consider a family of solutions $y_\epsilon(x)$ with the initial data $y_\epsilon(\alpha) = y_3(\alpha) - \epsilon$, $y'_\epsilon(\alpha) = y'_3(\alpha)$. We claim that for all rather small $\epsilon$, $y_2(x) > y_\epsilon(x)$ for $x \in (\alpha, \gamma)$. Otherwise, there is a sequence $\epsilon_n \to 0$ such that solutions $y_{\epsilon_n}(x)$ cross $y_2(x)$ at least twice, at the points: $x_2 > x_1 \in (\alpha, \gamma)$, say, as $y_{\epsilon_n}(x)$ have to be close to $y_3(x)$ and they cannot be tangent to $y_2(x)$. Before crossing $y_2(x)$ they have to cross $y_3(x)$ as $y_3(x)$ separated them from $y_2(x)$. Therefore, on the segment $[x_1, x_2]$ we have $y_2(x) > y_3(x)$ and $y_{\epsilon_n}(x) > y_3(x)$ for all large enough $n$. This contradicts the standard Moore–Nehari Lemma (see the first paragraph of this proof) as solutions $y_2(x)$ and $y_{\epsilon_n}(x)$ have two points of intersection in $[x_1, x_2]$. Thus, the triple of solutions $y_1(x)$, $y_2(x)$ and $y_\epsilon(x)$ for any sufficiently small $\epsilon$ satisfies the standard Moore–Nehari Lemma on $[\alpha, \gamma]$. This proves that $\gamma \notin (\alpha, \beta)$, since $y_1(\alpha) = y_2(\alpha)$.

We are left with the case when $y_1(x) > y_2(x) > y_3(x)$ for $x \in (\alpha, \beta)$ and $y_1(x) = y_2(x) = y_3(x)$ at $x = \alpha, \beta$. In this case consider a solution $y_4(x)$ with $y_4(\alpha) = y_k(\alpha)$ and $y'_4(\alpha) > y'_k(\alpha) > y'_3(\alpha)$. This solution cannot cross $y_2(x)$ for $x \in (\alpha, \beta)$ as follows from the previous paragraph (for a triple $y_2(x) \geq y_4(x) \geq y_3(x)$). Suppose $y_4(x)$ crosses $y_3(x)$ at $x = \gamma < \beta$. In this case $y_4(x) < y_3(x)$ for $x \in (\gamma, \beta)$ as it follows from Theorem 6.1. Therefore, we arrive at the situation considered in the second paragraph of this proof but for the triple $y_1(x) \geq y_2(x) \geq y_4(x)$ for $x \in [\alpha, \beta]$. The only situation that is left is that for all slopes $y'_4(\alpha)$: $y'_4(\alpha) \geq y'_3(\alpha)$ solution $y_4(x)$ is such that $y_4(\beta) = y_1(\beta)$, i.e., $y_4(\beta)$ is independent of the initial slope $y'_4(\alpha) \in [y'_3(\alpha), y'_2(\alpha)]$. Taking into account that $y_4(\beta)$ is actually a complex analytic function of $y'_4(\alpha)$ we get that $y_4(\beta) = y_1(\beta)$ for all $y'_4(\alpha) \in \mathbb{C}$. That cannot be the case as for $y'_4(\alpha) \to +\infty y_4(x)$ has a pole $x_p \to \alpha$.

Consider now the case $y_k(\beta) = y_1$ for $k = 1, 2, 3$. Without loss of generality we suppose now that $y_1(\alpha) > y_2(\alpha) > y_3(\alpha)$. According to the standard Moore–Nehari lemma $y_1(x)$ and $y_2(x)$ has at most one intersection point $\gamma \in (\alpha, \beta)$. Then we have $y_2(x) > y_1(x) > y_3(x)$ for $x \in (\gamma, \beta)$ and $y_1(\gamma) = y_2(\gamma) > y_3(\gamma)$. Now consider an auxiliary solution $y_4(x)$ with initial data $y_4(\gamma) = y_1(\gamma) = y_2(\gamma)$ and $y'_4(\gamma) > y'_3(\gamma) > y'_2(\gamma)$. The graph of this solution cannot cross either $y_1(x)$ or $y_2(x)$ on the interval $[\gamma, \beta)$ (by the standard Moore–Nehari lemma). Thus, $y_4(\beta) = y_1$ and we arrive to the situation considered in the previous paragraph.

Remark 6.1. In Lemma 6.1 “the point of intersection” is supposed to be a finite point of intersection. However this Lemma can be generalized to include the points of intersection at infinity in the sense of Definition 6.1. In the latter case we have to allow $\alpha$ (or $\beta$) to be the left (or right) bound of the interval of existence of one of
the functions $y_k(x)$, $k = 1, 2, 3$. Otherwise all points of intersection are finite. If $\alpha$ ($\beta$) is the left (right) bound of the function $y_3(x)$, then inequality $y_k(x) \geq y_3(x)$ is violated at $\alpha$ ($\beta$): we formally have $y_1(\alpha) = y_2(\alpha) = y_3(\alpha) = +\infty$ (the same condition at $\beta$). As proved below in this situation we have to count the point at infinity with abscissa $\alpha$ ($\beta$) as the intersection point. We call Lemma 6.1 equipped with the above extended conditions as Projective Lemma 6.1. The proof is given below after Theorem 6.2 and Lemma 6.2.

**Theorem 6.2.** For any $x_0 \leq 0$ there is a unique solution with the interval of existence $(X(x_0), x_0)$.

**Proof.** Suppose there are at least two such solutions $Y_1(x)$ and $Y_2(x)$. Denote by $Y_{k}$, $k = 1, 2$ their respective minimum values. Suppose $Y_{11} > Y_{12}$. From Lemmas 3.1, 4.1, and 4.3 it follows that solutions do not intersect. Therefore, $Y_1(x) > Y_2(x)$ for $x \in (X(x_0), x_0)$. Suppose there is a solution, $Y_0(x)$ with the pole at $x_0$ with the minimum value $Y_0 \in (Y_{11}, Y_{12})$, such that the left bound of its interval of existence is larger than $X(x_0)$. Therefore, the graph of $Y_0(x)$ crosses at some finite point, $z$, the graph of $Y_1(x)$, however, it does not cross the graph of $Y_2(x)$. This crossing point, $z$, may occur before, after, or exactly at the minimum of $Y_1(x)$.

Consider now a solution $y(x)$ with the finite initial value $y(x_0) = y_0 > Y_1$ and minimum value $Y_{11}$ at $x = x_{\min}$. Denote by $x_{\min}$ the minimum of $Y_1(x)$. Recall that $x_{\min} \setminus x_{\min}$ as $y_0 \to +\infty$. If $x_{\min} < z$, then increasing, if necessary $y_0$, we can assume that $x_{\min} < x_{\min} < z$. In this case, the graph of $y(x)$ first crosses the graph of $Y_0(x)$ at some point with abscissa $\beta$. Then it crosses the graph of $Y_0(x)$ at some point with abscissa $x_1 < \beta$. At $\beta$ and $x_1$ both solutions $y(x)$ and $Y_0(x)$ have positive derivatives. If one of these crossings does not occur, say $x_1$, then the graph of $y(x)$ for $x \in [x_{\min}, x_0]$ will be located below the graph of $Y_0(x)$. Therefore, the minimum point of $y(x)$ will be on finite distance from the minimum point of $Y_1(x)$, despite $y_0 \to +\infty$. At point $z$, however, $Y_0'(z) < 0$, as $Y_0(x)$ and $Y_1(x)$ have the same pole $x_0$: they cannot, according to Lemma 3.1, intersect both having positive derivatives. Solution $Y_0(x)$ has the minimum value $Y_0 < Y_{11}$, therefore, to have a negative derivative at $z$ it should cross the level $Y_{11}$ at some point with abscissa $z_1 > z$ with $Y_0'(z_1) < 0$. Thus, the graph of $y(x)$ crosses the graph of $Y_0(x)$ at a point with abscissa $x_2 \in (z, z_1)$. Moreover, $x_2 \neq x_1$ since $Y_0'(x_1)Y_0'(x_2) < 0$.

In case $z \leq x_{1\min}$ the second crossing point, with abscissa $x_2$, of $y(x)$ and $Y_0(x)$ occurs in the domain above the graph of $Y_1(x)$, where both derivatives $Y_0'(x)$ and $y'(x)$ are negative. Note, that the first intersection point always belongs to the domain below the graph of $Y_1(x)$. More precisely, in this case, the graph of $y(x)$ crosses the graph of $Y_1(x)$ at some point with abscissa located in the interval $(x_{1\min}, x_{\min})$. It cannot cross $Y_1(x)$ anymore according to Lemma 4.2. However, for any $\varepsilon_1, \varepsilon_2 > 0$ for all large enough $y_0$ we have $0 < y(x) - Y_1(x) < \varepsilon_1$ for $x \in [X(x_0) + \varepsilon_2, x_{1\min}]$. For all rather small $\varepsilon_2$, $z \in [X(x_0) + \varepsilon_2, x_{1\min}]$, thus $0 < y(z) - Y_0(z) < \varepsilon_1$. On the other hand, $Y_0(x)$ has a pole at $x = [X(x_0) + \varepsilon_2, x_{1\min}]$, therefore the difference $y(x) - Y_0(x) < 0$ as $x \not\in \alpha$. This proves existence of the second intersection point with abscissa $x_2$.

Thus, in all cases when $Y_0(x)$ has its interval of existence less than $(X(x_0), x_0)$, we constructed three solutions: $y(x)$, $Y_0(x)$, and $Y_2(x)$, such that $y(x) > Y_2(x)$ and $Y_0(x) > Y_2(x)$ for $x \in [\alpha, \beta - \varepsilon]$, with small enough $\varepsilon$, and the graphs of
solutions $y(x)$ and $Y_0(x)$ have two (finite) intersection points with abscissas $x_1$ and $x_2 \in [\alpha, \beta - \epsilon]$. This contradicts Lemma 6.1.

Therefore, all solutions with the pole at $x_0$ and minimum value $Y_0 \in [Y_2, Y_1]$ have the interval of existence $(X(x_0), x_0)$ and their graphs do not intersect. Recall that $\mathcal{X}(x_0, y_l)$ is a (complex) analytic function of the pole parameter $c$; it coincides with the function $x_1 = x_1(x_0, c_0)$ for $c = c_0$ introduced in Proposition 3.4. The fact that $\mathcal{X}(x_0, y_l(c)) = X(x_0)$ on some segment $c \in [c_1, c_2]$ where $c_1 < c_2 \in \mathbb{R}$ implies that $\mathcal{X}(x_0, y_l(x_0, c)) = X(x_0)$ for all $c \in \mathbb{C}$ and, in particular, for $c \in \mathbb{R}$. This contradicts the fact that $\mathcal{X}(x_0, y_l(x_0, c)) \to x_0$ as follows from Corollary 3.3 and Proposition 4.3.

**Corollary 6.1.** The function $X(x_0)$ is smooth. There exists a continuous function $\hat{y}_l(x_0)$, such that $X'(x_0) = \frac{\partial}{\partial x_0} \mathcal{X}(x_0, \hat{y}_l(x_0))$.

**Proof.** In Proposition 4.5 it was proved that for any $x_0 \leq 0$ there exists $\hat{y}_l$ such that $X(x_0) = \mathcal{X}(x_0, \hat{y}_l)$. Now by Theorem 6.2 we can define a single valued function $\hat{y}_l(x_0)$ with the property $X(x_0) = \mathcal{X}(x_0, \hat{y}_l(x_0))$. The function $\hat{y}_l(x_0)$ is continuous. Actually, consider a sequence $x_n \to x_0$. The sequence $\hat{y}_l(x_n)$ is bounded, since for an unbounded subsequence $\hat{y}_l(x_{n_k})$ we would have $X(x_{n_k}, \hat{y}_l(x_{n_k})) - x_{n_k} \to 0$ (see Proposition 4.3) rather than to a finite value about $2C/|x_0|^{1/4}$ (cf. Theorem 2.1). If we suppose that there is a subsequence $\hat{y}_l(x_{n_m})$ that does not converge to $\hat{y}_l(x_0)$, then we get that a subsequence $\hat{y}_l(x_{n_{m_k}})$ converges to some number $\hat{y} \neq \hat{y}_l(x_0)$. The functions $\mathcal{X}(x_0, y_l)$ are smooth with respect to both variables (see Proposition 4.1), therefore $X(x_0) = \mathcal{X}(x_0, \hat{y}_l(x_0)) = \mathcal{X}(x_0, \hat{y})$ in contradiction with Theorem 6.2.

In the following part of the proof we consider, as at the end of the proof of Theorem 6.2, the analytic functions $\mathcal{X}(x_0, y_l(x_0, c))$, which we denote for simplicity as $\mathcal{X}(x_0, c)$. Note that from the previous paragraph via the smooth bijection $c(x_0, y_l)$ (see Corollary 3.3), we have a continuous function $\hat{c}(x_0) = c(x_0, \hat{y}_l(x_0))$ with the property $X(x_0) = \mathcal{X}(x_0, \hat{c}(x_0))$. Consider variation of the function $X(x_0)$ under the shift $x_0 + \Delta x_0$,

$$
\Delta X(x_0) = \frac{\partial}{\partial x_0} \mathcal{X}(\xi, \hat{c}(x_0 + \Delta x_0)) \Delta x_0 + \frac{\partial}{\partial c} \mathcal{X}(x_0, \eta) \Delta \hat{c},
$$

where the numbers $\xi \in (x_0, x_0 + \Delta x_0)$, $\eta \in (\hat{c}(x_0 + \Delta x_0), \hat{c}(x_0))$, and $\Delta \hat{c} = \hat{c}(x_0 + \Delta x_0) - \hat{c}(x_0)$, where for definiteness we assume $\Delta x_0 > 0$ (the case $\Delta x_0 < 0$ differs only by an obvious modification of the notation). Since $\hat{c}(x_0)$ is a continuous function there is a point $x_0 + \Delta x_1 \in (x_0, x_0 + \Delta x_0)$ such that $\hat{c}(x_0 + \Delta x_1) = \eta$. From the definition of the function $\hat{c}(x_0)$ it follows that: $\frac{\partial}{\partial c} \mathcal{X}(x_0, \hat{c}(x_0)) = 0$ and $\frac{\partial^2}{\partial x_0 \partial c} \mathcal{X}(x_0 + \Delta x_1, \eta) = 0$. The last equation can be rewritten as

$$
\frac{\partial^2}{\partial x_0 \partial c} \mathcal{X}(\xi_1, \eta) \Delta x_1 + \frac{\partial}{\partial c} \mathcal{X}(x_0, \eta) = 0,
$$

where $\xi_1 \in (x_0, x_0 + \Delta x_1)$. Now substituting $\frac{\partial}{\partial c} \mathcal{X}(x_0, \eta)$ from the last equation into Equation 6.1 and taking into account that $0 < \Delta x_1 < \Delta x_0$, $\hat{c}(x_0)$ is a continuous function, and $\mathcal{X}(x_0, c)$ is analytic in both variables, and taking a limit $\Delta x_0 \to 0$, we arrive at the announced result. \qed

**Remark 6.2.** Only notational modifications to the above proof are required to prove the smoothness of $\Xi(x_0)$. If the uniqueness Conjectures 8.2 and 8.3 of § 8 are
Corollary 6.2. For all \( x_0 \leq 0 \):

\[
0 < X'(x_0) < 1, \quad \lim_{x_0 \to -\infty} X'(x_0) = 1,
\]

For all \( x_0 \leq X(0) \):

\[
0 < \Xi'(x_0) \leq L_\Xi, \quad \lim_{x_0 \to -\infty} \Xi'(x_0) = 1.
\]

Proof. The inequalities and the limit for \( X'(x_0) \) follows from the formula for this derivative in terms of the partial \( x_0 \)-derivative of the corresponding level function (see Corollary 6.1), and Inequalities (5.8) and double limit (4.8), respectively.

The limit of \( \Xi'(x_0) \) as \( x_0 \to -\infty \) is the same as the limit of \( X'(x_0) \) by virtue of the formula \( X'(\Xi(x_0))\Xi'(x_0) = 1 \) and the fact that \( \Xi(x_0) \to -\infty \) when \( x_0 \to -\infty \). To prove the inequalities we recall that, as mentioned in Remark 6.2, there is an analogous representation for \( \Xi'(x_0) \) as the one for \( X'(x_0) \): 

\[
\Xi'(x_0) = \frac{\partial}{\partial x_0} \Xi(x_0, \tilde{y}(x_0)),
\]

where \( \tilde{y}(x_0) \) is a continuous function. To finish the proof we refer to Propositions 5.8, 5.9.

Remark 6.3. We expect that the functions \( X'(x_0) \) and \( \Xi'(x_0) \) are monotonic.

Lemma 6.2. Any solution \( y(x) \) of Equation (1.1) with the interval of existence \((a, b)\) can be presented (non-uniquely) as a limit of a sequence of solutions \( y_n(x) \) that are regular at point \( a \) (or \( b \)) with initial data \( y_n(a) \to +\infty \) and \( y_n'(a) \to +\infty \) (\( y_n(b) \to -\infty \) and \( y_n'(b) \to +\infty \)). More precisely, this means that for any \( \epsilon, \delta > 0 \) and \( x \in [a + \delta, b - \delta], \) there is \( N \) such that for all \( n \geq N : |y(x) - y_n(x)| < \epsilon. \)

Proof. From Theorem 3.1 one deduces that the minimum of any solution with the pole at \( b \) and minimum value \( y_l \) can be presented as \( X_{\min}(b, y_l). \) Using Definition 3.1 and Proposition 3.1 one constructs, for any sequence \( y_n(b) \to +\infty \) as \( n \to +\infty, \) a sequence of solutions with the properties stated in this Lemma. These properties follow from Proposition 3.2, Corollary 3.1, and standard facts known about dependence of the solution on the initial data. Existence of the sequence under discussion at point \( a \) follows by analogous arguments from the results collected in §5.

Remark 6.4. In fact one can suggest many other constructions of the sequences satisfying the requirements of Lemma 6.2. Say, consider any point \((x_0, y(x_0))\) of the graph of \( y(x) \) with \( y'(x_0) > 0 \) \( (y'(x_0) < 0). \) Take any sequence of numbers \( a_n \to y'(x_0). \) Consider a sequence of solutions \( y_n(x) \) with initial data \( y_n(x_0) = y(x_0) \) and \( y_n'(x_0) = a_n. \) All these solutions are regular at \( b \) (or \( a \)), this follows from Lemma 3.1 (Lemma 4.1), and have the properties stated in Lemma 6.2.

Proof of Projective Lemma 6.1. We have to consider several cases:

1. We have three solutions \( y_k(x), \) with the graphs located as in Lemma 6.1. Solutions \( y_1(x) \) and \( y_2(x) \) have a pole at \( x = \beta \) and one more finite point of intersection at \( x_1 < \beta. \) Suppose, first, that \( y_3(\beta) \) is finite. Then, consider two sequences of solutions \( y^*_k(x), \) \( k = 1, 2 \) and \( n = 1, 2, \ldots, \) regular at \( x = \beta \) and approximating respectively the given solutions \( y_k(x), \) in the sense of Lemma 6.2. To shorten the proof, we suppose that \( y^*_1(\beta) = y^*_2(\beta) \to +\infty: \) it follows from Proposition 3.1 that it is always possible. (However, this assumption is not important and one can
modify this proof by considering more general sequences of solutions and using only Lemma 6.2.) For any \( n \), solutions \( y_1^n(x) \) and \( y_2^n(x) \) have one point of intersection \( \beta \). Then increasing \( n \), we see that there exist \( N \in \mathbb{N} \) such that every solution \( y_2^N(x) \) will be intersecting with all solutions \( y_2^m(x) \), \( n, m \geq N \) near the finite point of intersection \( x_1 \). We apply Lemma 6.1 to get a contradiction.

Suppose that \( y_2(x) \) also has a pole at \( \beta \). We first approximate it by a solution \( \hat{y}_2(x) \) finite at \( x = \beta \) such that the graph of the latter solution lies below the graphs of both solutions \( y_1(x) \) and \( y_2(x) \). After that we have a situation considered in the previous paragraph.

(2) In the case when \( y_1(x) \) and \( y_2(x) \) have an intersection at the point of infinity at the left bound of their interval of existence and a finite intersection point to the right of it, the proof is analogous to that in case (1).

(3) Consider the situation when solutions \( y_1(x) \) and \( y_2(x) \) have two points of intersection at infinity with the abscissas \( \alpha \) and \( \beta \), i.e., they have the common interval of existence \( (\alpha, \beta) \). The interval of existence of \( y_3(x) \) in this case includes \( (\alpha, \beta) \). According to Theorem 6.1 \( y_1(x) \) and \( y_2(x) \) have no other finite points of intersection. Suppose that \( y_1(x) > y_2(x) \). A solution \( y_0(x) \) with the pole at \( \alpha \) or \( \beta \) that have a minimum value (a pole parameter) between minimum values of \( y_1(x) \) and \( y_2(x) \) (pole parameters corresponding to \( y_1(x) \) and \( y_2(x) \)) does not intersect \( y_3(x) \) before it crosses \( y_2(x) \). If such intersection happens then for the triple, \( y_0(x), y_2(x) \) and \( y_3(x) \), we have a situation considered in the previous paragraphs. Therefore, for the triple \( y_0(x), y_1(x) \) and \( y_2(x) \) we meet a situation considered in the last paragraph of the proof of Theorem 6.2.

\[ \square \]

**Theorem 6.3.** Given any two (different) points, including the points at infinity, with non-positive abscissas, there are no more than two solutions of Equation (1.1) whose graphs pass through these points.

**Proof.** If we have three solutions whose graphs pass through the given points, then these graphs have no other points of intersection according to Theorem 6.1. Therefore, for \( x \in [\alpha, \beta] \), where \( \alpha \) and \( \beta \) are abscissas of the given points we have that \( y_1(x) > y_2(x) > y_3(x) \) and we can apply Projective Lemma 6.1. \( \square \)

**Remark 6.5.** Theorem 6.3 is dual to Theorem 6.1.

**Definition 6.2.** For \( x_0 \leq 0 \) (respectively, \( x_0 \leq X(0) \)), any solution having \( x_0 \) as the right (left) bound of its interval of existence is called the **left (right) solution** for \( x_0 \).

For any \( x_0 \leq 0 \) the **left maximum solution** for \( x_0 \) is the solution from Theorem 6.2. For any \( x_0 \leq X(0) \) the **right maximum solution** for \( x_0 \) is the left maximum solution for the point \( X^{-1}(x_0) \).

**Corollary 6.3.** Let \( x_0 \leq X(0) \) and \( \{(x_0, a)\} \) be the set of intervals of existence of the right solutions for \( x_0 \). Then \( X^{-1}(x_0) = \sup a \leq 0 \). The right maximum solution for \( x_0 \) is the unique solution with the interval of existence \( (x_0, X^{-1}(x_0)) \).

**Proof.** Since \( X(X^{-1}(x_0)) = x_0 \), the first statement follows from Proposition 4.5. The uniqueness is proved in Theorem 6.2. \( \square \)

**Theorem 6.4.** For \( x_0 \leq 0 \) (respectively, \( x_0 \leq X(0) \)), consider left (right) solutions for \( x_0 \) with the minimum values \( y_1 \geq y_1^\mu \), where \( y_1^\mu \) is the minimum value of \( y_\mu(x) \), the left (right) maximum solution for \( x_0 \). Any two such solutions, \( y_1(x) \) and
Proof. This is an immediate consequence of Projective Lemma 6.1 for the triple of solutions: \( y_1(x), y_2(x), \) and \( y_\mu(x) \). The only point of intersection that is allowed by this Lemma is the point of intersection at infinity with abscissa \( x_0 \). The properties of the monotonic function are stated in Propositions 3.4 and 4.3. \( \square \)

**Theorem 6.5.** For \( x_0 \leq 0 \) (respectively, \( x_0 \leq X(0) \)), consider left (right) solutions for \( x_0 \) with the minimum values \( y_l \leq y_\mu \), where \( y_\mu \) is the minimum value of \( y_\mu(x) \), the left (right) maximum solution for \( x_0 \). Any two such solutions, \( y_1(x) \) and \( y_2(x) \), have one finite point of intersection. The left (right) bound of their interval of existence is a continuous strictly monotonically decreasing (increasing) function of \( y_1 \) mapping \( (-\infty, y_\mu] \) onto \([X(x_0), x_0) \) ((\( x_0, X^{-1}(x_0) \))):

\[
y_\mu \geq y_2 > y_1 \quad \Rightarrow \quad X(x_0, y_2) < X(x_0, y_1) \quad (\Xi(x_0, y_2) > \Xi(x_0, y_1)).
\]

**Proof.** Suppose \( y_\mu \geq y_2 > y_1 \), but solutions \( y_1(x) \) and \( y_2(x) \) have no finite point of intersection. Let \( \alpha \) be an abscissa of the point of intersection of \( y_2(x) \) with \( y_\mu(x) \). Then, \( y_\mu(x) > y_1(x) \) and \( y_2(x) > y_1(x) \) for \( x \in [\alpha, x_0] \) (or \( x \in (x_0, \alpha] \)), the triple \( y_1(x), y_2(x), \) and \( y_\mu(x) \) have one point of intersection at infinity with abscissa \( x_0 \), and \( y_1(x) \) intersects \( y_2(x) \) at \( \alpha \). Therefore, we arrive at the contradiction by applying Projective Lemma 6.1 to the triple: \( y_\mu(x), y_2(x), \) and \( y_1(x) \) on the segment \( x \in [\alpha, x_0] \) (or \( x \in [x_0, \alpha] \)). The properties of the monotonic function are stated in Propositions 3.4 and 4.3. \( \square \)

**7. Boundary Value Problems**

**Theorem 7.1.** Let \( x_1 < x_0 \leq 0 \). If

1. \( x_1 < X(x_0) \), then there are no solutions with the interval of existence \( (x_1, x_0) \);
2. \( x_1 = X(x_0) \), then there is one solution with the interval of existence \( (x_1, x_0) \);
3. \( X(x_0) < x_1 < x_0 \), then there are two solutions with the interval of existence \( (x_1, x_0) \).

**Proof.** The first statement follows from Definition 2.2 and Theorem 2.1. The second statement is proved in Theorem 6.2. The third statement follows immediately from Theorems 6.4 and 6.5. \( \square \)

**Remark 7.1.** Theorem 7.1 can be viewed as a statement about a boundary value problem with infinite boundary conditions. To study finite boundary value problems we introduce the function \( Z(x_0, y_0, y^0) \) which plays the role of the function \( X(x_0) \) in Theorem 7.1. The properties of \( Z(x_0, y_0, y^0) \) are studied with the help of an auxiliary “level-function” \( Z(x_0, y_0, y^0; y_l) \).

**Definition 7.1.** Let \( x_0 \leq 0 \), \( y^0 \geq y_0 > y_l \). Denote by \( y_\pm(x) \) the solution with the initial value \( y_\pm(x_0) = y_0 \) and minimum value \( y_\pm(x_{\min} \pm) = y_1 \), where \( x_{\min} < x_0 \) and \( x_{\min} > x_0 \). Define the function, \( Z_\pm(x_0, y_0, y^0; y_l) \) as the (unique) solution of the equation \( y_\pm(Z_\pm) = y^0 \) satisfying inequalities: \( Z_- < x_0 \) and \( Z_+ > x_0 \). We also put \( Z_\pm(x_0, y_0, y^0; y_0) = x_0 \). (In case \( x_0 > X(0) \) the definition of the function \( Z_+ \) is limited to those solutions whose graphs do not cross the axis of ordinates below \( y^0 \).)
Proposition 7.1. The function \( Z_-(x_0, y_0, y^0; y_1) \) is a smooth function of all its variables. Moreover, if \( y_0 > y_1 \), then:

\[
(7.1) \quad \lim_{y_0 \to +\infty} Z_-(x_0, y_0, y^0; y_1) = X(x_0, y_1), \quad \lim_{y_0 \to +\infty} Z_-(x_0, y_0, y^0; y_1) = x_0.
\]

\[
(7.2) \quad y^1 > y^0 \Rightarrow Z_-(x_0, y_0, y^1; y_1) < Z_-(x_0, y_0, y^0; y_1),
\]

\[
(7.3) \quad y^0 \geq y_0 > y_1 \Rightarrow Z_-(x_0, y_0, y^0; y_1) < Z_-(x_0, y_0, y^0; y_1),
\]

\[
(7.4) \quad 0 < x_2 > x_1 \Rightarrow 0 < Z_-(x_2, y_0, y^0; y_1) - Z_-(x_1, y_0, y^0; y_1) < x_2 - x_1.
\]

Proof. The smoothness is a consequence of the implicit function theorem and the smooth behaviour of the solution \( y_-(x) \) on initial data \( x_0, y_0 \) and the minimum value \( y_1 \) (Remark 3.2). Inequalities (7.1) immediately follow from definitions of the functions \( X \) and \( Z_- \). To prove the first limit note that \( y^0 > y_0 \to +\infty \) therefore as \( y_0 \to +\infty \) the solution corresponding to the function \( Z_-(x_0, y_0, y^0; y_1) \) is approximating the one that corresponds to the function \( X(x_0, y_1) \) in the sense of Lemma 6.2 (see Proposition 4.1, Definition 3.1, and Proposition 3.2). The proof of the second limit follows from Inequalities (7.1) and Proposition 4.3. Inequality (7.2) is evident from the definition of \( Z_- \). Left Inequality (7.3) for \( Z_- \) follows from Proposition 3.2 and Lemma 4.2. The proof of the right is literally the same as the one of Proposition 3.5.

\( \square \)

Proposition 7.2. The function \( Z_+(x_0, y_0, y^0; y_1) \) is a smooth function of all its variables. Moreover, if \( y_0 > y_1 \), then:

\[
(7.5) \quad x_0 < Z_+(x_0, y_0, y^0; y_1) < \Xi(x_0, y_1),
\]

\[
(7.6) \quad \lim_{y_0, y_1 \to +\infty} Z_+(x_0, y_0, y^0; y_1) = \Xi(x_0, y_1), \quad \lim_{y_0 \to +\infty} Z_+(x_0, y_0, y^0; y_1) = x_0.
\]

\[
(7.7) \quad y^1 > y^0 \Rightarrow Z_+(x_0, y_0, y^1; y_1) > Z_+(x_0, y_0, y^0; y_1),
\]

\[
(7.8) \quad y^0 \geq y_0 > y_1 \Rightarrow Z_+(x_0, y_0, y^0; y_1) > Z_+(x_0, y_0, y^0; y_1),
\]

\[
X(0, y_1) \geq x_2 > x_1 \Rightarrow 0 < x_2 - x_1 < Z_+(x_2, y_0, y^0; y_1) - Z_+(x_1, y_0, y^0; y_1).
\]

Proof. The proof is very similar to the one of Proposition 7.1.

\( \square \)

Definition 7.2. For \( x_0 \leq 0 \) and \( y_0, y^0 \geq y_1 \) define

\[
Z(x_0, y_0, y^0; y_1) = \begin{cases} 
Z_-(x_0, y_0, y^0; y_1), & \text{for } y^0 \geq y_0 \geq y_1, \\
Z_+^{-1}(x_0, y^0, y_0; y_1), & \text{for } y_0 \geq y^0 \geq y_1
\end{cases}
\]

where the notation \( Z_+^{-1}(x_0, y^0, y_0; y_1) \) denotes a section of the fibre \( Z_+^{-1}(x_0) \) by the hyperplane parallel to the \( x \)-axis and passing through the point with the coordinates \( \{0, y^0, y_0, y_1\} \).

Proposition 7.3. The function \( Z(x_0, y_0, y^0; y_1) \) is a smooth function of all its variables. It has the following properties:

\[
(7.9) \quad Z(x_0, y_0, y^0; y_1) < X(x_0, y_1), \quad \lim_{y_0, y_1 \to +\infty} Z(x_0, y_0, y^0; y_1) = X(x_0, y_1), \quad \lim_{y_0 \to +\infty} Z(x_0, y_0, y^0; y_1) = x_0.
\]
Proof. The properties are the same as the ones for the function $Z_-(x_0, y_0, y^0; y_l)$, without the restriction $y^0 \geq y_0$. Continuity as well as differentiability, with respect to $x_0$ and $y_0$, at the “matching hyperplane” $y_0 = y^0$, follows from the identity, $Z_-(x_0, y_0; y_0; y_l) = Z^{-1}_+(x_0, y_0; y_l)$ and corresponding properties of the functions $Z_\pm$. Differentiability on $y_0$ and $y^0$ at the matching hyperplane is deduced from the observation that both functions are defined as the abscissas of the crossing point of the straight line $y = y^0$ by a unique family of solutions $y(x)$ that depend smoothly on their initial value $y_0$.

Monotonicity properties of the function $Z_+^{-1}(x_0, y^0, y_0; y_l)$ with respect to the $y$-variables do not follow from the corresponding properties of $Z_+(x_0, y^0, y_0; y_l)$ mentioned in Proposition 7.2, however, they can be established in analogous way on the basis of the results of §§3 and 5.

**Definition 7.3.** For $x_0 \leq 0$ and $y_0$, $y^0 \in \mathbb{R}$, denote by $Y$ the set of solutions $y(x)$ of Equation (1.1) with the initial value $y(x_0) = y_0$. Define the function

$$Z(x_0, y_0, y^0) = \inf_Y \{z : y(z) = y^0\}.$$ 

**Proposition 7.4.** For any given $x_0 \leq 0$ and $y_0$, $y^0 \in \mathbb{R}$, there is a unique solution $y(x)$ of Equation (1.1) that corresponds to the function $Z(x_0, y_0, y^0)$, i.e., $y(x_0) = y_0$ and $y(Z(x_0, y_0, y^0)) = y^0$. This solution achieves its (global) minimum in the segment $[Z(x_0, y_0, y^0), y_0]$.

Proof. The infimum in Definition 7.3 is achieved on the solutions with the minimum to the left of $x_0$. This follows from the monotonic growth of $Z(x_0, y_0, y^0; y_l)$ with respect to the $x$-variable (Proposition 7.3). Using the continuity of the function $x_0 - Z(x_0, y_0, y^0; y_l)$ and the fact that it vanishes as $y_l \to \pm \infty$ we arrive to the existence of $y_1$ such that

$$Z(x_0, y_0, y^0) = Z(x_0, y_0, y^0; y_1).$$  

Equation (7.5) verifies existence of the solution $y(x)$.

From Theorem 6.3 it follows that there exist at least two solutions that correspond to $Z(x_0, y_0, y^0)$. Suppose that there are two maximal solutions, $y_1(x)$ and $y_2(x)$, passing through the points $\{x_0, y_0\}$ and $\{Z(x_0, y_0, y^0), y^0\}$. Theorem 6.1 implies that these solutions do not have any other points of intersection, therefore $y_1(x) > y_2(x)$ for $x \in (Z(x_0, y_0, y^0), x_0)$. Consider a solution $y_3(x)$ passing through the point $\{x_0, y_0\}$ and having the slope between the slopes of the maximal solutions. The third solution cannot intersect $y_1(x)$ on the segment $x \in [Z(x_0, y_0, y^0), x_0]$ as this contradicts Lemma 6.1. Thus, $y_3(x)$ intersects $y_2(x)$ at some $x_1 \in (Z(x_0, y_0, y^0), x_0)$. Solutions $y_3(x)$ and $y_2(x)$ do not have any other points of intersection. This means that $y_3(x) < y_2(x)$ for $x \in [Z(x_0, y_0, y^0), x_1]$. Therefore, $y_3(Z(x_0, y_0, y^0)) < y_2(Z(x_0, y_0, y^0)) = y^0$ and solution $y_3(x)$ should cross the level $y^0$ to the left of $Z(x_0, y_0, y^0)$. That contradicts the definition of $Z(x_0, y_0, y^0)$ and establishes uniqueness of $y(x)$.

Suppose that the solution $y(x)$ has no minimum in the segment $[Z(x_0, y_0, y^0), x_0]$. Denote by $x_{\text{min}}$ the global minimum of $y(x)$ and suppose first that $y'(x) > 0$ for $x \in$
Graphs of any two such solutions, corresponding solutions with the initial data set. It is also bounded. To prove it, consider inequality of the maximal solution 0 \equiv y(x_0) = y_0 \equiv y'(x_0), and \mu - \epsilon < y'(x_0) would cross the straight line \( x = Z(x_0, y_0, y^0) \) below \( y^0 \). By virtue of Lemma 3.1, \( y_\mu(x) \) would not cross the graph of \( y(x) \) at any \( x < x_0 \) until they get to their minimum point, which, by continuity, for small \( \epsilon \) is close enough to \( x_{\min} \). Therefore, the graphs of all these solutions cross the line \( y = y^0 \) to the left of the point with the abscissa \( Z(x_0, y_0, y^0) \). That contradicts the definition of the function \( Z(x_0, y_0, y^0) \). Suppose now that \( y'(x) < 0 \) for \( x \in [Z(x_0, y_0, y^0), x_0] \). Consider the solution \( y_0(x) \) with the following initial data \( y_0(x_0) = y(x_0) = y_0 \) and \( y'_0(x_0) = 0 \). As follows from Lemma 4.1 \( y_0(x) < y(x) \) for \( x < x_0 \). Thus its graph intersects with the line \( y = y^0 \) to the left of the point with the coordinates \( (Z(x_0, y_0, y^0), y^0) \). That contradicts the definition of \( Z(x_0, y_0, y^0) \).

Definition 7.4. We call the solution from Proposition 7.4 the maximal solution corresponding to the segment \([Z(x_0, y_0, y^0), x_0] \).

Proposition 7.5. The function \( Z(x_0, y_0, y^0) \) is a smooth function of all its variables with the following properties:

\[
X(x_0) < Z(x_0, y_0, y^0) < x_0,
\lim_{y_0, y^0 \to \infty} Z(x_0, y_0, y^0) = X(x_0),
\lim_{y_0 \to -\infty} Z(x_0, y_0, y^0) = x_0,
\lim_{y_0 \to -\infty} Z(x_0, y_0, y^0) = x_0,
\]

\( y^0 > y^0 \Rightarrow Z(x_0, y_0, y^0) < Z(x_0, y_0, y^0),\ y_0 > y_0 \Rightarrow Z(x_0, y_0, y^0) < Z(x_0, y_0, y^0),\)

\( 0 \geq \tilde{x}_0 > x_0 \Rightarrow 0 < Z(\tilde{x}_0, y_0, y^0) - Z(x_0, y_0, y^0) < \tilde{x}_0 - x_0. \)

Proof. The proof of smoothness is analogous to the one for \( X(x_0) \) given in Corollary 6.1; it is based on the uniqueness of representation \((7.5)\) proved in Proposition 7.4 and smoothness of the corresponding level function \( Z(x_0, y_0, y^0, y_0) \), see Proposition 7.3. The other properties are proved analogously as the ones in Proposition 3.7 with the help of representation \((7.5)\).

Remark 7.2. In Appendix A we give estimates for the function \( Z(x_0, y_0, y^0) \) similar to those obtained for the function \( X(x_0) \) in § 2.

Proposition 7.6. Let \( x_0 \leq 0, y_0, y^0 \in \mathbb{R} \). Consider solutions \( y(x) \) of Equation 1.1 with the initial data \( y(x_0) = y_0 \) and \( y'(x_0) = y_1 \leq y_1\text{, where }y_1\text{ is the initial slope at }x_0\text{ of the maximal solution }y_m(x)\text{ corresponding to the segment }[Z(x_0, y_0, y^0), x_0].\) Graphs of any two such solutions, \( y_1(x) \) and \( y_2(x) \), have no (finite/infinite) points of intersection with abscissa \( x \in [Z(x_0, y_0, y^0), x_0] \) other than \( \{x_0, y_0\} \). In particular, for any fixed \( x_1 \in (Z(x_0, y_0, y^0), x_0) \), \( y(x) \equiv y(x_1, y_1) \) is a smooth monotonically decreasing function of \( y_1 \in [c(x_1), y_1]\) with the image \( [y_m(x_1), +\infty) \), where the function \( c(x_1) \) takes values in \(( -\infty, y_1\text{)} \).

Proof. The absence of intersections and monotonicity is a direct consequence of Projective Lemma 6.1. Denote by \( Y_1 \) the set of initial slopes \( \{y_1\} \) such that corresponding solutions \( y(x, y_1) \) cross the straight line \( x = x_1 \) with finite values \( y(x_1, y_1) \). It is clear that \( Y_1 \) is nonempty \( (y_1 \in Y_1) \) and open (by smoothness of \( y(x, y_1) \) on \( y_1 \)) set. It is also bounded. To prove it, consider inequality \( y'' > 6y^2 \) for \( y = y(x, y_1) \),
Proposition 7.7. Let $x_0 \leq 0$, $y_0, y^0 \in \mathbb{R}$. Consider solutions $y(x)$ of Equation (1.1) with the initial value $y(x_0) = y_0$ and minimum value $y_l \leq y_m$, where $y_m$ is the minimum value of the maximal solution $y_m(x)$ corresponding to the segment $[Z(x_0, y_0, y^0), x_0]$. Graphs of any two such solutions, $y_1(x)$ and $y_2(x)$, have one more (finite) point of intersection in the segment $x \in [Z(x_0, y_0, y^0), x_0]$ different from $x_0$. The function $Z(x_0, y_0, y^0; y_1)$ is a strictly monotonically decreasing function from $x_0$ to $Z(x_0, y_0, y^0)$ as $y_1$ increases from $-\infty$ to $y_m$.

Proof. Without loss of generality assume that the minimum values of the solutions are ordered as follows, $y_{1m} > y_{1l} > y_{2l}$. Denote by $P$ the point of intersection of the graphs of $y_m(x)$ and $y_1(x)$ with abscissa in $(Z(x_0, y_0, y^0), x_0)$. This point exists due to the uniqueness of the maximal solution $y_m(x)$. If we suppose that the graph of $y_2(x)$ does not intersect the graph of $y_1(x)$ to the right of $P$, then we arrive at a contradiction with Lemma 6.1. Denote by $Q$ the point of intersection of the graphs $y_1(x)$ and $y_2(x)$. To prove strict monotonicity of $Z(x_0, y_0, y^0; y_1)$ for $y_1 \leq y_{1m}$, let us notice that, according to the definition, it is the abscissa of the point of intersection of the graphs of solutions with the level $y = y^0$ located to the left of their minima. The graphs of $y_1(x)$ and $y_2(x)$ have no points of intersection other than $\{x_0, y(x_0)\}$ and $Q$. This means that to the left of $Q$ $y_2(x) > y_1(x)$; therefore the graph of $y_2(x)$ intersects the level $y = y_0$ at a point with larger abscissa than it does the graph of $y_1(x)$. \[ \square \]

Theorem 7.2. Let $x_1 < x_0 \leq 0$. The Dirichlet boundary value problem $y(x_0) = y_0$ and $y(x_1) = y^0$ has

1. No solution if $x_1 < Z(x_0, y_0, y^0)$;
2. A unique solution if $x_1 = Z(x_0, y_0, y^0)$;
3. Two solutions if $Z(x_0, y_0, y^0) < x_1 < x_0$.

Proof. The first statement follows from Definition 7.3 of the function $Z(x_0, y_0, y^0)$ and its boundedness (Proposition 7.5). The second statement is proved in Proposition 7.4.

Consider the third statement. Suppose first that the straight lines $y = y_0$ and $x = x_1$ cross above the maximal solution $y_m(x)$. As follows from Proposition 7.6 there is exactly one solution $y_1(x)$ of the Dirichlet boundary problem with the initial slope $y_1'(x_0) < y_m'(x_0)$. It satisfies the inequality $y_1(x) > y_m(x)$ for all $x < x_0$ in the interval of existence of $y_1(x)$.

On the other hand, from Proposition 7.7 it follows that there is exactly one solution $y_2(x)$ of the Dirichlet problem in the class of solutions of Equation (1.1) with the (global) minimal value $y_2 < y_{1m}$, where $y_{1m}$ is the minimum value of $y_m(x)$. It is clear that $y_2(x)$ achieves its minimum value to right of $x_1$, as the solutions with
minimum values less than \( y_{lm} \) and (global) minima located to the left of \( x_1 \) cross the line \( x = x_1 \) below \( y_{m}(x_1) < y^0 \). The solution \( y_2(x) \) crosses the maximal solution at some point \( x_2: x_1 < x_2 < x_0 \).

Consider the case when the lines \( y = y^0 \) and \( x = x_1 \) cross below the maximal solution. According to Proposition 7.7 we still have exactly one solution \( y_2(x) \) to the Dirichlet problem in the class of the functions with the (global) minimum values \( y_{2l} < y_{lm} \) whose minima are located to the right of \( x_1 \). We claim that we also have the second solution, \( y_1(x) \), which is the unique solution of the Dirichlet problem in the class of functions with the (global) minimum value \( y_{1l} < y_{lm} \) whose minima are located to the left of \( x_1 \). Existence of such a solution is established by considering a family of solutions of Equation (1.1), \( y_\mu(x) \), defined by the initial data: \( y_\mu(x_0) = y_0 \) and \( y'_\mu(x_0) \equiv \mu \geq y'_m(x_0) \). As follows from Corollary 3.1, the minimum values of \( y_\mu(x) \), satisfy the inequality \( y_{\mu l} < y_{lm} \leq y^0 \). Therefore, the graphs of \( y_\mu(x) \) cross the line \( y = y^0 \) to the right of their minima. As \( \mu \to +\infty \) the crossing point approaches \( x_0 \). This fact follows from Corollary 3.1 and Proposition 3.6. Actually, the function \( y_\mu(\mu) \) is the inverse to \( \mu = f(x_0; y_0, y_{\mu l}) \) for fixed \( x_0 \) and \( y_0 \), thus Corollary 3.1 implies that \( y_\mu(\mu) \downarrow -\infty \) as \( \mu \nearrow +\infty \). For \( \mu \to y'_m(x_0) \), the crossing point approaches the crossing point of \( y_m(x) \) with the line \( y = y^0 \), which is located to the left of the point \( (x_1, y^0) \). Thus, existence of \( y_1(x) \) follows by standard continuity arguments. Uniqueness is the consequence of the fact that the graphs of any two (different) solutions \( y_{\mu 1}(x) \) and \( y_{\mu 2}(x) \) in the right half-plane of the largest of their minima have only one point of intersection, \( (x_0, y_0) \), (cf. Lemma 3.1). In this case \( y_k(x) < y_m(x) \), \( k = 1, 2 \), for \( x \in (x_1, x_0) \).

Finally we have to consider the case when the point \( (x_1, y^0) \) lies on the graph of the maximal solution. This is a limit of the first case considered in this proof, namely, the “upper” solution \( y_1(x) \) coincides with \( y_m(x) \). It can also be considered as a limit of the second case considered in the proof, when again \( y_1(x) \) coincides with \( y_m(x) \). Either case leads to existence of two solutions: the first coincides with \( y_m(x) \) and the second is \( y_2(x) \) which is the same in both cases. Note, that in this case the requirement \( x_1 > Z(x_0, y_0, y^0) \) implies that \( y_{lm} < y^0 \).

\[ \square \]

8. Numerics

All calculations for this section, except those presented in Table 3, are done with the help of MAPLE 6. The results of Table 3 are calculated with MAPLE 8.

We begin with the illustration of Corollary 3.1 and Remark 3.3, namely, in Table 1 we present some numerical values of the functions \( f(x_0; y_0, y_l) \) and \( \Delta(x_0; y_0, y_l) \). Corresponding calculations are performed twice with the MAPLE parameter Digits set to 16 and 18. These settings allow one to calculate values of the functions \( f \) and \( \Delta \) up to 6 decimal digits. Table 1, together with Remark 3.3 allow us to conjecture that the upper bound of function \( \Delta \) is very close to zero, most probably, it is less than \( 10^{-1} \). This suggests the following interesting property, an approximate symmetry, of solutions of Equation (1.1) defined in Corollary 3.1.

**Conjecture 8.1.** Consider two solutions: the first one defined via the initial value \( y_0 \in \mathbb{R} \) and minimum value \( y_l \leq y_0 \), the second one via \( -y_l \) and \( -y_0 \), respectively. The integer part of the difference between their initial slopes is zero, i.e., \( [\Delta] = 0 \).
Numerical studies also confirm an analogous conjecture for the function
\[ \Delta^+(x_0; y_0, y_l) = f_+^+(x_0; y_0, y_l) - f^+(x_0; -y_l, -y_0). \]

Table 1

<table>
<thead>
<tr>
<th>( x_0 )</th>
<th>( y_0 )</th>
<th>( y_l )</th>
<th>( f(x_0; y_0, y_l) )</th>
<th>( \Delta(x_0; y_0, y_l) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.2</td>
<td>0.1</td>
<td>0.270736...</td>
<td>0.004050...</td>
</tr>
<tr>
<td>0.0</td>
<td>2.0</td>
<td>1.0</td>
<td>5.319113...</td>
<td>0.006387...</td>
</tr>
<tr>
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<td>46.0</td>
<td>2.0</td>
<td>624.047258...</td>
<td>0.004173...</td>
</tr>
<tr>
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<td>-300.0</td>
<td>31176.973126...</td>
<td>0.000401...</td>
</tr>
<tr>
<td>-3.0</td>
<td>10.0</td>
<td>-5.0</td>
<td>67.798893...</td>
<td>0.022622...</td>
</tr>
<tr>
<td>-5.0</td>
<td>262.0</td>
<td>1.0</td>
<td>8481.836260...</td>
<td>0.001160...</td>
</tr>
</tbody>
</table>

As proved in Section 2, \( X(0) \) and \( X_{\min}(0) \) are finite, however, the results obtained in that section allow us to find only rough estimates of their numerical values (see Remark 2.3). In Table 2, we present numerical calculations of some important data characterizing three distinguished solutions of Equation (1.1) that have a pole at \( x_0 = 0 \). In the second and last lines of this table we give the values of \( v(\hat{x}) = c/\hat{x} \), where \( c \) is the parameter in the corresponding Laurent expansion (3.6) of the solutions at \( \hat{x} = 0 \) and \( x_p \), the right and, respectively, left bounds of their intervals of existence. The solutions have two zeroes \( z_1 \) and \( z_2 \). In the second column we present numerical data characterizing \( y_{\text{max}}(0; x) \), the solution with the maximum interval of existence, i.e., \( x_p = X(0) \). In the third column, the data are given for \( y_{\text{min}}(0; x) \), the solution with \( x_{\text{min}} = X_{\text{min}}(0) \). Finally, in the last column of Table 2, we list the data for \( y_{\text{s-sym}}(x) \), the symmetric solution that has a pole at 0. Equation (1.1) has actually two symmetric solutions which were distinguished long ago by Boutroux [3]. They are uniquely characterized by the following symmetry condition, \( y(x) = \varepsilon^2 y(\varepsilon x) \), where \( \varepsilon^5 = 1 \), and behaviour at \( x = 0 \): one of them, \( y_{r-sym}(x) \) is regular at 0 and the other, \( y_{s-sym}(x) \), has a pole. The notation in

Table 2

<table>
<thead>
<tr>
<th>( v(0) )</th>
<th>( y_{\text{max}}(0; x) )</th>
<th>( y_{\text{min}}(0; x) )</th>
<th>( y_{\text{s-sym}}(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.110489160...</td>
<td>0.125565964...</td>
<td>0.0</td>
<td>0.125565964...</td>
</tr>
<tr>
<td>-1.528999716...</td>
<td>-1.537495773...</td>
<td>-1.476591053...</td>
<td>-1.476591053...</td>
</tr>
<tr>
<td>1.113243043...</td>
<td>1.082837664...</td>
<td>1.313083166...</td>
<td>1.313083166...</td>
</tr>
<tr>
<td>-2.05505831...</td>
<td>-2.05505831...</td>
<td>-2.04984309...</td>
<td>-2.04984309...</td>
</tr>
<tr>
<td>-0.322633511...</td>
<td>-0.307468294...</td>
<td>-0.423460899...</td>
<td>-0.423460899...</td>
</tr>
<tr>
<td>-2.546577118...</td>
<td>-2.539078168...</td>
<td>-2.575983030...</td>
<td>-2.575983030...</td>
</tr>
<tr>
<td>-1.292743827...</td>
<td>-1.256197484...</td>
<td>-1.527257430...</td>
<td>-1.527257430...</td>
</tr>
<tr>
<td>-3.915285797...</td>
<td>-3.914972029...</td>
<td>-3.902470099...</td>
<td>-3.902470099...</td>
</tr>
<tr>
<td>-0.916786830...</td>
<td>-0.892041655...</td>
<td>-1.091093248...</td>
<td>-1.091093248...</td>
</tr>
</tbody>
</table>

the first column of Table 3 has the same meaning as in the corresponding column of Table 2; the only difference is that in Table 2 the right pole of the solutions, \( x = 0 \), is known, while in Table 3 we add an additional line, labeled \( x_{p_1} \), with the information about this pole. In the second column the data for the symmetric
solution $y_{r-sym}(x)$ mentioned above are given. In the third and fourth columns we list the data for solutions $y_-(x_0; x)$ for $x_0 = 0$ and $x_0 = -1$, respectively. These solutions correspond to $X_-(x_0)$, i.e., they have a minimum at $x_0$ and the left pole $x_{p2} = X_-(x_0)$. A comment, probably should be given to the line labeled $x_{min}$. The zero is the global minimum of the restrictions of $y_{r-sym}(x)$ and $y_-(0; x)$ on the non-positive semi-axis. If we consider this solutions, as we should, on the maximum interval of existence, then zero is the (local) minimum of $y_-(0; x)$ and an inflection point for $y_{r-sym}(x)$. In particular, from Tables 2 and 3 it follows that:

**Table 3**

<table>
<thead>
<tr>
<th>$x_{p1}$</th>
<th>$y_{r-sym}(x)$</th>
<th>$y_-(0; x)$</th>
<th>$y_-(1; x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v(x_{p1})$</td>
<td>regular for $x \geq 0$</td>
<td>regular for $x \geq 0$</td>
<td>1.848036525...</td>
</tr>
<tr>
<td>$z_1$</td>
<td>N/A</td>
<td>N/A</td>
<td>0.161869969...</td>
</tr>
<tr>
<td>$y'(z_1)$</td>
<td>no zeroes for $x &gt; 0$</td>
<td>no zeroes for $x &gt; 0$</td>
<td>-0.303329968...</td>
</tr>
<tr>
<td>$x_{min}$</td>
<td>0 for $x \leq 0$</td>
<td>0 for $x \leq 0$</td>
<td>-1</td>
</tr>
<tr>
<td>$y_{min}$</td>
<td>0</td>
<td>-0.124293080...</td>
<td>-0.249902470...</td>
</tr>
<tr>
<td>$z_2$</td>
<td>no zeroes for $x &lt; 0$</td>
<td>-0.838060764...</td>
<td>-1.582985950...</td>
</tr>
<tr>
<td>$y'(z_2)$</td>
<td>N/A</td>
<td>-0.398459416...</td>
<td>-0.870257802...</td>
</tr>
<tr>
<td>$x_{p2}$</td>
<td>-2.615571209...</td>
<td>-2.677058361...</td>
<td>-3.121759948...</td>
</tr>
<tr>
<td>$v(x_{p2})$</td>
<td>-0.371644061...</td>
<td>-0.438744582...</td>
<td>-0.639793560...</td>
</tr>
</tbody>
</table>

$X_{min}(0) = -2.055703500...$, $X_-(0) = -2.677058361...$, $X(0) = -3.915285797...$

It is interesting to compare how these numerical values fit the bounds given in Section 2; namely, in Remark 2.3 and Inequalities (2.11), (2.14):

$$-3.239042305... = -5 \left( \frac{C}{4} \right)^{4/5} < X_-(0) < X_{min}(0) < -C^{4/5} = -1.963788033...,$$

and second Inequalities (2.17), (2.18):

$$-3.997162138... = X_{min}(0) - \frac{C}{|X_{min}(0)|^{1/4}} < X(0) < -(2C)^{4/5} = -3.419153556...$$

Another observation is that $y_{s-sym}(x)$ appear to have the least interval of existence comparing to the other solutions in Table 2, however, in comparison to the solutions with the pole at $x_0 = 0$ the interval of existence is considerably larger spacing between the zeroes. The solutions presented in Table 2 have the following spacing between their zeroes: $1.017587402...$, $1.001582395...$, and $1.099407250...$, respectively. For a comparison we calculated the spacing between the zeroes for solutions with the pole at $x_0 = 0$ and the interval of existence to the left of it, i.e., the function $\delta_-(0, +\infty)$ in the notation of Appendix B; $\delta_-(0, +\infty) = 1.1808499889180...$. The corresponding solution has the following data: $v(0) = -0.518045...$, $z_1 = -1.3362856...$, $x_{min} = -1.9417146...$, $y_{min} = -0.741427...$, $z_2 = -2.5171356...$, $x_p = -3.7427412...$, $v(x_p) = -1.798000...$. Numerical calculation shows that one gets more than a twofold increase in accuracy for the spacing function compared to the accuracy of calculation of other data, in particular, the zeroes themselves.

It is also important to mention that our numerical studies support the conjecture of Remark 3.8 that the solution with $x_{min} = X_{min}(0)$ is unique.
The notation used in the first and fourth columns of Table 4 has the same
meaning as the one in the first column of Table 2. In the second and third columns
of Table 4 we present numerical data for solutions $y_{\text{max}}(-1; x)$ and $y_{\text{min}}(-1; x)$
which are analogous to $y_{\text{max}}(0; x)$ and $y_{\text{min}}(0; x)$, but with the right pole at $x_0 = -1$.

\textbf{Table 4}

<table>
<thead>
<tr>
<th></th>
<th>$y_{\text{max}}(-1; x)$</th>
<th>$y_{\text{min}}(-1; x)$</th>
<th>$v(X(0))$</th>
<th>$y_{\text{min}}(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v(-1)$</td>
<td>-0.064360748...</td>
<td>-0.045841066...</td>
<td>-0.943704177...</td>
<td></td>
</tr>
<tr>
<td>$z_1$</td>
<td>-2.374548413...</td>
<td>-2.379585633...</td>
<td>$z_2$</td>
<td>-2.554014196...</td>
</tr>
<tr>
<td>$y'(z_1)$</td>
<td>1.468791556...</td>
<td>1.441765379...</td>
<td>$y'(z_2)$</td>
<td>-1.331432779...</td>
</tr>
<tr>
<td>$x_{\text{min}}$</td>
<td>-2.853609725...</td>
<td>-2.853690013...</td>
<td>$x_{\text{min}}$</td>
<td>-2.055297172...</td>
</tr>
<tr>
<td>$y_{\text{min}}$</td>
<td>-0.381066130...</td>
<td>-0.369105745...</td>
<td>$y_{\text{min}}$</td>
<td>-0.338834697...</td>
</tr>
<tr>
<td>$z_2$</td>
<td>-3.111108506...</td>
<td>-3.306469422...</td>
<td>$z_1$</td>
<td>-1.52059470...</td>
</tr>
<tr>
<td>$y'(z_2)$</td>
<td>-1.62081956...</td>
<td>-1.590390526...</td>
<td>$y'(z_1)$</td>
<td>1.145676637...</td>
</tr>
<tr>
<td>$x_p$</td>
<td>-4.589970403...</td>
<td>-4.589834999...</td>
<td>$x_p$</td>
<td>-0.000570546...</td>
</tr>
<tr>
<td>$v(x_p)$</td>
<td>-1.187365438...</td>
<td>-1.161843481...</td>
<td>$v(x_p)$</td>
<td>0.093928571...</td>
</tr>
</tbody>
</table>

In particular, from Tables 3 and 4 it follows that: $X(-1) = -4.589970403...$, $X_{\text{min}}(-1) = -2.853690013...$, $X_-(1) = -3.121759948...$. Comparison of these results with the estimations obtained in Corollary (2.1) and Theorem (2.1) gives:

$$-3.324707204... = -1 - C < X_-(1) < X_{\text{min}}(-1) < -1 - \frac{C}{\eta(-1)} = -2.797524387...,$$

$$-4.642303753... = X_{\text{min}}(-1) - \frac{C}{|X_{\text{min}}(-1)|^{1/4}} < X(-1) < -1 - \frac{2^{4/5}C}{\eta(-2^{-4/5})} = -4.240073805...,$$

where, $\eta(-1) = 1.293282706...$ and $\eta(-2^{-4/5}) = 1.249215473...$. Experience with these calculations and numerical verification of Conjecture 8.4 (see below) support the following conjecture made in Remark 3.8:

\textbf{Conjecture 8.2.} For any $x_0$ the solution with the pole at $x_0$ and minimum at $X_{\text{min}}(x_0)$ is unique.

In the last column of Table 4 we present numerical data for solution $\hat{y}_{\text{min}}(x)$ which has a pole at $X(0)$, interval of existence to the right of $X(0)$, and the extremal property: $x_{\text{min}} = \Xi_{\text{min}}(X(0)) = X_-(x_0)$. So, it serves as an illustration of the content of Section 5 and supports, together with the check of Conjecture 8.4 below, the conjecture made in Remark 5.5:

\textbf{Conjecture 8.3.} For any $x_0$ the solution with the pole at $x_0$ and minimum at $\Xi_{\text{min}}(x_0)$ is unique.

Comparing the numerical data given in Tables 2 and 4 for $y_{\text{min}}(x_0; x)$ with that for $y_{\text{max}}(x_0; x)$, $x_0 = 0, -1$, we see that they are very close, though distinct.
Moreover, in both cases, \( x_0 = 0 \) and \( x_0 = -1 \),
\[
(8.1) \quad y_{\text{max}}(x_0; x) < y_{\text{min}}(x_0; x),
\]
where \( x \) belongs to the interval of existence of \( y_{\text{min}}(x_0; x) \). We actually, numerically checked Inequality (8.1) for \( x_0 \in [-3, 0] \), so that it is natural to make the following conjecture.

**Conjecture 8.4.** *Inequality (8.1) holds for any \( x_0 \leq 0 \).*

Moreover, it is easy to see that the leading term of asymptotics as \( x_0 \to -\infty \) of both minimum values, \( y_m := y_{\text{max}}(x_{\text{min}}; x_0) \) or \( y_{\text{min}}(x_{\text{min}}; x_0) \) is given by the same formula, \( y_m = v_{\text{min}}^{\text{max}} \sqrt{|x_0|} + o\left(\sqrt{|x_0|}\right) \), where \( v_{\text{min}}^{\text{max}} \) is defined in Remark 2.2 (for numerical estimates it is better to use in this formula \( |x_{\text{min}}| \) instead of \( |x_0| \)).

It would be interesting to find the next correction term to this asymptotics and prove that both minima decrease monotonically with the decrease of \( x_0 \), as numerical calculations also confirm. Another interesting question related with these solutions is to prove or disprove the following inequality:

**Conjecture 8.5.** *For \( x_1 < x_2 < 0 \), \( 0 < X(x_2) - X(x_1) < X_{\text{min}}(x_2) - X_{\text{min}}(x_1) \).*

We have established an analogous inequality for the level functions (see Proposition 4.2, however, it does not imply Conjecture 8.5.

The data presented in Tables 2–4 require precise calculations, as some of the (different) quantities coincide up to four digits after the decimal point. An important ingredient of these calculations is a special representation, the so-called PRP \( I \)-system, of solutions of Equation (1.1) in the neighbourhood of a pole. Namely, let \( z(x) \) and \( v(x) \) be a solution of the system,
\[
\begin{align*}
\frac{dz}{dx} &= 1 - \frac{x}{4} z^4 - \frac{z^5}{4} - \frac{1}{2} v z^6, \\
\frac{dv}{dx} &= \frac{x^2}{8} z + \frac{3}{8} x z^2 + \left(\frac{1}{4} + x v\right) z^3 + \frac{5}{4} v z^4 + \frac{3}{2} v^2 z^5,
\end{align*}
\]
with initial data \( z(x_0) = 0 \), \( v(x_0) = c/7 \). This is Painlevé’s regularised polar patch (PRP) system \([16]\). Note that
\[
(8.3) \quad y(x) = \frac{1}{z(x)^2}, \quad y'(x) = -\frac{2}{z(x)^3} + \frac{x z(x)}{2} + \frac{z(x)^2}{2} + v(x) z(x)^3,
\]
are compatible and the function \( y(x) \) solves \( P_1 \).

To define a solution of Equation (1.1) in a neighbourhood of a right pole we choose some parameter \( c \) in the initial data for the PRP\( _I \)-system. In neighbourhoods of zeroes of solutions , \( y(x) = 0 \), the corresponding functions \( z(x) \) and \( v(x) \to \infty \). Therefore, to go beyond the zeroes we choose a connection point in the segment between the first zero and the right pole and continue our calculation using our original Equation (1.1). To reach the left pole of the solution we have to choose another connection point after the second zero and continue our calculations again with the help of the PRP\( _I \)-system. Note that in the first segment \( z(x) < 0 \) whilst in the second it is positive. Numerical results depend, of course, on the choice of the connection points, however, the better the precision the lesser the dependence. We believe that the nine digits after the decimal point for the data
presented in Tables 2–4 coincide with those for the corresponding true solutions. Our belief is based on calculations performed for Tables 2 and 4 with MAPLE 6 by setting the parameter Digits to 20, 22, and 24, and for Table 3 with MAPLE 8 by setting the absolute and relative errors to $10^{-16}$, $10^{-18}$, and $10^{-20}$, and observation of the stability of the numbers obtained under variation of the connection points.

APPENDIX A. Estimates for $Z(x_0, y_0, y^0)$

**Definition A.1.** For $v_0 \in \mathbb{R}$ define the function

$$C(v_0) := \sup_{x>0, v_0 \sqrt{x} > -1} C(v_0, x), \quad C(v_0, x) := (2x)^{1/4} \int_0^{\sqrt{1+v_0 \sqrt{2x}}} \frac{dw}{\sqrt{w^4 - 3w^2 + 3 + x}}.$$ 

**Proposition A.1.** $C(v_0)$ is a strictly monotonic continuous function, $0 < C(v_0) < C$, where $C$ is defined in Equation (2.8) of § 2. Moreover,

$$\lim_{v_0 \to -\infty} C(v_0) = 0, \quad \lim_{v_0 \to +\infty} C(v_0) = C.$$ 

**Proof.** The boundedness and strict monotonicity of $C(v_0)$ follow immediately from Definition A.1.

If $v_0 \leq 0$, then the upper limit of the integral $C(v_0, x)$ is $\leq 1$ and $\sqrt{2x} \leq -1/v_0$. Thus, the integral is uniformly bounded, with respect to $x$. If now $v_0 \to -\infty$, then the factor $(2x)^{1/4} \to 0$ and we arrive at the first limit. Note that $C(v_0, 0) = 0$, therefore, since $C(v_0, x)$ is a continuous function, for $v_0 \leq 0$ the supremum in Definition A.1 is achieved at some finite value of $x > 0$ which is a local maximum of $C(v_0, x)$.

For $v_0 > 0$, $C(v_0, +\infty) := \lim_{x \to +\infty} C(v_0, x) = \sqrt{2} \int_0^{1/\sqrt{2}} \frac{du}{\sqrt{u^4 + 1}}$. Now, we make a change of the variable of integration in $C(v_0, x)$, $w = x^{1/4}u$ and consider the difference, $C(v_0, x) - C(v_0, +\infty)$. For all $x$ this difference is strictly larger than the following sum of two integrals, $(\int_0^{1/\sqrt{2}} + \int_1^{1/\sqrt{2}}) Ddu$, where $D$ is the difference of the corresponding integrands. The first integral is negative, whilst the second one is positive. For large $x$ the first integral can be estimated as $O\left(1/x^{3/4}\right)$ and the second behaves as $O\left(1/x^{1/2}\right)$. Thus the supremum of $C(v_0, x)$ in Definition A.1 is achieved at some finite local maximum of $C(v_0, x)$. For each $v_0$ we denote any such maxima as $x_{\max}(v_0)$. It is easy to observe that $\frac{\partial}{\partial x} C(v_0, x) > 0$ for $v_0 \geq 0$ and $x \leq 3/4$, thus for $v_0 \geq 0$, $x_{\max}(v_0) > 3/4$. The above proof also works for $v_0 = +\infty$. In the last case we denote $x_{\max}(+\infty) = x_{\max}$. (According to the numerical studies it is unique and its numerical value is given in Remark 2.2, however, we do not use this uniqueness in the rest of the proof.)

The numbers $x_{\max}(v_0)$ are uniformly bounded, i.e., there exists $\tilde{x}_{\max}$ that for all $v_0$: $x_{\max}(v_0) \leq \tilde{x}_{\max}$. Actually suppose that for $v_n \to +\infty$ the corresponding sequence $x_{\max}(v_n) \to +\infty$. Then according to the above considerations we have that $C(+\infty, x_{\max}) > C(+\infty, +\infty)$. It follows from this that for all sufficiently large $n$: $C(v_n, x_{\max}) > C(v_n, x_{\max}(v_n))$, which contradicts the definition of $x_{\max}(v_n)$. If $v_n \to v_0 > 0$, the proof is analogous; for $v_n \to v_0 < 0$ $\sqrt{2x_{\max}(v_0)} < -1/v_0$. If $v_n \to 0$ and $x_{\max}(v_n) \to +\infty$, then $\lim C(v_n, x_{\max}(v_n)) = 0 < C(0, x_{\max}(0))$. This again leads to a contradiction with the definition of $x_{\max}(v_n)$. 
The proof of continuity of \( C(v_0) \) is analogous to the above proof of the uniform boundedness of \( x_{\text{max}}(v_0) \). In fact, suppose \( v_n \to v_0 \) and \( C(v_n, x_{\text{max}}(v_n)) \neq C(v_0, x_{\text{max}}(v_0)) \). Then there is a subsequence \( v_{n_k} \) such that \( x_{\text{max}}(v_{n_k}) \to \hat{x} \) and \( C(v_{n_k}, x_{\text{max}}(v_{n_k})) \to C(v_0, \hat{x}) < C(v_0, x_{\text{max}}(v_0)) \). The last inequality means that for all sufficiently large \( n_k \): \( C(v_{n_k}, x_{\text{max}}(v_0)) > C(v_{n_k}, x_{\text{max}}(v_{n_k})) \), which is a contradiction.

Now the second limit can be proved as follows,

\[
C = C(+\infty, x_{\text{max}}) = \lim_{v_0 \to +\infty} C(v_0, x_{\text{max}}(v_0)) \leq \lim_{v_0 \to +\infty} C(v_0, x_{\text{max}}(v_0)).
\]

On the other hand, for all \( v_0 \in \mathbb{R} \) and \( x \geq 0 \): \( C(v_0, x) < C \), that implies \( \lim_{v_0 \to +\infty} C(v_0, x_{\text{max}}(v_0)) \leq C \).

**Remark A.1.** Most probably the functions \( x_{\text{max}}(v_0) \) from the above proof are single-valued. This leads to the smoothness of \( C(v_0) \) in the spirit of Corollary 6.1.

**Definition A.2.** For \( x_0 \leq 0 \) and \( y_0 \in \mathbb{R} \)

\[
X_{\text{min}}(x_0, y_0) := \inf_{y} \{ x_{\text{min}} : \text{the minimum of solution } y(x; x_0, y_0, y) \}.
\]

**Proposition A.2.** For \( x_0 < 0 \) and \( y_0 \in \mathbb{R} \), define \( \eta_{y_0}(x_0) \) as the unique positive solution of the equation \( \eta^5 - |x_0|\eta = C(y_0/\eta^2) \). Then

\[
(\text{A.1}) \quad \eta_{y_0}^4(x_0) = |x_0| + \frac{C\left(\frac{y_0}{\eta_{y_0}(x_0)}\right)}{\eta_{y_0}(x_0)} < |X_{\text{min}}(x_0, y_0)| < |x_0| + \frac{C\left(\frac{y_0}{\sqrt{|x_0|}}\right)}{|x_0|^{1/4}}.
\]

**Proof.** The proof is similar to the one for the analogous estimates of \( X_{\text{min}}(x_0) \) in Corollary 2.1 with the help of Proposition A.1.

**Remark A.2.** Left Inequality (A.1) remains valid if instead of the function \( \eta_{y_0}(x_0) \) satisfying the transcendental equation one substitutes the function \( \hat{\eta}_{y_0}(x_0) \) which is defined as the unique positive solution of the algebraic equation, \( \hat{\eta}^5 - |x_0|\hat{\eta} = C(v_0) \), where

\[
v_0 = \begin{cases} \frac{y_0}{\sqrt{|x_0|}}, & y_0 \leq 0 \\ \frac{y_0}{\sqrt{|x_0|} + \kappa(x_0, y_0)}, & y_0 > 0 \end{cases}, \quad \kappa(x_0, y_0) = \frac{C\left(\frac{y_0}{\sqrt{|x_0|}}\right)}{|x_0|^{1/4}}.
\]

This makes the inequality less accurate.

**Remark A.3.** It is easy to see in a similar way as for \( X_{\text{min}}(x_0) \), that there is at least one solution \( y(x; x_0, y_0, y) \) whose minimum equals \( X_{\text{min}}(x_0, y_0) \). However, we can only conjecture the uniqueness of this solution.

**Remark A.4.** Concerning the behaviour of \( X_{\text{min}}(x_0, y_0) \) for \( x_0: -1 < x_0 \leq 0 \) we can make an analogous comment as in Remark 2.3 with the change \( C \) on \( C(y_0/\eta_{y_0}^3(x_0)) \).

**Proposition A.3.**

\[
|Z(x_0, y_0, y^0)| < |X_{\text{min}}(x_0, y_0)| + \frac{C(v^0)}{|X_{\text{min}}(x_0, y_0)|^{1/4}}, \quad v^0 = \frac{y^0}{\sqrt{|X_{\text{min}}(x_0, y_0)|}}.
\]

If \( y^0 \geq \tilde{y}_m \), where \( \tilde{y}_m \) is the minimum value of the solution \( y(x; x_0, y_0, \tilde{y}_m) \) with the minimum at \( X_{\text{min}}(x_0, y_0) \), then \( |X_{\text{min}}(x_0, y_0)| \leq |Z(x_0, y_0, y^0)| \), where the equality is only possible if \( y^0 = \tilde{y}_m \).
Proof. Denote $y_{lm}$ the minimum value of the maximal solution. Consider the solution $y(x; X_{\text{min}}(x_0, y_0), y_{lm}, y_{lm})$. It follows from Lemma 4.2 that the graph of $y(x; X_{\text{min}}(x_0, y_0), y_{lm}, y_{lm})$ crosses the straight line $y = y^0$ at some point $P$ which is located to the left of the point where the maximal solution crosses this line. This means that $|Z(x_0, y_0, y^0)| - |X_{\text{min}}(x_0, y_0)| < |P_x| - |X_{\text{min}}(x_0, y_0)|$, where $P_x$ is the abscissa of $P$. The estimation of the last difference is similar to the one for $X_-(x_0)$ in Lemma 2.2 and yields the announced result.

The second statement is evident as the graph of $y(x; x_0, y_0, y_{lm})$ crosses the straight line $y = y^0$ to the left of $X_{\text{min}}(x_0, y_0)$.

\textbf{Proposition A.4.} Let $y_{m0} = \min\{y_0, y^0\}$ and $\eta_{y_{m0}}(x_0)$ be the unique positive solution of the equation $\eta^5 - |x_0|\eta = 2C(y_{m0}/\eta^2)$. Then

\begin{equation}
\eta^4_{y_{m0}}(x_0) = |x_0| + \frac{2C(\eta_{y_{m0}}(x_0))}{\eta_{y_{m0}}(x_0)} < |Z(x_0, y_0, y^0)|. \tag{A.2}
\end{equation}

Proof. The proof is very similar to the one for the second Inequality (2.17) for $X(x_0)$ in Theorem 2.1 with the help of monotonicity of the function $C(v_0)$. \hfill \Box

\textbf{Remark A.5.} As in Remark A.2 we can, with a loss of accuracy, substitute the function $\eta_{y_{m0}}(x_0)$ satisfying the transcendental equation with the function $\hat{\eta}_{y_{m0}}(x_0)$ which is defined as the unique positive solution of the algebraic equation, $\eta^5 - |x_0|\hat{\eta} = 2C(v_0)$, where now

\[ \hat{\eta}_{y_{m0}}(x_0) = \begin{cases} \frac{y_{m0}}{\sqrt{|X_{\text{min}}(x_0, y_0)|}}, & y_0 \leq 0 \\ \frac{y_{m0}}{\sqrt{|X_{\text{min}}(x_0, y_0)|} + \kappa(x_0, y_0)}, & y_0 \geq 0 \end{cases} \quad \text{and} \quad \hat{\kappa}(x_0, y_0) = \frac{C\left(\frac{y_{m0}}{\sqrt{|X_{\text{min}}(x_0, y_0)|}}\right)}{|X_{\text{min}}(x_0, y_0)|^{1/4}}. \]

\section*{Appendix B. Spacing of Zeroses}

\textbf{Definition B.1.} Let $x_0 \leq 0$, $y_1 < 0$, and $y_0 \geq 0$. Denote $z_1^- < z_2^- < x_0$ the zeroes of the solution $y(x; x_0, y_0, y_1)$ and $x_0 < z_1^+ < z_2^+$ the zeroes of the solution $y_+(x; x_0, y_0, y_1)$ with the initial slope $y_1 < 0$ at $x_0 \leq X(0)$. Define

\[ \delta_\pm(x_0, y_0, y_1) = z_1^\pm - z_2^\pm, \quad \delta_\pm(x_0, y_0) = \sup_{y_1 < 0} \delta_\pm(x_0, y_0, y_1). \]

\textbf{Proposition B.1.} The functions $\delta_\pm(x_0, y_0)$ are finite for all $x_0$ and $y_0$, in their domains of definition. For given $x_0$ and $y_0$ there exists a value of $y_1 = \hat{y}_1$ such that the solution $y(x; x_0, y_0, \hat{y}_1)$ has the zero spacing $\delta_\pm(x_0, y_0)$. In the case $y_0 > 0$ the number $\hat{y}_1$ for $\delta_\pm(x_0, y_0)$ is less than the minimum value of the maximal solution corresponding to the segment $[Z(x_0, y_0, 0), x_0]$ (see § 7, Proposition 7.4), and $\hat{y}_1$ for $\delta_\pm(x_0, y_0)$ is less than the minimum value of the maximal solution corresponding to the segment $[x_0, Z^{-1}(x_0, 0, y_0)]$, where $x_0 = Z^{-1}(x_0, 0, y_0)$ means that $x_0 = Z_0(0, y_0)$.

Proof. It is clear that $0 < \delta_-(x_0, y_0) < x_0 - Z(x_0, y_0, 0)$ and $0 < \delta_+(x_0, y_0) < Z^{-1}(x_0, 0, y_0) - x_0$.

The functions $\delta_\pm(x_0, y_0, y_1)$ are smooth since they are differences of two smooth functions, $z_1^\pm(x_0, y_0, y_1)$ and $z_2^\pm(x_0, y_0, y_1)$. That the functions $z_1^\pm$ and $z_2^\pm$ are smooth follows from the implicit function theorem, due to the fact that $y'(z_k^\pm; x_0, y_0, y_1) \neq 0$ for $k = 1, 2$. 


The functions $\delta_\pm(x_0, y_0, y_1)$ are bounded as $y_1 \to -\infty$; for $\delta_-(x_0, y_0, y_1)$ it follows from Proposition 4.3, for $\delta_+(x_0, y_0, y_1)$ an analogous estimate as in Proposition 4.3 clearly holds. Thus, we get existence of finite value of $y_1 = \hat{y}_1$, which, of course, depends on $x_0$, $y_0$, and on the sign $\pm$, such that $\delta_\pm(x_0, y_0) = \delta_\pm(x_0, y_0, \hat{y}_1)$.

Now we prove that $\hat{y}_l$ for $\delta_-(x_0, y_0, y_l)$ is less than the minimum value of the solution corresponding to $Z(x_0, y_0, 0)$. The proof of the corresponding statement for $\delta_+(x_0, y_0, y_l)$ is absolutely analogous. Let $y_{lm}$ be the minimum value of the maximal solution, i.e., $Z(x_0, y_0, 0) = Z_-(x_0, y_0, 0, y_{lm})$. The graphs of solutions $y(x; x_0, y_0, y_l)$ with $y_l : y_{lm} < y_l < 0$ have only one point of intersection, $(x_0, y_0)$. Therefore, their first zero, $z_1^-$, is monotonically increasing and the second, $z_2^-$, is monotonically decreasing as $y_l \searrow y_{lm}$. These mean that for such values of $y_l$ the function $\delta_-(x_0, y_0, y_l)$ is monotonically increasing. Thus $\hat{y}_l \leq y_{lm}$. For $y_l < y_{lm}$, $z_1^-$ continues its monotonic increase, while $z_2^-$ begins a monotonic increase too. However since $y_{lm}$ is the minimum of $z_2^-$, for some $\epsilon > 0$ we have $\Delta z_2^- = O((\Delta y_l)^2)$, while $\Delta z_1^- = O(\Delta y_l)$, where $\Delta y_l = y_{lm} - y_l < \epsilon$. Therefore, $\Delta z_1^- - \Delta z_2^- > 0$ in some lower neighbourhood of $y_{lm}$, which means that $\hat{y}_l < y_{lm}$.

\begin{proposition}

The functions $\delta_\pm(x_0, y_0)$ are continuous functions of two variables.

\end{proposition}

\begin{proof}

Let $\hat{y}_l$ be defined as in Proposition B.1, i.e., $\delta_\pm(x_0, y_0) = \delta_\pm(x_0, y_0, \hat{y}_l)$. Consider a sequence of points $(x_n, y_n) \to (x_0, y_0)$ and denote by $y_{ln}$ a corresponding sequence of the levels, such that $\delta_\pm(x_n, y_n) = \delta_\pm(x_n, y_n, y_{ln})$. The sequence $\hat{y}_{ln}$ is bounded, since for any unbounded subsequence $\hat{y}_{ln_k}$ the corresponding values of $\delta_\pm(x_{n_k}, y_{n_k}, \hat{y}_{ln_k}) \to 0$ (cf. Proposition 4.3).

Suppose that there is a subsequence $\delta_\pm(x_{n_k}, y_{n_k}, \hat{y}_{ln_k})$ that does not converge to $\delta_\pm(x_0, y_0)$. Changing the notation, if necessary, we can assume that it converges to some $\delta < \delta_\pm(x_0, y_0)$. Consider another subsequence $\delta_\pm(x_{n_k}, y_{n_k}, \hat{y}_l)$. It converges to $\delta_\pm(x_0, y_0)$. Therefore, for all $n > N$, $\delta_\pm(x_{n_k}, y_{n_k}, \hat{y}_l) > \delta_\pm(x_{n_k}, y_{n_k}, y_{ln_k})$, which contradicts the definition of the numbers $\hat{y}_{ln_k}$.

\end{proof}

\begin{proposition}

For $\hat{x}_0 < x_0 \leq 0$,

$0 < \delta_-(x_0, y_0, y_1) - \delta_-(\hat{x}_0, y_0, y_1) < x_0 - \hat{x}_0, \quad 0 < \delta_-(x_0, y_0) - \delta_-(\hat{x}_0, y_0) < x_0 - \hat{x}_0.$

For $\hat{y}_0 > y_0 \geq 0 > y_l$,

\begin{equation}
0 < \delta_-(x_0, y_0, y_1) - \delta_-(x_0, \hat{y}_0, y_1) < \frac{\hat{y}_0 - y_l}{f(x_0; y_0, y_1)},
\end{equation}

where the function $f(x_0; y_0, y_1)$ is defined in Corollary 3.1. Moreover,

\begin{equation}
0 < \delta_-(x_0, y_0) - \delta_-(x_0, \hat{y}_0) < \frac{x_0 - X_{\min}(x_0, y_0)}{y_0 - y_{lm}(x_0, y_0)}(\hat{y}_0 - y_0),
\end{equation}

where the function $y_{lm}(x_0, y_0)$ is the minimum value of the maximal solution corresponding to the segment $[Z(x_0, y_0, 0), x_0]$ and $X_{\min}(x_0, y_0)$ is given in Definition A.2.

\begin{proof}

Consider solutions $y_1(x) = y(x; x_0, y_0, y_1)$ and $y_2(x) = y(x; \hat{x}_0, y_0, y_1)$. Denote by $x_1(y)$ and $x_2(y)$, respectively, their inverse functions on the segments bounded by minima and $x_0$. The difference $x_1(y) - x_2(y)$ is monotonically decreasing with $y \searrow y_l$, see the proof of Proposition 3.5. Monotonic decrease of the difference continues with $y \searrow +\infty$, if we consider the inverse functions on the
segments bounded by the left poles and the minima. Thus we have the following inequalities:

\[ 0 < Z_-(x_0, y_0, y^0, y_1) - \delta_-(\hat{x}_0, y_0, y^0, y_1) < z^-_2 - \hat{z}_2 < z^-_1 - \hat{z}_1 < x_0 - \hat{x}_0, \]

where \( y^0 \geq y_0 \) is arbitrary and the "hat-notation" is used to distinguish the zeroes of solution \( y_2(x) \). This proves both inequalities for the difference \( \delta_-(x_0, y_0, y_1) - \delta_-(\hat{x}_0, y_0, y_1) \). Now choosing in the last difference \( y_1 = \hat{y}_1 \) such that \( \delta_-(\hat{x}_0, y_0, \hat{y}_1) = \delta_-(\hat{x}_0, y_0) \) (see Proposition B.1) and increasing then, if necessary, \( \delta_-(x_0, y_0, y_1) \) to \( \delta_-(x_0, y_0) \) we arrive at the left inequality for \( \delta_-(x_0, y_0) - \delta_-(\hat{x}_0, y_0) \). The right inequality is obtained by choosing \( y_1 = \hat{y}_1 \) such that \( \delta_-(x_0, y_0, \hat{y}_1) = \delta_-(x_0, y_0) \) and increasing, if necessary, \( \delta_-(\hat{x}_0, y_0, \hat{y}_1) \).

Consider now solutions \( y_1(x) = y(x; x_0, y_0, y_1) \) and \( y_2(x) = y(x; x_0, \hat{y}_0, y_1) \). Denote \( \hat{x}_0 \) the right point of intersection of the graph of \( y_3(x) \) with the straight line \( y = y_0 \). Then we can apply already proved inequalities to the zero spacing functions of \( y_1(x) \) and \( y_2(x) \). Using the concavity of graphs of the solutions and monotonicity of the function \( f(x_0, y_0, y_1) \) one writes,

\[ f(x_0; y_0, y_1)(x_0 - \hat{x}_0) < f(\hat{x}_0; y_0, y_1)(x_0 - \hat{x}_0) < \hat{y}_0 - y_0 < f(x_0; \hat{y}_0, y_1)(x_0 - \hat{x}_0). \]

This completes the proof of Inequality (B.1). Inequality (B.2) is a consequence of Inequality (B.1) and the estimate \( f(x_0; y_0, y_1)(x_0 - X_{min}(x_0, y_0)) > y_0 - y_{min}(x_0, y_0) \), which follows from the concavity and Proposition B.1.

Remark B.1. It is interesting to note that the function \(|Z(x_0, y_0, +\infty)|\) introduced in § 7 (cf. Definition 7.3) that measures a distance to the farthest (to the left) pole of the solutions with initial value \( y_0 \) at \( x_0 \) is monotonically increasing with the increase of \( y_0 \) (see Proposition 7.5), while the behaviour of \( \delta_-(x_0, y_0) \) is directly opposite. It is natural to expect that the solutions with larger intervals of existence have a larger spacing between their zeroes, and it is actually true. However, it is true only if we consider zeroes of solutions whose intervals of existence contain the point \( x_0 \) rather than having it as their right bound.

Proposition B.4. For \( \hat{x}_0 < x_0 \leq \mathcal{X}(0, y_1) \),

\[ 0 < \delta_+(x_0, y_0, y_1) - \delta_+(\hat{x}_0, y_0, y_1) < (L_\Xi(y_1) - 1)(x_0 - \hat{x}_0), \]

where \( L_\Xi(y_1) := \sup_{x_0 \leq \mathcal{X}(0, y_1)} \frac{\partial}{\partial x_0} \Xi(x_0, y_1) \) satisfies the following inequalities

\[ 1 < L_\Xi(y_1) \leq L_\Xi \]

with \( L_\Xi \) defined in Definition 5.3 of § 5.

For \( \hat{x}_0 < x_0 \leq X(0) \) and \( \hat{y}_0 > y_0 \geq 0 > y_1 \):

\[ 0 < \delta_+(x_0, y_0) - \delta_+(\hat{x}_0, y_0) < (L_\Xi - 1)(x_0 - \hat{x}_0); \]

\[ 0 < \delta_+(x_0, \hat{y}_0, y_1) - \delta_+(x_0, y_0, y_1) < \frac{L_\Xi(y_1) - 1}{f^+(x_0; y_0, y_1)}(\hat{y}_0 - y_0), \]

where the function \( f^+(x_0; y_0, y_1) \) is defined in Remark 3.3;

\[ 0 < \delta_+(x_0, \hat{y}_0) - \delta_+(x_0, \hat{y}_0) < (L_\Xi - 1)\frac{\Xi_{min}(x_0) - x_0}{y_0 - y_{min}(x_0, y_0)}(\hat{y}_0 - y_0), \]

where the function \( y_{min}(x_0, y_0) \) is the minimum value of the maximal solution corresponding to the segment \([x_0, Z^{-1}(x_0, 0, y_0)]\).
Remark whose maximum equals $Z$.

Proof. Consider solutions $y_1(x) = y_+(x; x_0, y_0, y_0)$ and $y_2(x) = y_+(x; x_0, y_0, y_i)$. Denote by $x_1(y)$ and $x_2(y)$, respectively, their inverse functions on the segments bounded by minima and $x_0$. The difference $x_1(y) - x_2(y)$ is monotonically increasing with $y \searrow y_i$, see the proof of Proposition 3.5 (cf. Proposition 5.5). Monotonic increase of the difference continues with $y \nearrow +\infty$ if we consider the inverse functions on the segments bounded by the left poles and the minima. Thus we have the following inequalities:

$$0 < x_0 - \hat{x}_0 < z^*_2 - \hat{z}^*_2 < Z_+(x_0, y_0, +\infty, y_i) - Z_+(\hat{x}_0, y_0, +\infty, y_i),$$

where the “hat-notation” is used to distinguish the zeroes of solution $y_2(x)$. Now, we notice that $Z_+(x_0, y_0, +\infty, y_i) - Z_+(\hat{x}_0, y_0, +\infty, y_i) = \Xi(\hat{x}_0, y_i) - \Xi(\hat{x}_0, y_i)$, for some $\hat{x}_0$ and $\hat{x}_0$, such that $\hat{x}_0 - \hat{x}_0 < x_0 - \hat{x}_0$, by virtue of the monotonicity. This proves both Inequalities (B.3) and (B.4), since according to Proposition 5.9 $L_\Xi$ is finite.

Consider solutions $y_1(x) = y_+(x; x_0, y_0, y_i)$ and $y_2(x) = y(x; x_0, \hat{y}_0, y_i)$. Denote by $\hat{x}_0$ the left point of intersection of the graph of $y_1(x)$ with the straight line $y = \hat{y}_0$. Then we can apply already proved Inequalities (B.3) for the zero spacing functions of $y_1(x)$ and $y_2(x)$ with $y_0$ changed to $\hat{y}_0$. Now, taking into account that $\delta_+(\hat{x}_0, \hat{y}_0, y_i) = \delta_+(x_0, y_0, y_i)$ and, by concavity of the graph of $y_1(x)$, $|f^+(x_0, y_0, y_i)|(x_0 - \hat{x}_0) < (\hat{y}_0 - y_0)$ we arrive at Inequalities (B.5). Inequalities (B.6) are the consequence of the ones for the level functions (B.5) and the estimate $|f^+(x_0; y_0, y_i)|(\Xi_{\min}(x_0) - x_0) > y_0 - y_{\lim}(x_0, y_0)$, which follows from the concavity and Proposition B.1.\hfill $\square$

Corollary B.1. The functions $\delta_+(x_0, 0)$ are smooth and have the following properties:

$$\delta_-(x_0, 0) = \sup_{y_0 \geq 0} \delta_-(x_0, y_0), \quad \delta_+(x_0, 0) = \inf_{y_0 \geq 0} \delta_+(x_0, y_0).$$

For all $x_0 < 0$

$$\delta_+(x_0, 0) = \frac{2C(0)}{|x_0|^{1/4}} + o\left(\frac{1}{|x_0|^{3/2}}\right) > \delta_-(x_0, 0) = \frac{2C(0)}{|x_0|^{1/4}} + O\left(\frac{1}{|x_0|^{3/2}}\right),$$

where $C(0)$ is given in Definition A.1 for $v_0 = 0$ and $-\frac{C^2(0)}{|x_0|^{3/2}} < C\left(\frac{1}{|x_0|^{3/2}}\right) < 0$.

Proof. The extremal properties of $\delta_+(x_0, 0)$ follows from Propositions B.3 and B.4. The smoothness is the consequence of the representations

(B.7)

$$\delta_\pm(x_0, 0) = |Z^{\pm 1}(x_0, 0, 0) - |x_0||,$$

where $Z^{+1}(x_0, 0, 0) \equiv Z(x_0, 0, 0)$ and $Z^{-1}(x_0, 0, 0)$ is the the section of the fiber $Z^{-1}(x_0)$ with the straight line passing through the origin in direction of the vector $(1, 0, 0)$. Now the uniqueness of the maximal solutions (see Proposition 7.4) allows one to prove smoothness in the spirit of Corollary 6.1. Of course, smoothness of $\delta_-(x_0, 0)$ follows directly from representation (B.7) and Proposition 7.5. The estimates are special cases of the estimates proved in Propositions A.3 and A.4.\hfill $\square$

Remark B.2. Numerical calculations shows that the integral (see Definition A.1) whose maximum equals $C(0)$ has only one extremum at

$$x_{\max} = 2.04124321493909675 \ldots, \quad \nu_{\min} = \frac{1}{\sqrt{2x_{\max}}} = -0.4949229867007907 \ldots$$
Remark it is possible to define their continuation into the lower half-plane, so we consider solutions with initial values at the point \((x_0, 0, 0)\).

Functions for \(x\) can be immediately established by the analogous proof as the one for smoothness of \(y\) of \(x\) such that the minima of solutions corresponding to \(0\) are located to the left of \(x_0\) and, respectively, to the right of \(x_0\) for \(\delta_+ (x_0, y_0)\). Since we know that \(\lim_{y_0 \to -\infty} \delta_+ (x_0, y_0) = 0\), both minima have to reach abscissa \(x_0\) at some finite values (maybe different) \(y_- (x_0)\) and \(y_+ (x_0)\). It is a further interesting question whether actually \(y_- (x_0) = y_+ (x_0) = y_0 (x_0)\) and for \(y_0 \leq y_0 (x_0)\), \(\delta_- (x_0, y_0) = \delta_+ (x_0, y_0)\)?

**References**


The accuracy of 18 digits in \(x_{\text{max}}\) allows one to determine 36 digits of \(C(0)\),

\[
C(0) = 0.69663587640019346382935052393992493\ldots
\]

The minimum value \(y_{\text{lim}}\) of the maximal solution corresponding to the segment \([Z(x_0, 0, 0), x_0]\) has the following asymptotics,

\[
y_{\text{lim}} = \lim_{x_0 \to -\infty} y_{\text{max}}^{x_0} \sqrt{|x_0|} + o \left( \sqrt{|x_0|} \right).
\]

For a better numerical approximation in the last formula is preferable to use instead of \(x_0\) the corresponding minimum of the maximal solution.

**Remark B.3.** If the solution with the zero spacing equal to \(\delta_- (x_0, y_0)\) (cf. Proposition B.1) is unique, then smoothness of \(\delta_- (x_0, y_0)\) with respect to both variables can be immediately established by the analogous proof as the one for smoothness of \(X(x_0)\) in § 6 Corollary 6.1.

**Remark B.4.** We define functions \(\delta_+ (x_0, y_0)\) in the natural domain \(y_0 \geq 0\), however it is possible to define their continuation into the lower half-plane, \(y_0 < 0\). To do so we consider solutions with initial values at the point \((x_0, y_0)\) with the zeroes, \(z_0 < x_0 < z_1\) for \(\delta_- (x_0, y_0)\), and \(z_1 < x_0 < z_2\) for \(\delta_+ (x_0, y_0)\) and the supremum in Definition B.1 is taken over non-negative slopes for \(\delta_- (x_0, y_0)\) and non-positive for \(\delta_+ (x_0, y_0)\). Then in some neighbourhood of \(y_0 = 0\) function \(\delta_- (x_0, y_0)\) continues its monotonic increase, respectively function \(\delta_+ (x_0, y_0)\) its decrease, with the decrease of \(y_0\). This follows by the same arguments as in the proof of monotonicity of these functions for \(y_0 \geq 0\) in Propositions B.3 and B.4. The proof works for all values of \(y_0\) such that the minima of solutions corresponding to \(\delta_- (x_0, y_0)\) are located to the left of \(x_0\) and, respectively, to the right of \(x_0\) for \(\delta_+ (x_0, y_0)\).


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