The Quintessential Option Pricing Formula

Max Skipper & Peter Buchen

School of Mathematics and Statistics
University of Sydney, Australia

(Version 3: September 2003)

Abstract
It might be argued that nothing more can be said about pricing options under the Black-Scholes paradigm. We express the opposite view by presenting in this paper a new formula that unifies much of the existing literature on pricing exotic options within the Black-Scholes framework. The formula gives the arbitrage-free price of an $\mathbb{M}$-binary (a generalised multi-asset, multi-period binary option), which is a fundamental building block for more complex exotic options. To demonstrate the utility of the formula, we apply it to pricing several well known exotics and also to a new option: a discretely monitored call barrier option on the maximum of several assets.

Keywords: Exotic options, binaries, digitals, static replication.

1 Introduction

Any option or derivative that is not a *plain vanilla* call or put is generally referred to as an *exotic option.* One class of single asset exotics are those with path-dependent payoffs. Examples include: Asian options, barrier options, lookback options, multi-period digitals, compound options, chooser options and many others. Multi-asset exotics, sometimes called *rainbow options* have also become popular in the last couple of decades. Examples of these include: exchange options, basket options, min/max and best/worst options. Closed form price solutions have been found for most of the above [1] in the Black-Scholes framework. While the following is by no means an exhaustive list, it nevertheless shows the wide range of published option price solutions within the Black-Scholes framework. Merton (1973), Reiner and Rubinstein (1991), Rich (1994),Heynen and Kat (1994), Buchen (2001) analyse barrier options, Geske (1979) and Rubinstein (1992) provide pricing formulae for compound options; Margrabe (1978) and Carr (1988) price exchange options; Stulz (1982), Johnson (1987), Rich and Chance (1993) price options on the maximum and minimum of several assets and their variants; Kemna and Vorst (1990) price Geometric Asian options; Longstaff (1990) prices extendable options; Conze and Viswanathan (1991) analyse lookback options; Rubinstein and Reiner (1991) and Heynen and Kat (1996) price single asset and two-asset binaries; Rubinstein (1991) introduced and analysed chooser options; Zhang (1995) prices correlated digital options; while Buchen (2003) analyses the class of dual-expiry exotics.

Pricing for non Black-Scholes dynamics (including stochastic volatility) will generally require numerical schemes such as Monte Carlo simulation (*e.g.* Boyle 1977). Since most exotic options these days do require a stochastic volatility model, one might ask the point of continuing to price in the Black-Scholes world. Apart from its intrinsic and academic value, one practical application of the formula presented in this paper is that it can provide Monte Carlo schemes with good control variates, used for variance reduction.

It might seem implausible that many of the above options can be priced by a single universal formula. We present in this paper precisely such a formula within the Black-Scholes framework. In particular we derive the arbirage-free price of a generalised multi-asset, multi-period exotic binary option [2]. We

[1] Arithmetic Asian and basket options are the exception.

[2] Binary options are also called digital options.

shall henceforth refer to these fundamental derivatives as $\mathbb{M}-$binaries (see Eq(16) for a formal definition).

We demonstrate that $\mathbb{M}-$binaries are building blocks for a wide class of exotic options. In this regard our approach is similar to that of Ingersoll (2000) who showed how digital options can be used to price more complex options including two-asset exotics. While Ingersoll derived several expressions for digitals under varying exercise conditions, we adopt a very different approach in this paper. We are formally concerned with pricing only a single option: the $\mathbb{M}$-binary , which includes cash digitals, asset digitals and all possible power (or turbo) digitals. Many exotic options can be expressed as simple static portfolios of these digitals. It follows from the principle of static replication that the arbitrage-free price of these exotics must then be the corresponding value of their replicating portfolios.

While the formula we derive can price multi-variate, discretely monitored barrier and lookback options, certain enhancements to the theory presented here must be made to price continuously monitored versions of these options. Since these enhancements are not trivial we do not consider them in this paper. However preliminary results can be found in Skipper (2003).

The remainder of the paper is organised as follows. In Section 2 we define the multi-asset, multi-period framework and because the formulation is much more involved than usual, we give considerable attention to establishing a descriptive and unifying notation. We develop the multi-asset, multi-period price dynamics in Section 3 and state the Main Theorem and universal formula in Section 4. In Section 5 we illustrate the use of the formula through several examples including: asset and bond type binaries, compound options, strike reset options, geometric mean Asian options, quality options [3] includ-

[3]We coin the term *quality options* to include any option whose payoff depends on the

ing best and worst options, options on the maximum and minimum of several assets, discretely monitored lookback and barrier options. Section 6 offers a short conclusion and the Appendix gives details of the proofs of the two Theorems stated in the main text.

2 Set-up, Definitions and Notation

1. Let

$$\begin{aligned}
\mathcal{I}_n &= \text{The index set } \ \{1, 2, \ldots, n\} \\
\mathcal{V}_n &= \text{Any } n\text{−dimensional column vector} \\
\mathcal{A}_{mn} &= \text{Any } \ m \times n \text{ matrix} \\
\mathcal{D}_n &= \text{Any } \ n \times n \text{ diagonal matrix} \\
\mathcal{C}_n, \mathcal{C}_n^* &= \text{Any } \ n \times n \text{ covariance/correlation matrix} \\
\mathcal{G}_n(R) &= \text{Any zero-mean Gaussian } \ n - \text{vector} \\
&\quad\ \text{with positive definite correlation matrix } R \in \mathcal{C}_n^*.
\end{aligned}$$

2. The *asset parameter set* $\mathbb{A}$ is the set of (constant) parameters:

$$\mathbb{A} = [r,\, x_i,\, q_i,\, \sigma_i,\, \rho_{ij}]; \qquad (i, j \in \mathcal{I}_N) \tag{1}$$

that underlie the $N-$asset price dynamics. In the above: r is the risk free interest rate, x_i, q_i, σ_i are the present value, dividend yield and volatility of asset i and ρ_{ij} is the correlation coefficient of the instantaneous log-returns of asset i with asset j.

3. The *tenor set* $\mathbb{T}$ for an $\mathbb{M}$-binary is the set of times:

$$\mathbb{T} = [t,\, T_1, T_2, \cdots, T_M,\, T] \tag{2}$$

maximum or minimum price of a given set of assets.

where t is the current time, $T_k,\ k \in \mathcal{I}_M$ are M fixed asset price monitoring times which occur in the $\mathbb{M}$-binary payoff function V_T, and T is the expiry date of the option (*i.e.* the date at which the payoff is actually made). We assume that $t < T_1 < T_2 < \cdots < T_M \le T$. Often the last monitoring time T_M coincides with T, but they may also be distinct.

4. The *dimension set* $\mathbb{D}$ for an $\mathbb{M}$-binary is the set of integers:

$$\mathbb{D} = [N,\, M,\, n,\, m]; \qquad (m \in \mathcal{I}_n) \tag{3}$$

where N is the number of assets, M is the number of monitoring periods; n is the 'payoff dimension' and m is the 'exercise dimension'. The payoff and exercise dimensions are defined later.

5. The *payoff parameter set* $\mathbb{P}$ for an $\mathbb{M}$-binary is the 4-parameter vector and matrix set

$$\mathbb{P} = [\boldsymbol{\alpha},\, \boldsymbol{a},\, S,\, A] \tag{4}$$

where $\boldsymbol{\alpha} \in \mathcal{V}_n$ is the 'payoff index vector', $\boldsymbol{a} \in \mathcal{V}_m$ is the 'exercise price vector', $S \in \mathcal{D}_m$ is the 'exercise indicator matrix' and $A \in \mathcal{A}_{mn}$ is the 'exercise condition matrix'. These parameters, which determine the expiry payoff and exercise conditions of the $\mathbb{M}$-binary are defined in Section 4.

6. The *payoff vector* $\boldsymbol{X}$.
Let $X_i(s), i \in \mathcal{I}_N$ denote the price of asset-i at any time s where $t < s \le T$. The $X_i(s)$ are stochastic processes, which in the Black-Scholes framework, follow correlated multi-variate geometric Brownian motions. We shall be concerned with options whose expiry T payoffs V_T depend on the random prices [4] $X_{ik} = X_i(T_k)$ for some subset of monitoring times T_k. To allow flexible payoff structures we do not require that X_{ik} contains every asset at

[4] We shall generally use indices i, j to denote different assets and indices k, l to denote different monitoring times.

all monitoring times – only that each asset involves at least one of these times.

We assemble all the relevant components X_{ik} into a *payoff vector* $\boldsymbol{X}$. The arrangement of these components is arbitrary, but each component corresponds to a unique pair $(i, k) \in \mathcal{I}_N \times \mathcal{I}_M$. It should be clear that for each index $i \in \mathcal{I}_N$, k may assume any subset of values in $\mathcal{I}_M$; similarly for each index $k \in \mathcal{I}_M$, i may take a corresponding subset of values in $\mathcal{I}_N$. Because it is more informative, rather than re-label the components with a new index, we retain both indices (i, k) and call such vectors: *dual-index* vectors. Clearly, dual-index vectors (and matrices) become single-index when there is only one underlying asset, or when there is only one future monitoring time. We refer to the dimension n of $\boldsymbol{X}$ as the *payoff dimension*. In general it differs from both N (the number of assets) and M (the number of time periods) and may take any value in the range $N \leq n \leq NM$.

Example 1
To illustrate the type of payoff structures we shall be pricing, consider a derivative whose expiry T payoff is $V_T = f[X_1(T_1), X_1(T_2), X_2(T_1), X_3(T_2)]$ for some function f, with $T_1 < T_2 \leq T$. This derivative is multi-asset, multi-period in the sense we have defined above with $N = 3$ assets, $M = 2$ periods and payoff dimension $n = 4$. One representation of the dual-index payoff vector (any ordering of the components is permissible) is $\boldsymbol{X} = (X_{11}, X_{12}, X_{21}, X_{32})'$. The payoff function for this example would then be written as $V_T(\boldsymbol{X}) = f(\boldsymbol{X})$. □

It should be clear that multi-asset, multi-period binaries are very complex derivatives requiring many parameters to define them in full. We therefore adopt a notation that is designed to both simplify mathematical formulae and aid understanding. It is evident from the preceding discussion, that this notation is vector and matrix oriented. As described below, we also employ some deliberate abuse of standard matrix rules to enhance the clarity of the exposition.

- If $\boldsymbol{x} \in \mathcal{V}_n$ is a vector (we use the notation $\boldsymbol{x}'$ to represent its transpose) then we shall freely use expressions like $e^{\boldsymbol{x}}$, $\log \boldsymbol{x}$, $\boldsymbol{x}^2$, $\sqrt{\boldsymbol{x}}$, ... to denote similar vectors obtained by component-wise function evaluation. Thus $\log \boldsymbol{x}$ for example, denotes the vector with components $(\log x_1, \log x_2, \ldots, \log x_n)'$.

- We often ignore the difference between a vector (or matrix) and its components. Hence we shall feel free to write $\boldsymbol{x} = x_i$ or $\boldsymbol{X} = X_{ik}$ for single and dual-index vectors and $R = \rho_{ij}$ or $R = \rho_{ij}\tau_{kl}$ for matrices. The second matrix here is defined in Eq(11) and is a dual-index matrix with components drawn from the asset-time index pairs (i, k) and (j, l).

- Let $\boldsymbol{x} \in \mathcal{V}_n$ be a column vector of dimension n and let $A \in \mathcal{A}_{mn}$ be an $m \times n$ matrix. Then define the $m-$dimensional vector $\boldsymbol{x}^A \in \mathcal{V}_m$ by [5]

$$\boldsymbol{x}^A = \exp(A \log \boldsymbol{x}). \tag{5}$$

  The matrix A is the exercise condition matrix referred to earlier. A special case of Eq(5) is the choice $A = \boldsymbol{\alpha}$ where $\boldsymbol{\alpha} \in \mathcal{V}_n$ is an $n-$dimensional column vector. The result is the scalar quantity

$$\boldsymbol{x}^{\boldsymbol{\alpha}} = e^{\boldsymbol{\alpha}' \log \boldsymbol{x}} = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}. \tag{6}$$

  The parameter $\boldsymbol{\alpha}$ is the payoff index vector referred to in Eq(4).

- We let $\mathbf{1}$ and $\mathbf{0}$ denote column vectors with every component equal to 1 and 0 respectively. Also $\mathbf{1}_j$ denotes the column vector with $j-$component equal to 1 and all others equal to zero, and I_n denotes the $n-$dimensional identity matrix.

[5] We also allow A to be $n \times m$, in which case $\boldsymbol{x}^A = \exp(A' \log \boldsymbol{x})$ for compatibility.

- We define the $m-$dimensional indicator function by

$$\mathbb{I}_m(\boldsymbol{x} > \boldsymbol{a}) = \prod_{i=1}^{m} \mathbb{I}(x_i > a_i). \tag{7}$$

  Since $\mathbb{I}(x < a) \equiv \mathbb{I}(-x > -a)$ it is a simple matter to change 'less-than' indicators into 'greater-than' indicators as follows. Let $S \in \mathcal{D}_m$ be a diagonal matrix with all diagonal components equal to ± 1. Then the indicator $\mathbb{I}_m(S\boldsymbol{x} > S\boldsymbol{a})$ has $i-$component $\mathbb{I}(x_i > a_i)$ if $S_{ii} = 1$, and has $i-$component $\mathbb{I}(x_i < a_i)$ if $S_{ii} = -1$. We shall say that an option has *exercise dimension* m, if its expiry payoff depends on some $m-$dimensional indicator function. In practice, for any $\mathbb{M}$-binary we require $1 \leq m \leq n$ where n is the payoff dimension. Matrix S is the exercise indicator matrix of Eq(4).

- Option prices in the multi-variate Black-Scholes framework often involve the multi-variate normal distribution function. This is denoted here by $\mathcal{N}_m(\boldsymbol{d}; C)$, where $\boldsymbol{d} \in \mathcal{V}_m$ is an $m-$dimensional column vector and $C \in \mathcal{C}_m^*$ a positive definite $m-$dimensional correlation matrix. Thus if $\boldsymbol{Z} \in \mathcal{G}(C)$ is an $m-$dimensional Gaussian random vector with zero mean, unit variances and correlation matrix C, *i.e.* $\boldsymbol{Z} \sim N(\boldsymbol{0}, C)$, then

$$\mathcal{N}_m(\boldsymbol{d}; C) \stackrel{\text{def}}{=} \mathbb{E}\{\mathbb{I}_m(\boldsymbol{Z} < \boldsymbol{d})\}. \tag{8}$$

  Special cases of Eq(8) are the uni-variate ($m = 1$) and bi-variate ($m = 2$) normal distribution functions. These will be written in their more usual notations $\mathcal{N}(d)$ and $\mathcal{N}(d_1, d_2; \rho)$ respectively, where ρ is a correlation coefficient.

Of course a correlation matrix is just a normalised covariance matrix. In particular, if C is a positive definite covariance matrix with $D^2 = \mathrm{diag}(C)$ containing the variances on the main diagonal, then the corresponding correlation matrix is $C^* = D^{-1}CD^{-1}$.

3 Multi-Variate Asset Dynamics

We adopt the standard Black-Scholes framework extended to multiple assets with parameters defined by Eq(1). Thus the asset price process $X_i(s)$ for $i \in \mathcal{I}_N$ and $s \in (t, T]$ defined in the previous Section, satisfies the risk-neutral stochastic differential equation (SDE) for all $s > t$

$$dX_i(s) = X_i(s)[(r - q_i)ds + \sigma_i dW_i(s)]; \qquad X_i(t) = x_i \tag{9}$$

where $W_i(s)$ are correlated Brownian motions with $\mathbb{E}\{dW_i(s)dW_j(s)\} = \rho_{ij}ds$. Using Itô's Lemma and the stationarity of $W_i(s)$, this SDE has a solution that can be expressed as

$$\log X_i(s) \stackrel{\mathrm{d}}{=} \log x_i + (r - q_i - \tfrac{1}{2}\sigma_i^2)(s - t) + \sigma_i W_i(s - t) \tag{10}$$

It follows from this representation and the property $\mathbb{E}\{W_i(\tau_k)W_i(\tau_l)\} = \min(\tau_k, \tau_l)$ that $\log X_i(T_k)$ is Gaussian with correlation coefficients determined by

$$R = \mathrm{corr}\{\log X_{ik}, \log X_{jl}\} = \rho_{ij}\tau_{kl}; \qquad \tau_{kl} = \frac{\min(\tau_k, \tau_l)}{\sqrt{\tau_k \tau_l}} \tag{11}$$

where $\tau_k = T_k - t$.

Further, since the $W_i(\tau_k)$ are Gaussian with zero mean and variance equal to τ_k, we express Eq(10) at the monitoring times $s = T_k$ in the equivalent form

$$\log X_i(T_k) \stackrel{\mathrm{d}}{=} \log x_i + (r - q_i - \tfrac{1}{2}\sigma_i^2)\tau_k + \sigma_i \sqrt{\tau_k}\, Z_i(\tau_k). \tag{12}$$

$Z_i(\tau_k) \in \mathcal{G}_n(R)$ are Gaussian random variables with (dual-index) correlation matrix $R = \rho_{ij}\tau_{kl}$. The ρ_{ij} are correlation coefficients of instantaneous log-returns across different assets at the same time; the τ_{kl} are correlation coefficients across different times for the same asset. In matrix form, Eq(12) has the representation

$$\log \boldsymbol{X} \stackrel{\mathrm{d}}{=} \log \boldsymbol{x} + \boldsymbol{\mu} + \Sigma \boldsymbol{Z} \tag{13}$$

where $\boldsymbol{x}$ is the present value of $\boldsymbol{X}$ (evaluated at $\tau_k = 0$ for all $k \in \mathcal{I}_M$),

$$\boldsymbol{\mu} = (r - q_i - \tfrac{1}{2}\sigma_i^2)\tau_k \quad \in \mathcal{V}_n \tag{14}$$

$$\Sigma = \mathrm{diag}(\sigma_i\sqrt{\tau_k}) \quad \in \mathcal{D}_n \tag{15}$$

and $\boldsymbol{Z} \in \mathcal{G}_n(R)$ is a Gaussian vector with correlation matrix R defined by Eq(11).

4 The Main Result

In this Section we give a precise definition of the generalised multi-asset, multi-period $\mathbb{M}$-binary in terms of its expiry payoff $V_T(\boldsymbol{X})$ and state its arbitrage-free present value $V(\boldsymbol{x}, t)$ assuming the multi-variate Black-Scholes representation of the payoff vector $\boldsymbol{X}$ as stated in Eqs(12)-(15). This is the main result of the paper. The proof, which depends on three well known Lemmas is relegated to the Appendix and the notation used is that of Section 2 and Section 3.

Definition ($\mathbb{M}$-binary)
An $\mathbb{M}$-binary with payoff parameter set $\mathbb{P}$, dimension set $\mathbb{D}$, tenor set $\mathbb{T}$ and payoff vector $\boldsymbol{X} \in \mathcal{V}_n$ is a multi-asset, multi-period binary option with expiry T payoff function

$$\boxed{V_T(\boldsymbol{X}) = \boldsymbol{X}^{\boldsymbol{\alpha}}\, \mathbb{I}_m(S\boldsymbol{X}^A > S\boldsymbol{a})} \tag{16}$$

Theorem 1 (Main Theorem)
The arbitrage-free present value of an $\mathbb{M}$-binary as defined above, under log-normal asset price dynamics with asset parameter set $\mathbb{A}$ is given by

$$\boxed{V(\boldsymbol{x}, t) = \beta \boldsymbol{x}^{\boldsymbol{\alpha}}\, \mathcal{N}_m(S\boldsymbol{d}; SCS)} \tag{17}$$

where

$$
\begin{aligned}
\beta &= \exp\left(-r\tau + \boldsymbol{\alpha}'\boldsymbol{\mu} + \tfrac{1}{2}\boldsymbol{\alpha}'\Gamma\boldsymbol{\alpha}\right); \quad \tau = T - t && (18)\\
\boldsymbol{d} &= D^{-1}\left\{\log\left(\boldsymbol{x}^{A}/\boldsymbol{a}\right) + A(\boldsymbol{\mu} + \Gamma\boldsymbol{\alpha})\right\} \quad \in \mathcal{V}_m && (19)\\
C &= D^{-1}(A\Gamma A')D^{-1} \quad \in \mathcal{C}_m^* && (20)
\end{aligned}
$$

and

$$
\begin{aligned}
\Gamma &= \Sigma R \Sigma' \qquad \in \mathcal{C}_n && (21)\\
D^2 &= \mathrm{diag}(A\Gamma A') \in \mathcal{D}_m && (22)
\end{aligned}
$$

Proof: See the Appendix. □

The payoff function in Eq(16) is very general. It includes cash digitals (when $\boldsymbol{\alpha} = \mathbf{0}$ giving $\boldsymbol{X}^{\boldsymbol{\alpha}} = 1$), asset digitals (when $\boldsymbol{\alpha} = \mathbf{1}_{jk}$ resulting in $\boldsymbol{X}^{\boldsymbol{\alpha}} = X_{jk}$) and general power (or turbo) payoffs otherwise. These payoffs are contingent on a wide selection of possible exercise scenarios determined principally by the exercise condition matrix A. The importance of the matrix A is illustrated in the example below and the many applications that follow. The term $\log\left(\boldsymbol{x}^{A}/\boldsymbol{a}\right)$ in Eq(19) is simply a shorthand notation for the vector $(A\log\boldsymbol{x} - \log\boldsymbol{a})$.

Example 2

As an example of an $\mathbb{M}$-binary consider the derivative with expiry payoff

$$V_T(\boldsymbol{X}) = \sqrt{X_1X_2}\ \mathbb{I}(\sqrt{X_1X_3} > X_2)\mathbb{I}(X_4 < a)$$

This is an $\mathbb{M}$-binary with $n = 4$, $m = 2$, payoff vector $\boldsymbol{X} = (X_1, X_2, X_3, X_4)'$ and payoff parameter set $\mathbb{P} = [\boldsymbol{\alpha}, \boldsymbol{a}, S, A]$ where

$$\boldsymbol{\alpha} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix}; \quad \boldsymbol{a} = \begin{pmatrix} 1 \\ a \end{pmatrix}; \quad S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad A = \begin{pmatrix} \frac{1}{2} & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ \square$$

It is also worth noting that calculations are considerably reduced if $A = I_n$ is an $n-$dimensional identity matrix. This leads to the direct equivalences $D = \Sigma$ and $C = R$. In such cases the exercise conditions are simple: *i.e.* exercise is determined by whether asset prices at certain monitoring times are above or below a corresponding exercise price. We shall meet several examples (as in Example 2 above) where A is not an identity matrix and the corresponding exercise condition is non-simple.

5 Applications

In order to apply Theorem 1 to price a multi-asset, multi-period exotic option it is necessary to carry out some preparatory steps, as described below.

1. **Static Replication**
   The first step is to assign the parameter set $\mathbb{A}$ for the given asset dynamics and construct the payoff vector $\boldsymbol{X}$. In all the ensuing applications we shall assume that $\mathbb{A}$ has a structure of the form of Eq(1) and for simplicity we set all dividend yields $q_i = 0$. Then represent the option payoff at expiry T as a portfolio (or linear combination) of $\mathbb{M}-$binaries each of the algebraic form of Eq(16).

2. **Specify Binary Inputs**
   For each such $\mathbb{M}$-binary obtain the dimension, tenor and payoff parameter sets: $\mathbb{D}, \mathbb{T}$ and $\mathbb{P}$ defined in Section 2. Many of these quantities will be dual-index vectors and matrices based on the structure of the payoff vector $\boldsymbol{X} = X_{ik}$. Recall that it doesn't matter how the components are assembled into $\boldsymbol{X}$, as long as all similar quantities are assembled the same way.

   The asset parameter set $\mathbb{A}$ determines the additional quantities $[\boldsymbol{\mu}, \Sigma, \Gamma]$ defined respectively by Eqs(14), (15) and (21). Similarly, the tenor set

$\mathbb{T}$ allows calculation of the time dependent correlation coefficients τ_{kl} defined by Eq(11).

3. **Compute the Outputs**
   The third step is to compute the output parameters $[\beta,\, D,\, \boldsymbol{d},\, C]$ from Eqs (18), (22), (19) and (20) and obtain the price $V(\boldsymbol{x}, t)$ from Eq(17). Finally, the exotic option price will be equal to the combined price of the replicating portfolio of these $\mathbb{M}-$binaries .

We basically adopt this three step procedure in all the applications that follow.

5.1 Asset and Bond Binaries

Consider first a class of generalised asset and bond type binary options [6] with simple exercise conditions. These have often be used as building blocks for more complex exotics. Assume the tenor and dimension structures of Eqs(2) and (3) with $T = T_M$.

On the expiry date T, the bond binary pays one unit of cash provided the prices $\boldsymbol{X} = X_{ik} = X_i(T_k)$, are above (or below) corresponding exercise prices $\boldsymbol{a} = a_{ik}$. The payoff and exercise dimensions are equal and assumed to have value n, not necessarily equal to NM. For these binaries, two of the four payoff parameters are fixed: $\boldsymbol{\alpha} = \boldsymbol{0}$ and $A = I_n$. We therefore use the symbol $B(\boldsymbol{x}, t; S, \boldsymbol{a})$ to refer to this bond binary.

The asset binary, similarly referred to by the symbol $A_j(\boldsymbol{x}, t; S, \boldsymbol{a})$, is identical to the above except that it pays one unit of asset-j at the expiry date T. Its payoff differs only in the value of the parameter $\boldsymbol{\alpha} = \boldsymbol{1}_{jM}$.

[6] These are also known as asset-or-nothing and cash-or-nothing options.

Calculations for the bond-type $\mathbb{M}$-binary :

<u>*Inputs*</u>:

$$\begin{aligned} V_T(\boldsymbol{X}) &= B(\boldsymbol{X},T;S,\boldsymbol{a}) = \mathbb{I}_n(S\boldsymbol{X}>S\boldsymbol{a}) \\ \mathbb{D} &= [N,\ M,\ n,\ n] \\ \mathbb{P} &= [\mathbf{0},\ \boldsymbol{a},\ S,\ I_n] \\ [\boldsymbol{\mu},\ \Sigma,\ \Gamma] &= [(r-\tfrac{1}{2}\sigma_i^2)\tau_k,\ \mathrm{diag}(\sigma_i\sqrt{\tau_k}),\ \rho_{ij}\sigma_i\sigma_j\min(\tau_k,\tau_l)] \end{aligned}$$

where $i,j\in\mathcal{I}_N,\ \ k,l\in\mathcal{I}_M$ and $\tau_k=T_k-t$.

<u>*Outputs*</u>:

$$\begin{aligned} \beta &= e^{-r\tau}; \qquad \tau=T-t \\ D &= \Sigma \\ \boldsymbol{d} &= [\log(x_i/a_{ik})+(r-\tfrac{1}{2}\sigma_i^2)\tau_k]/\sigma_i\sqrt{\tau_k} = d'_{ik}\ \text{(say)} \\ C &= R=\rho_{ij}\tau_{kl} \end{aligned}$$

Hence, the present value of this $\mathbb{M}$-binary given by Eq(17), is

$$B(\boldsymbol{x},t;S,\boldsymbol{a}) = e^{-r\tau}\,\mathcal{N}_n(Sd'_{ik};\ SRS). \tag{23}$$

For the corresponding asset binary, the payoff is

$$V_T = A_j(\boldsymbol{X},T;S,\boldsymbol{a}) = X_{jM}\mathbb{I}_n(S\boldsymbol{X}>S\boldsymbol{a})$$

We assume that X_{jM} is also one of the components of the exercise condition; it is a simple matter to handle the alternative case, by making an appropriate change in the exercise condition matrix A. This has payoff index vector $\boldsymbol{\alpha}=\mathbf{1}_{jM}$, otherwise all other input parameters are as for the bond binary above. A similar calculation then leads to the expression

$$A_j(\boldsymbol{x},t;S,\boldsymbol{a}) = x_j\,\mathcal{N}_n(Sd_{ik};\ SRS) \tag{24}$$

where $d_{ik}=d'_{ik}+\rho_{ij}\sigma_j\sqrt{\tau_k}$.

(a) **Multi-asset, Single-period bond and asset binaries**

When there is only a single period ($M = 1$) we may drop all the time indices k, l and the above formulae reduce to

$$
\begin{aligned}
B(\boldsymbol{x}, t; S, \boldsymbol{a}) &= e^{-r\tau} \mathcal{N}_n(Sd'_i;\, SRS) && (25) \\
A_j(\boldsymbol{x}, t; S, \boldsymbol{a}) &= x_j \mathcal{N}_n(S[d'_i + \rho_{ij}\sigma_j\sqrt{\tau}];\, SRS) && (26)
\end{aligned}
$$

where $\tau = (T - t)$, $d'_i = [\log(x_i/a_i) + (r - \frac{1}{2}\sigma_i^2)\tau]/\sigma_i\sqrt{\tau}$ and $R = \rho_{ij}$.

(b) **Single-asset, multi-period bond and asset binaries**

When there is only a single asset ($N = 1$) we may drop all the asset indices i, j to get

$$
\begin{aligned}
B(x, t; S, \boldsymbol{a}) &= e^{-r\tau} \mathcal{N}_n(Sd'_k; SRS) && (27) \\
A(x, t; S, \boldsymbol{a}) &= x \mathcal{N}_n(Sd_k; SRS) && (28)
\end{aligned}
$$

with $[d_k, d'_k] = [\log(x/a_k) + (r \pm \frac{1}{2}\sigma^2)\tau_k]/\sigma\sqrt{\tau_k}$ and $R = \tau_{kl}$.

(c) **Single-asset, single-period bond and asset binaries**

In the simplest case where $N = M = 1$ we reduce to 1-asset, 1-period binaries which are the well known Black-Scholes components of standard European call and put options:

$$
A(x, t; s, a) = x\mathcal{N}(sd) \qquad \text{and} \qquad B(x, t; a) = e^{-r\tau}\mathcal{N}(sd') \tag{29}
$$

respectively with $s = \pm 1$ and $[d, d'] = [\log(x/a) + (r \pm \frac{1}{2}\sigma^2)\tau]/\sigma\sqrt{\tau}$.

(d) **Compound Options**

The bond and asset type binaries derived above can be used to price related derivatives. We illustrate this idea with a standard call-on-call compound option. Let c denote the strike price of the compound option exercised at T_1, and a_2 the strike price of the underlying call option exercised at $T_2 > T_1$. Let $x = a_1$ denote the solution of: $C(x, T_2 - T_1; a_2) = c$ where the left side of this equation denotes the standard Black-Scholes price of a strike a_2 call option with time $(T_2 - T_1)$ remaining to expiry. The compound option will be exercised at T_1 if and only if $X_1 > a_1$. Hence the final payoff of the compound option at time $T = T_2$ can be written in the form [7]

$$\begin{aligned} V(x, T_2) &= [(X_2 - a_2)^+ - ce^{r(\tau_2 - \tau_1)}]\,\mathbb{I}(X_1 > a_1) \\ &= (X_2 - a_2)\mathbb{I}(X_1 > a_1)\mathbb{I}(X_2 > a_2) - ce^{r(\tau_2 - \tau_1)}\mathbb{I}(X_1 > a_1). \end{aligned} \tag{30}$$

This is a portfolio of asset and bond type binaries considered in parts (b) and (c) above with $m = 1$ and $m = 2$. We can therefore write down the present value of this portfolio in terms of uni-variate and bi-variate normals:

$$V(x, t) = x\mathcal{N}(d_1, d_2; \rho) - a_2 e^{-r\tau_2}\mathcal{N}(d_1', d_2'; \rho) - ce^{-r\tau_1}\mathcal{N}(d_1') \tag{31}$$

where $\rho = \tau_{12} = \sqrt{\tau_1/\tau_2}$ and the notation is that of Eqs(27) and (28). This formula is the well-known result first derived by Geske (1979).

(e) **Multiple strike reset options**

These options have recently been priced in the Black-Scholes framework by Liao and Wang (2003). A reset call option involves a single-asset and a multi-period payoff of the form $V_T = [X(T) - K]^+$ where

$$K = \begin{cases} k_0 & \text{if } X_{\min} > a_1 \\ k_i & \text{if } a_{i+1} < X_{\min} \le a_i; \qquad i = 1, 2, \cdots, (p-1) \\ k_p & \text{if } X_{\min} \le a_p \end{cases} \tag{32}$$

$X_{\min} = \min(X_1, X_2, \cdots, X_{n-1})$ and $X_i = X(T_i)$ are the monitored asset prices at $n-1$ pre-determined reset dates T_i with $T_1 < T_2 < \cdots < T_{n-1} < T$. The k_i are the reset strike prices and the a_i are a decreasing set of discrete 'ladder prices'. It follows that

$$V_T = \sum_{i=0}^{p} (X_T - k_i)^+\,\mathbb{I}(a_{i+1} < X_{\min} \le a_i)$$

[7] The exponential multiplying c is the future value factor at time T_2 for a cash payment at time T_1.

with $a_0 = \infty$ and $a_{p+1} = 0$. Let $T_n = T$ and define $\boldsymbol{X}$ as the $n-$dimensional payoff vector with components $X_i,\ i \in \mathcal{I}_n$. Then it is a relatively straight forward matter to express this payoff as a portfolio of single-asset, multi-period asset and bond type binaries defined in part(b) above. This is readily achieved using the identity

$$\mathbb{I}(a_{i+1} < X_{\min} \le a_i)\mathbb{I}(X_T > k) = \mathbb{I}_n(\boldsymbol{X} > \hat{\boldsymbol{a}}_i) - \mathbb{I}_n(\boldsymbol{X} > \boldsymbol{a}_i) \tag{33}$$

where $\boldsymbol{a}_i = (a_i, a_i, \ldots, a_i, k_i)'$ and $\hat{\boldsymbol{a}}_i = (a_i, a_i, \ldots, a_i, k_{i-1})'$.

Eqs(27) and (28) then give the price of the multiple reset call option as the present value of this portfolio. We do not write the explicit formula down but remark that the representation we get appears to be considerably simpler than the one stated by Liao and Wang (2003).

5.2 Discrete Geometric Mean Asian Options

Prices for these options in the Black-Scholes framework are also well known, for both discrete and continuous time monitoring. We shall derive here two formulas for discretely monitored geometric mean Asian call options (one a fixed strike, the other a floating strike). These examples provide strong evidence of the utility of the fundamental formula for pricing exotics. Furthermore, they illustrate perhaps the simplest cases where the exercise condition matrix A is not an identity matrix.

While it is common to have the monitoring times equally spaced, this is by no means essential. We therefore assume there is a single asset and that the n monitored time periods $T_k, (1 \le k \le n)$ are arbitrarily spaced. Geometric mean Asian options have payoffs at time $T \ge T_n$ which depend on the geometric mean

$$G_n = \sqrt[n]{X_1 X_2 \cdots X_n} \tag{34}$$

where $X_k = X(T_k)$ is the observed asset price at time T_k. Since $t < T_1$ in our set-up, the first monitored price has yet to be observed. It is not difficult to include monitored prices prior to t, and this leads to only slightly more complicated expressions.

(a) **Fixed Strike Call**

The payoff for a fixed strike k, geometric mean Asian call option is given by $V_T(\boldsymbol{X}) = (G_n - k)^+$. This expression can be decomposed into the difference of two $\mathbb{M}-$binaries : $V_1 = G_n\mathbb{I}(G_n\!>\!k)$ and $V_2 = k\mathbb{I}(G_n\!>\!k)$. We investigate V_1 in detail.

Inputs:

$$
\begin{aligned}
V_1(\boldsymbol{X}, T) &= G_n\mathbb{I}(G_n\!>\!k)\\
\mathbb{D} &= [1, n, n, 1]\\
\mathbb{P} &= [\frac{1}{n}\mathbf{1},\ k,\ 1,\ \frac{1}{n}\mathbf{1}']\\
[\boldsymbol{\mu},\, \Sigma,\, \Gamma] &= [(r - \tfrac{1}{2}\sigma^2)\tau_k,\ \sigma\mathrm{diag}(\sqrt{\tau_k}),\ \sigma^2\min(\tau_k, \tau_l)]; \qquad \tau_k = T_k - t
\end{aligned}
$$

Outputs:

$$
\begin{aligned}
\beta &= e^{-r(\tau-\bar{\tau})-\frac{1}{2}\sigma^2(\bar{\tau}-\hat{\tau})}; \qquad \bar{\tau} = \frac{1}{n}\sum_k \tau_k; \quad \hat{\tau} = \frac{1}{n^2}\sum_{k,l}\min(\tau_k, \tau_l)\\
D &= \sigma\sqrt{\hat{\tau}}\\
d &= d' + \sigma\sqrt{\hat{\tau}}; \qquad d' = [\log(x/k) + (r - \tfrac{1}{2}\sigma^2)\bar{\tau}]/\sigma\sqrt{\hat{\tau}}\\
C &= 1\\
V_1(x, t) &= \beta x\mathcal{N}(d).
\end{aligned}
$$

The calculation for the second term $V_2(x, t)$ is very similar, with $\boldsymbol{\alpha}$ now being replaced by a zero vector. This results in $V_2(x, t) = ke^{-r\tau}\mathcal{N}(d')$. The final expression for the fixed strike, geometric mean Asian call option price is therefore:

$$V(x,t) = xe^{-r(\tau-\bar{\tau})-\frac{1}{2}\sigma^2(\bar{\tau}-\hat{\tau})}\mathcal{N}(d) - ke^{-r\tau}\mathcal{N}(d'). \tag{35}$$

(b) **Floating Strike Call** [8]

The floating strike geometric mean Asian call option has an expiry $T \geq T_n$ payoff given by $V_T = (X_n - G_n)^+$. Hence the geometric mean acts as the strike price of an otherwise standard call option. This too can be decomposed into two $\mathbb{M}-$binaries :

$$V_T = X_n\mathbb{I}(G_nX_n^{-1}<1) - G_n\mathbb{I}(G_nX_n^{-1}<1).$$

Of particular interest here are the associated expressions for the parameters S, A and $\boldsymbol{a}$ so that the exercise condition is equivalent to: $S\boldsymbol{X}^A > S\boldsymbol{a}$. We also find: $S = -1$, $A = (\frac{1}{n}\mathbf{1} - \mathbf{1}_n)'$, $\boldsymbol{a} = 1$ and $\boldsymbol{x} = x\mathbf{1}$. With these parameters it is evident that the term $\log(\boldsymbol{x}^A/\boldsymbol{a})$ vanishes identically. Straight forward calculations then lead to the expression:

$$V(x,t) = x\left[\mathcal{N}(-d) - e^{-r(\tau_n-\bar{\tau})+\frac{1}{2}\sigma^2(\hat{\tau}-\bar{\tau})}\mathcal{N}(-d')\right] \tag{36}$$

where

$$d = (r + \tfrac{1}{2}\sigma^2)(\bar{\tau} - \tau_n)/\sigma\sqrt{s}; \quad d' = [(r - \tfrac{1}{2}\sigma^2)(\bar{\tau} - \tau_n) + \sigma^2(\hat{\tau} - \bar{\tau})]/\sigma\sqrt{s}$$

and $s = (\tau_n + \hat{\tau} - 2\bar{\tau})$.

5.3 Quality Options

We define a *quality option* as any derivative whose payoff depends on the maximum or minimum of multiple asset prices at a fixed expiry T. Assume there are n correlated assets and a single period T coinciding with the expiry date of the option. In particular, a *quality binary* option $Q_p^s(\boldsymbol{x}, t; k)$ pays one

[8]This particular derivative does not seem to have been priced in the literature.

unit of asset p (say), with $p \in \mathcal{I}_n$, provided this is the maximum (if $s = 1$), or the minimum (if $s = -1$), of all the asset values at time T and a given amount of cash k. In order to price these binaries Theorem 2 is very useful, as it shows how to construct the required exercise condition matrix A and associated exercise price vector $\boldsymbol{a}$.

Theorem 2
Let $\boldsymbol{X} \in \mathcal{V}_n$ and let k be a positive scalar. Then for any integer $p \in \mathcal{I}_n$, the statement $X_p = \max / \min(\boldsymbol{X}, k)$ is equivalent to $s\boldsymbol{X}^{A_p} > s\boldsymbol{a}_p$ where $s = +1$ corresponds to the maximum, $s = -1$ to the minimum and $A_p \in \mathcal{A}_{nn}$, $\boldsymbol{a}_p \in \mathcal{V}_n$ have components

$$A_{ij} = \begin{cases} \;\;1 & \text{if } j = p \\ -1 & \text{if } i = j \neq p \\ \;\;0 & \text{otherwise} \end{cases} \qquad a_i = \begin{cases} k & \text{if } i = p \\ 1 & \text{if } i \neq p \end{cases} \qquad i, j \in \mathcal{I}_n \tag{37}$$

The proof of this result is also relegated to the Appendix. □

Theorem 2 allows us to write the quality binary payoff in the form of a simple $\mathbb{M}$-binary :

$$Q_p^s(\boldsymbol{X}, T; k) = X_p \, \mathbb{I}_n(s\boldsymbol{X}^{A_p} > s\boldsymbol{a}_p). \tag{38}$$

The most interesting part of the calculations is that for the correlation matrix. With $\Gamma = \rho_{ij}\sigma_i\sigma_j\tau$, $C_p = D_p^{-1}(A_p\Gamma A_p')D_p^{-1}$ and $D_p^2 = \text{diag}(A_p\Gamma A_p')$ we obtain the symmetric correlation matrix C_p with components

$$C_{ij} = \begin{cases} 1 & \text{for} \quad i = j \\ \frac{\sigma_p - \rho_{ip}\sigma_i}{\sigma_{ip}} & \text{for} \quad i \neq j = p \\ \frac{\sigma_{ip}^2 + \sigma_{jp}^2 - \sigma_{ij}^2}{2\sigma_{ip}\sigma_{jp}} & \text{for} \quad i \neq j \neq p \end{cases} \tag{39}$$

and $\sigma_{ij}^2 = \sigma_i^2 + \sigma_j^2 - 2\rho_{ij}\sigma_i\sigma_j$ for all i, j.

Then with the choices $\boldsymbol{\alpha} = \mathbf{1}_p$ and $S = sI_n$, the general binary option formula (17), for any $t < T$, results in the expression

$$Q_p^s(\boldsymbol{x}, t; k) = x_p \, \mathcal{N}_n(s\boldsymbol{d}_p; \, C_p) \tag{40}$$

where $\boldsymbol{d}_p$ is the vector with components

$$d_i = \begin{cases} \left[\log(x_p/k) + (r + \frac{1}{2}\sigma_p^2)\tau\right] / \sigma_p\sqrt{\tau} & \text{for} \quad i = p \\ \\ \left[\log(x_p/x_i) + \frac{1}{2}\sigma_{ip}^2\tau\right] / \sigma_{ip}\sqrt{\tau} & \text{for} \quad i \neq p. \end{cases} \tag{41}$$

(a) **Best and Worst Options**

Two well known quality options are the 'best' and 'worst' options on n assets and k units of cash, with expiry payoff:

$$Q^s(\boldsymbol{X}, T; k) = \begin{cases} \max(\boldsymbol{X}, k) & \text{if } s = 1 \\ \min(\boldsymbol{X}, k) & \text{if } s = -1. \end{cases} \tag{42}$$

Each of these payoffs can also be written as a portfolio of quality binaries and a bond type $\mathbb{M}$-binary (considered in Section 5.1):

$$\begin{aligned} Q^s(\boldsymbol{X}, T; k) &= \sum_{p=1}^{n} X_p \mathbb{I}_n(s\boldsymbol{X}^{A_p} > s\boldsymbol{a}_p) + k\mathbb{I}_n(s\boldsymbol{X} < sk\mathbf{1}) \\ &= \sum_{p=1}^{n} Q_p^s(\boldsymbol{X}, T; k) + kB(\boldsymbol{X}, T; -sI_n, k\mathbf{1}). \end{aligned} \tag{43}$$

It follows that the present value of these options can be expressed as

$$Q^s(\boldsymbol{x}, t; k) = \sum_{p=1}^{n} Q_p^s(\boldsymbol{x}, t; k) + kB(\boldsymbol{x}, t; -sI_n, k\mathbf{1}). \tag{44}$$

Eq(40) can be substituted into the first term while the last term is a special case of Eq(25) with $S = -sI_n$ and $\boldsymbol{a} = k\mathbf{1}$.

(b) **Call/Put on the max/min of several assets**

A call (put) option on the maximum (minimum) of several assets and its relatives have been considered by several authors including (Johnson 1987 and Rich and Chance 1993). Our analysis above provides expressions for these prices in terms of quality binaries as follows. The payoff for a call option on the maximum is given by

$$C_{\max}(\boldsymbol{X}, T; k) = [\max(\boldsymbol{X}) - k]^+ = \max(\boldsymbol{X}, k) - k. \tag{45}$$

Hence for any $t < T$ we can write down from Eq(44) the pricing formula

$$C_{\max}(\boldsymbol{x}, t; k) = \sum_{p=1}^{n} Q_p^+(\boldsymbol{x}, t; k) + kB(\boldsymbol{x}, t; -I_n, k\mathbf{1}) - ke^{-r\tau}. \qquad (46)$$

Similarly, the corresponding results for a put option on the minimum are:

$$P_{\min}(\boldsymbol{X}, T; k) = [k - \min(\boldsymbol{X})]^+ = k - \min(\boldsymbol{X}, k) \qquad (47)$$

and

$$P_{\min}(\boldsymbol{x}, t; k) = ke^{-r\tau} - \sum_{p=1}^{n} Q_p^-(\boldsymbol{x}, t; k) - kB(\boldsymbol{x}, t; I_n, k\mathbf{1}). \qquad (48)$$

These expressions of course agree with previously published formulations of the problem, albeit in a very different notation, and obtained by a very different method.

Formulae for a call option on the minimum and a put option on the maximum are easily obtained by a similar construction, so we are content to skip the details.

5.4 Discrete lookback options

Suppose there is a single asset, whose price $X_i = X(T_i)$ is monitored at the increasing sequence of times T_i for $i \in \mathcal{I}_n$. Derivatives whose payoffs at expiry $T \geq T_n$ depend on the maximum or minimum value of X_i are called discrete *lookback options*. It is clear that these options are the single-asset, multi-period counterparts of the multi-asset, single-period quality options considered in the previous sub-section. We shall consider here the case of a binary option that pays X_p provided $X_p = \max(\boldsymbol{X}, k)$ where k is a fixed amount of cash [9]. The payoff for this lookback binary is then

$$L(x, T; k) = X_p \, \mathbb{I}_n(\boldsymbol{X}^{A_p} > \boldsymbol{a}_p) \qquad (49)$$

[9]The fixed cash k can also be thought of as the currently (*i.e.* time t) observed maximum asset price.

where $A_p, \boldsymbol{a}_p$ are defined in Theorem 2 with indices referring to times rather than assets.

Since the analysis using Eq(17) of the Main Theorem is very similar to that for the quality binary, we leave out the details and simply state the result. The main difference occurs in the covariance matrix Γ which must now be replaced by $\Gamma = \sigma^2 \min(\tau_i, \tau_j)$. We ultimately obtain the result:

$$L(x, t; k) = xe^{-r(\tau-\tau_p)} \mathcal{N}_n(\boldsymbol{d}_p;\, C_p) \tag{50}$$

where $\boldsymbol{d}_p$ is the vector with components

$$d_i = \begin{cases} \left[\log(x/k) + (r + \frac{1}{2}\sigma^2)\tau_p\right]/\sigma\sqrt{\tau_p} & \text{for} \quad i = p \\ \frac{1}{\sigma}(r + \frac{1}{2}\sigma^2)\sqrt{\tau_p - \tau_i} & \text{for} \quad i < p \\ -\frac{1}{\sigma}(r - \frac{1}{2}\sigma^2)\sqrt{\tau_i - \tau_p} & \text{for} \quad i > p \end{cases} \tag{51}$$

and C_p is the symmetric correlation matrix with components

$$C_{ij} = \begin{cases} 1 & \text{for} \quad i = j \\ (1 - \tau_i/\tau_p)^{\frac{1}{2}} & \text{for} \quad i < j = p \\ [(\tau_i - \tau_p)/(\tau_j - \tau_p)]^{\frac{1}{2}} & \text{for} \quad p < i < j \\ [(\tau_p - \tau_j)/(\tau_p - \tau_i)]^{\frac{1}{2}} & \text{for} \quad i < j < p \\ 0 & \text{for} \quad i < p < j \end{cases} \tag{52}$$

Related lookback options, such as fixed strike lookback calls and puts can now be priced in terms of portfolios of these lookback binaries.

5.5 Discrete Barrier Options

As a final illustration of our fundamental pricing formula Eq(17) we consider a multi-asset, multi-period down-and-out discrete barrier call option. At expiry T the strike a option is assumed to pay $[\max\{X_i(T)\} - a]^+$ provided $X_{ik} > b_i$ for a given set of discrete barrier prices b_i. Thus if any of the asset prices $X_{ik} = X_i(T_k)$ ($i \in \mathcal{I}_N,\ k \in \mathcal{I}_M$ as usual) falls below the barrier level, the contract is immediately knocked out; if X_{ik} stays above the barrier level for all monitoring times T_k the contract pays a call option on the maximum asset price recorded at the expiry date $T > T_M$.

This option is certainly not a simple one and its pricing would, under standard methods of analysis, present a great challenge. The fundamental formula derived in this paper allows its calculation in just a few steps.

First we write its expiry payoff in the form

$$V(\boldsymbol{X}, T; a, \boldsymbol{b}) = [\max\{X_i(T)\} - a]^+ \, \mathbb{I}_m(X_i(T_k) > b_i) \tag{53}$$

where m is the barrier exercise dimension. The Main Theorem tells us that the only task is to write this expression as the sum of $\mathbb{M}-$binaries and identify the associated parameters.

In order to do this, define the payoff vector $\boldsymbol{X}$ as a collection of the m asset prices $\boldsymbol{X}_b = X_i(T_k)$ used for barrier monitoring and the N prices $\boldsymbol{X}_T = X_i(T)$ that determine the call option payoff. The partitioned payoff vector $\boldsymbol{X} = [\boldsymbol{X}_b, \boldsymbol{X}_T]'$ has dimension $n = (m + N)$.

Using the identity $(\max \boldsymbol{X}_T - a)^+ \equiv \max(\boldsymbol{X}_T, a) - a$ and the results of Section 5.3 on pricing quality options, we can write the payoff (53) in the required form as

$$V(\boldsymbol{X}, T; a, \boldsymbol{b}) = \sum_{p=1}^{N} \boldsymbol{X}^{\alpha_p^*} \, \mathbb{I}_n(\boldsymbol{X}^{A_p^*} > \boldsymbol{a}_p^*) + a\mathbb{I}_n(S^*\boldsymbol{X} > S^*\boldsymbol{c}^*) - a\mathbb{I}_m(\boldsymbol{X}_b > \boldsymbol{b}) \tag{54}$$

where

$$\begin{aligned}
\boldsymbol{\alpha}_p^* &= \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{1}_p \end{pmatrix}; \quad A_p^* = \begin{pmatrix} I_m & 0 \\ 0 & A_p \end{pmatrix}; \quad \boldsymbol{a}_p^* = \begin{pmatrix} \boldsymbol{b} \\ \boldsymbol{a}_p \end{pmatrix} \\
S^* &= \begin{pmatrix} I_m & 0 \\ 0 & -I_N \end{pmatrix}; \quad \boldsymbol{c}^* = \begin{pmatrix} \boldsymbol{b} \\ a\boldsymbol{1} \end{pmatrix}
\end{aligned}$$

and $\boldsymbol{b} = b_{ik} = b_i$ for all k. The first term in Eq(54) is the sum of general $\mathbb{M}-$binaries considered in the Main Theorem, while the second and third terms are bond type $\mathbb{M}-$binaries considered in Section 5.1. The present value of this payoff can therefore be obtained by employing the principle of static replication and results in the expression

$$V(\boldsymbol{x}, t; a, \boldsymbol{b}) = \sum_{p=1}^{N} x_p \mathcal{N}_n(\boldsymbol{d}_p^*;\ C_p^*) + aB(\boldsymbol{x}, t; S^*, \boldsymbol{c}^*) - aB(\boldsymbol{x}, t; I_m, \boldsymbol{b}) \tag{55}$$

where $(\boldsymbol{d}_p^*,\ C_p^*)$ are determined from Theorem 1 with $(A, \boldsymbol{a})$ replaced by $(A_p^*, \boldsymbol{a}_p^*)$; and the bond type B terms are given explicitly by Eq(23). Little would be served by expanding the details of this representation any further, so we are content to leave the solution in the stated form of Eq(55).

6 Conclusion

Theorem 1 stated in Section 4 is the main result of this paper. It provides a versatile formula for pricing a very general multi-asset, multi-period exotic binary option (our $\mathbb{M}$-binary) which we have demonstrated is a fundamental component of many of the exotics published in the literature. The basic idea, very much in the spirit of Ingersoll (2000), is to decompose an option's expiry payoff into portfolios of $\mathbb{M}-$binaries and invoke the principle of static replication to obtain an arbitrage-free price for the present value of the option. Each $\mathbb{M}$-binary expiry payoff has the very specific 4-parameter form: $\boldsymbol{X}^{\boldsymbol{\alpha}}\, \mathbb{I}_m(S\boldsymbol{X}^A > S\boldsymbol{a})$. These parameters, $[\boldsymbol{\alpha}, A, \boldsymbol{a}, S]$ together with the dual asset-time correlation matrix R of Eq(11) and monitoring time structure T_k, provide sufficient scope to match a great variety of exotic option payoffs. The exercise condition matrix A plays a particularly important role. If it is different from an identity matrix, it is responsible for generating changes in the correlation structure, which in turn has an important bearing on the eventual pricing formula. An interesting observation is that the dimension of the multi-variate normal for each component binary is determined only by the exercise dimension m, and not by the payoff dimension n or the number of assets N or time periods M.

The pricing formula itself Eq(17), is quite complex and may in certain circumstances involve considerable calculation. However, although we have concentrated in this paper on analytical solutions, the formula can readily be coded up for numerical computation, particularly with matrix oriented programming languages such as Matlab and Mathematica.

To illustrate the power and versatility of the formula, we have priced many well known exotic options and also a new one of very great complexity: namely, a discretely monitored call barrier option on the maximum of N assets. This option includes both a multi-asset payoff and multi-period monitoring.

Extensions of the general formula Eq(17) are underway to allow similar pricing of continuously monitored multi-asset barrier and lookback options.

7 References

1. Boyle P 1977 Options: a Monte Carlo approach *Journal of Financial Economics* **4** 323-338

2. Buchen P 2001 Image options and the road to barriers *Risk Magazine* **14**(9) 127-130

3. Buchen P 2003 The pricing of dual expiry options, *Journal of Quantitative Finance*, IOP (to appear)

4. Carr P 1988 The valuation of sequential exchange opportunities *Journal of Finance*, **43** 1235-1256

5. Conze A and Viswanathan 1991 Path dependent options: the case of lookback options *Journal of Finance* **46** 1893-1907

6. Geske R 1979 The valuation of compound options *Journal of Financial Economics* **7** 63-81

7. Harrison J and Pliska S 1981 Martingales and stochastic integrals in the theory of continuous trading *Stochastic Processes and their Applications* **11** 215-260

8. Heynen R and Kat H 1994 Partial barrier options, *Journal of Financial Engineering* **3** 253-274

9. Heynen R and Kat H 1996 Brick by brick *Risk Magazine* **9**(6)

10. Ingersoll J 2000 Digital contracts: simple tools for pricing complex derivatives *The Journal of Business* **73**(1) 67-88

11. Johnson H 1987 Options on the maximum or minimum of several assets *Journal of Financial and Quantitative Analysis* **22** 277-283

12. Kemna A and Vorst A 1990 A pricing method for options based on average asset values *Journal of Banking and Finance* **14** 113-129

13. Liao S-Z and Wang C-W 2003 The valuation of reset options with multiple strike resets and reset dates *Journal of Futures Markets* **23**(1) 87-107

14. Longstaff F 1990 Pricing options with extendable maturities: analysis and applications *Journal of Finance* **45**(3) 935-957

15. Margrabe W 1978 The value of an option to exchange one asset for another *Journal of Finance* **33** 177-186

16. Merton R 1973 Theory of rational option pricing *Bell Journal of Economics and Management Science* **4** 141-183

17. Reiner E and Rubinstein M 1991 Breaking down the barriers *Risk Magazine* **4**(8)

18. Rich D and Chance D 1993 An alternative approach to the pricing of options on multiple assets *Journal of Financial Engineering* **2** 271-285

19. Rich D 1994 The mathematical foundation of barrier option pricing theory *Advances in Futures and Options Research* **7** 267-311

20. Rubinstein M 1991 Options for the undecided *Risk Magazine* **4**(4) 43

21. Rubinstein M 1991 Double trouble *Risk Magazine* **5**(1)

22. Skipper M 2003 Inside exotic barrier options *Sydney University School of Mathematics & Statistics Research Report*

23. Stulz R 1982 Options on the minimum or the maximum of two risky assets *Journal of Financial Economics* **10** 161-185

24. Zhang P 1995 Correlation digital options *Journal of Financial Engineering* **4** 75-96

A Appendix

In this Appendix we prove the two Theorems stated in Sections 4 and 5.3.

Theorem 1 (Section 4)
The proof of the Main Theorem depends on the three Lemmas listed below. The first deals with option pricing theory; the second and third deal with specific properties of Gaussian random vectors.

Lemma 1 In a complete, arbitrage-free, multi-asset market the discounted price of any derivative contract is a martingale with respect to the risk-neutral measure. This leads to the well-known pricing formula (Harrison and Pliska 1981)

$$V(\boldsymbol{x}, t) = e^{-r(T-t)}\mathbb{E}\{V_T(\boldsymbol{X})|\boldsymbol{X}(t) = \boldsymbol{x}\}. \tag{A1}$$

Lemma 2 Let $\boldsymbol{c} \in \mathcal{V}_n$ be a constant vector, $\boldsymbol{Z} \in \mathcal{G}_n(R)$ a Gaussian random vector with positive definite correlation matrix R and let $F(\boldsymbol{Z})$ be a scalar function of n variables. Then

$$\mathbb{E}\{e^{\boldsymbol{c}'\boldsymbol{Z}}\, F(\boldsymbol{Z})\} = e^{\frac{1}{2}\boldsymbol{c}'R\boldsymbol{c}}\,\mathbb{E}\{F(\boldsymbol{Z} + R\boldsymbol{c})\}. \tag{A2}$$

We like to term this result the *Gaussian Shift Theorem* because the mean has been 'shifted' from zero to $R\boldsymbol{c}$. The proof is fairly elementary and involves little more than the following identity for multi-variate Gaussian pdf's:

$$\phi(\boldsymbol{z} - R\boldsymbol{c}) \equiv e^{\boldsymbol{c}'\boldsymbol{z} - \frac{1}{2}\boldsymbol{c}'R\boldsymbol{c}}\phi(\boldsymbol{z})$$

where $\phi(\boldsymbol{z}) = \exp(-\frac{1}{2}\boldsymbol{z}'R^{-1}\boldsymbol{z})$ and K is the usual normalising constant. This Lemma can also be shown to be equivalent to expectation under a change of measure.

Lemma 3 For any matrix $B \in \mathcal{A}_{mn}$ of rank $m \leq n$, a vector $\boldsymbol{b} \in \mathcal{V}_m$ and a Gaussian random vector $\boldsymbol{Z} \in \mathcal{G}_n(R)$ with correlation matrix R,

$$\mathbb{E}\{\mathbb{I}_m(B\boldsymbol{Z} < \boldsymbol{b})\} = \mathcal{N}_m(D^{-1}\boldsymbol{b};\ D^{-1}(BRB')D^{-1}) \tag{A3}$$

where $D^2 = \mathrm{diag}(BRB')$. This Lemma is also easily proved noting that $B\boldsymbol{Z} \in \mathcal{G}_m$ with covariance matrix BRB'. The diagonal matrix D provides the normalisation to an $m-$dimensional correlation matrix.

Proof of Main Theorem

From the payoff function Eq(16) for an $\mathbb{M}$-binary and Lemma 1 we have

$$\begin{aligned} V(\boldsymbol{x}, t) &= e^{-r\tau}\mathbb{E}\{\boldsymbol{X}^{\boldsymbol{\alpha}}\, \mathbb{I}_m(S\boldsymbol{X}^A > S\boldsymbol{a})|\boldsymbol{X}(t) = \boldsymbol{x}\}; \quad \tau = T - t \\ &= e^{-r\tau}\mathbb{E}\{\boldsymbol{X}^{\boldsymbol{\alpha}}\, \mathbb{I}_m(SA\log\boldsymbol{X} > S\log\boldsymbol{a})|\boldsymbol{X}(t) = \boldsymbol{x}\}. \end{aligned}$$

For log-normal asset price dynamics described by Eq(13) we first obtain

$$\boldsymbol{X}^{\boldsymbol{\alpha}} = \boldsymbol{x}^{\boldsymbol{\alpha}} \exp\{\boldsymbol{\alpha}'\boldsymbol{\mu} + (\Sigma\boldsymbol{\alpha})'\boldsymbol{Z}\}$$

Then using Lemma 2 with $\boldsymbol{c} = \Sigma\boldsymbol{\alpha}$, the last equation for $V(\boldsymbol{x}, t)$ can be written as

$$\begin{aligned} V(\boldsymbol{x}, t) &= \boldsymbol{x}^{\boldsymbol{\alpha}} e^{-r\tau + \boldsymbol{\alpha}'\boldsymbol{\mu}} \mathbb{E}\{e^{\boldsymbol{c}'\boldsymbol{Z}} \mathbb{I}_m(SA(\log \boldsymbol{x} + \boldsymbol{\mu} + \Sigma\boldsymbol{Z}) > S \log \boldsymbol{a})\} \\ &= \beta \boldsymbol{x}^{\boldsymbol{\alpha}}\, \mathbb{E}\{\mathbb{I}_m(SA\Sigma(\boldsymbol{Z} + R\Sigma\boldsymbol{\alpha}) > -S[\log(\boldsymbol{x}^A/\boldsymbol{a}) + A\boldsymbol{\mu}])\} \\ &= \beta \boldsymbol{x}^{\boldsymbol{\alpha}}\, \mathbb{E}\{\mathbb{I}_m(-(SA\Sigma)\boldsymbol{Z} < S[\log(\boldsymbol{x}^A/\boldsymbol{a}) + A(\boldsymbol{\mu} + \Gamma\boldsymbol{\alpha})])\} \end{aligned}$$

Finally, from Lemma 3 with $B = -(SA\Sigma)$ we get

$$V(\boldsymbol{x}, t) = \beta \boldsymbol{x}^{\boldsymbol{\alpha}} \mathcal{N}_m(S\boldsymbol{d}; SCS)$$

where $\beta, \Gamma, \boldsymbol{d}$ and C have all been defined in Section 4.
This completes the proof of Theorem 1. $\square$

Theorem 2 (Section 5.3)
Partition vector $\boldsymbol{X} \in \mathcal{V}_n$ in the form $\boldsymbol{X} = (X_p, \bar{\boldsymbol{X}}_p)'$ where $(\bar{X}_p)_i = X_i \in \mathcal{V}_{n-1}$ for all $i \neq p$. That is, asset$-p$ is moved to the first element of $\boldsymbol{X}$. Partitioning matrix A_p and vector $\boldsymbol{a}_p$ in the same way, we write:

$$A_p = \begin{pmatrix} 1 & \mathbf{0}' \\ \mathbf{1} & -I_{n-1} \end{pmatrix}; \qquad \boldsymbol{a}_p = \begin{pmatrix} k \\ \mathbf{1} \end{pmatrix}. \tag{A4}$$

It follows that

$$\begin{aligned} & s\boldsymbol{X}^{A_p} > s\boldsymbol{a}_p \\ \Rightarrow\quad & sA_p \log \boldsymbol{X} > s \log \boldsymbol{a}_p \\ \Rightarrow\quad & s\begin{pmatrix} 1 & \mathbf{0}' \\ \mathbf{1} & -I_{n-1} \end{pmatrix}\begin{pmatrix} \log X_p \\ \log \bar{\boldsymbol{X}}_p \end{pmatrix} > s\begin{pmatrix} \log k \\ \mathbf{0} \end{pmatrix} \\ \Rightarrow\quad & s\begin{pmatrix} \log X_p \\ \log X_p\mathbf{1} - \log \bar{\boldsymbol{X}}_p \end{pmatrix} > s\begin{pmatrix} \log k \\ \mathbf{0} \end{pmatrix} \\ \Rightarrow\quad & s\begin{pmatrix} X_p \\ X_p\mathbf{1} \end{pmatrix} > s\begin{pmatrix} k \\ \bar{\boldsymbol{X}}_p \end{pmatrix} \end{aligned}$$

The last line demonstrates that $X_p = \max(\boldsymbol{X}, k)$ if $s = 1$ and $X_p = \min(\boldsymbol{X}, k)$ if $s = -1$ and Theorem 2 is proved. $\square$

—oOo—