On the injectivity of the Vassiliev homomorphism
of singular Artin monoids

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Abstract

We prove general combinatorial properties which apply to singular Artin monoids and examine their relationship with the Vassiliev homomorphism $\eta$. We show that $\eta$ preserves the Intermediate Property which holds in positive singular Artin monoids of finite type; namely, if $\eta(\tau_i U) = \eta(\tau_j V)$ then $m_{ij} = 2$ or $i = j$. From this it follows that $\eta$ is injective for a class of monoids which include singular Artin monoids of type $I_2(p)$.

1 Introduction and preliminaries

Let $\Gamma^M$ be a labelled graph with $n$ vertices in one-to-one correspondence with a finite indexing set $I$, and with edge labels from the set $\{3, 4, 5, \ldots, \infty\}$. For $i \neq j$ let $m_{ij}$ denote the label of the edge between nodes $i$ and $j$, or set $m_{ij} = 2$ if there is no such edge. Let $\langle xy \rangle^m$ denote the alternating product $xyx \ldots$ of length $m$. Let $S = \{\sigma_i \mid i \in I\}$, $T = \{\tau_i \mid i \in I\}$ and let $S^{-1} = \{\sigma_i^{-1} \mid i \in I\}$, the set of formal inverses of $S$. If $X$ is a set then $X^*$ denotes the free monoid generated by $X$. The Artin group of type $M$, denoted $G_M$, is the group generated by $S$ subject to the relations

$$\langle \sigma_i \sigma_j \rangle^{m_{ij}} = \langle \sigma_j \sigma_i \rangle^{m_{ij}} \quad \text{for } i, j \in I, \quad \text{and} \quad m_{ij} \neq \infty$$

denoted $\mathcal{R}_1$ and called the braid relations. The positive Artin monoid of type $M$, $G_M^+$, is the monoid generated by $S$ subject to the braid relations $\mathcal{R}_1$. The Coxeter group of type $M$ is the group generated by $S$ subject to the preceding relations and the relations $\sigma_i^2 = 1$ for every $i$ in $I$. In this way we see that Coxeter groups arise as quotient groups of Artin groups. If the Coxeter group of type $M$ is finite, then $M$ is said to be of finite type. The singular Artin monoid of type $M$, denoted $SG_M$, is the monoid generated by $S \cup S^{-1} \cup T$ and has as its defining relations $\mathcal{R}_1$, the free group relations $\sigma_i \sigma_i^{-1} = \sigma_i^{-1} \sigma_i = 1$ and the
relations $R_2$ listed below,

\[ \tau_i \tau_j = \tau_j \tau_i \quad \text{if } m_{ij} = 2, \]
\[ \tau_i \langle \sigma_j \sigma_i \rangle^{m_{ij}^{-1}} = \begin{cases} \langle \sigma_j \sigma_i \rangle^{m_{ij}^{-1}} \tau_j & \text{if } m_{ij} < \infty \text{ and is odd, or} \\ \langle \sigma_j \sigma_i \rangle^{m_{ij}^{-1}} \tau_i & \text{if } m_{ij} < \infty \text{ and is even,} \end{cases} \]
\[ \tau_i \sigma_i = \sigma_i \tau_i \quad \text{for all } i \text{ in } I. \]

In arguments below we may regard a relation formally as an ordered pair of words. If $X$ is a set of ordered pairs of words then $X^\Sigma = \{(U, V) \mid (U, V) \text{ or } (V, U) \in X \}$. The special case when $I = \{1, \ldots, n\}$, $m_{ij} = 3$ when $|i - j| = 1$ and $m_{ij} = 2$ when $|i - j| \geq 2$ is the singular Artin monoid of type $A_n$ and may be familiar to some readers as the singular braid monoid on $n + 1$ strings, denoted $SB_{n+1}$, which was introduced by Baez and Birman in [Bae] and [Bir] respectively.

If $A$ and $B$ are words in the above generators, we write $A \approx B$ if $A$ can be transformed into $B$ by the use of the defining relations of $SG_M$, and $A = B$ if the two words are equal letter by letter.

We define the positive singular Artin monoid denoted by $SG_M^+$, to be the monoid generated by $S \cup T$ and defining relations $R$ comprising of both $R_1$ and $R_2$ listed above. Where it does not cause confusion we denote elements of $G_M$, $G_M^+$, $SG_M$ and $SG_M^+$ by words which represent them. If $A$ and $B$ are words in the generators $S$ and $T$, we write $A \sim B$ if $A$ can be transformed into $B$ by the use of $R$. The following result is a restatement of Theorem 20 of [Cor1]:

**Theorem 1.1.** Let $M$ be of finite type. If $A$, $B$ are words over $S \cup T$ and $A \approx B$ then $A \sim B$.

Thus $SG_M^+$ embeds into $SG_M$ whenever $M$ is of finite type. We denote by $\ell(A)$ the length of any word $A$. It is easy to see, by inspection of the defining relations, that $SG_M^+$ is homogenous so the length of an element is defined to be the length of any word representing it. Corran [Cor1, p. 258] defined the reduction property and showed [Cor1, Lemma 15] that cancellativity and the reduction property also hold in $SG_M^+$. By reduction we mean an application of the reduction property.

Let $A$ and $B$ be words in $(S \cup T)^*$. We say $A$ divides $B$ or $B$ is a multiple of $A$ if there exists a word $X$ in $SG_M^+$ such that $B \sim AX$ in which case we write $A \prec B$. We say $A$ right divides $B$, or $B$ is a right multiple of $A$, if there is a word $X$ in $SG_M^+$ such that $B \simXA$, in which case we write $B \succ A$.

Let $\Omega = \{A_1, A_2, \ldots, A_k\}$ be a set of words in $(S \cup T)^*$. If $\Omega$ has a common multiple then by Corollary 13 of [Cor1], $\Omega$ has a common multiple (unique up to equivalence) of minimal length which we denote by $\text{lcm}(A_1, A_2, \ldots, A_k)$ or $\text{lcm}(\Omega)$. If $\Omega$ has no common multiple then we write $\text{lcm}(A_1, A_2, \ldots, A_k) = \infty$.

In Section 2 we discover properties pertaining to fundamental elements in addition to general results regarding divisibility in $SG_M^+$. In Section 3 the Vassiliev homomorphism $\eta$ is defined, we state Birman’s conjecture and show that if Birman’s conjecture is true
for $SG^+_M$, where $M$ is of finite type, then it is true for $SG_M$; this is followed by some observations regarding $\eta$. The results of Sections 4 and 5 hold for finite type $M$. In Section 4 we study the relationship between divisibility in $SG^+_M$ and the support of $\eta$. In particular we show that if $U, V \in SG^+_M$, $C \in G^+_M$ and $\eta(U) = \eta(V)$ then $C$ divides $U$ if and only if $C$ divides $V$. Finally, in Section 5, we prove that $\eta$ preserves the Intermediate Property which holds in $SG^+_M$, namely if $\eta(\tau_i U) = \eta(\tau_j V)$ then $m_{ij} = 2$ or $i = j$. From this it follows that $\eta$ is injective for a class of monoids which include singular Artin monoids of type $I_2(p)$.

2 The fundamental word $\Delta$

The following refers to a construction developed in Section 2 of [Cor1]. For every generator $\alpha$ and word $W$ in $(S \cup T)^*$, the word $K_{\alpha}(W)$ is defined and begins with $\alpha$ if and only if $W$ is divisible by $\alpha$, in which case we write $W_{\alpha} = (W/\alpha)$ for the word obtained by removing the letter $\alpha$ from $K_{\alpha}(W)$. Then the word $(W/V)$ is defined recursively and exists precisely when $V \prec W$ and has the property that $W \sim V(W/V)$.

2.1 Properties of $\Delta$

Suppose in this subsection that $M$ is of finite type. Let $\Delta = \text{lcm}(\sigma_1, \ldots, \sigma_n)$. We call $\Delta$, in accordance with Garside [Gar, Section 2], the fundamental word of $SG_M$ and write $\zeta = \Delta^2$. Theorem 5.6 of [BS] tells us that $\Delta$ exists precisely when $M$ is of finite type whilst by [Cor1, p. 280] the following holds:

**Theorem 2.1.** For any word $A$ in $(S \cup S^{-1} \cup T)^*$, there exists an integer $p$ and a word $\bar{A}$ in $(S \cup T)^*$ such that $A \cong \Delta^p \bar{A}$.

In Section 4 of [Cor1], Corran showed that there exists a uniquely determined involution-ary automorphism, which we denote by $R$, of $SG_M$ with the following property:

- $R$ sends letters to letters so that, for any $i \in I$, $\alpha = \sigma$ or $\tau$, $R(\alpha_i) = \alpha_{\sigma(i)}$. Hence $R$ arises from a permutation $\phi$ of $I$ with $\phi^2 = \text{id}$ and $m_{\phi(i)\phi(j)} = m_{ij}$.

(See also Lemma 5.2 of [BS]). We write $\alpha_i' = \alpha_{\sigma(i)}$ for $R(\alpha_i)$. By Lemma 18 of [Cor1] we have the following property of $\Delta$:

**Lemma 2.1.** Let $W$ be any word in $(S \cup T)^*$. Then $W \Delta \sim \Delta R(W)$. In particular $W$ is divisible by $\Delta$ if and only if $W$ is right divisible by $\Delta$.

Given that $R$ is an automorphism of $SG^+_M$, the previous lemma tells us that $\Delta$ acts almost like a central element of the monoid, but not quite, as Lemma 2.2 below shows. The proof of the first part of the lemma is in [BS, Lemmas 5.2(ii), 5.1(ii)] whilst the second part of the result is a combination of discoveries made in [Cor1, Lemma 18] and [BS, Theorem 7.2] respectively.
Lemma 2.2.  (1) \( \mathcal{R}(\Delta) \sim \Delta \) and \( \text{Rev}(\Delta) \sim \Delta \).

(2) The centre of the singular Artin monoid, \( SG_M \), is generated by the fundamental element \( \Delta \), if the associated involution \( \mathcal{R} \) is trivial. The involution \( \mathcal{R} \) is non-trivial only for types \( A_n \), when \( n \geq 2 \), \( D_{2k+1} \), \( E_6 \) and \( I_2(2q+1) \), in which case \( \Delta^2 \) represents the generator of the centre.

Thus, since \( \Delta \) is the lowest common multiple of the set \( S \), and \( \zeta = \Delta^2 \), the words \( \Delta_a \) and \( \zeta_a \), are defined for every \( a \) in \( S \).

Lemma 2.3.  (1) For any \( a \) in \( S \), \( \mathcal{R}(\Delta_a) \sim \Delta_a' \).

(2) For \( \alpha = \sigma \) or \( \tau \) and \( i \in I \), \( \alpha_i \Delta_{\sigma_i} \sim \Delta_{\sigma_i} \alpha_i' \). In particular, \( a \Delta_a \sim \Delta \sim \Delta_a a' \), whenever \( a \in S \).

Proof: (1) Let \( a \) be a generator in \( S \). Then by Lemma 2.2(1) and since \( b \mathcal{R} b \) for any \( b \) in \( S \), it follows that
\[
a' \mathcal{R}(\Delta_a) \sim \mathcal{R}(a \Delta_a) \sim \mathcal{R}(\Delta) \sim \Delta \sim a' \Delta_a'.
\]
By cancellativity we deduce that \( \mathcal{R}(\Delta_a) \sim \Delta_a' \), proving the result.

(2) Let \( \sigma_i \) be any generator in \( S \) and let \( \alpha_i = \sigma_i \) or \( \tau_i \). Then \( \alpha_i \sigma_i \sim \sigma_i \alpha_i \) and, by Lemma 2.1, \( \alpha_i \Delta \sim \Delta \alpha_i' \), whence,
\[
\sigma_i \alpha_i \Delta_{\sigma_i} \sim \alpha_i \sigma_i \Delta_{\sigma_i} \sim \alpha_i \Delta \sim \Delta \alpha_i' \sim \sigma_i \Delta_{\sigma_i} \alpha_i'.
\]
The result then follows by cancellativity. \( \square \)

We note here that variations of the preceding Lemmas 2.2 and 2.3 can be found in Lemma 2.3 of [Cha].

2.2 Divisibility theory

The results of this subsection hold for positive singular Artin monoids of any (not necessarily finite) type. The ensuing definitions are obtained from Section 2 of [Cor1]. Let \( C \) be a non-empty word and \( a, b \in S \cup T \). We say \( C \) is a simple \( a \)-chain with source \( a \) and target \( b \) if there is a (non-empty) word \( P \) and (possibly empty) word \( Q \) such that \( (aP,CbQ) \) is a relation in \( \mathcal{R}^\Sigma \). We call \( C \) an \( a \)-chain if \( C = C_1 \ldots C_k \) for simple chains \( C_1, \ldots, C_k \) where the source of \( C_1 \) is \( a \) and the source of \( C_{i+1} \) is the target of \( C_i \) for \( i = 1, \ldots, k - 1 \). In this case, the source and target of \( C \) are defined to be the source of \( C_1 \) and the target of \( C_k \), respectively.

Remark 1. In \( G^+_M \), if \( C \) is an \( a \)-chain to \( b \) then \( \text{Rev}(C) \) is a \( b \)-chain to \( a \). However this does not always hold in \( SG^+_M \). For example, if \( m_{ab} \geq 3 \), \( \sigma_b \) is a \( \tau_a \)-chain to \( \sigma_a \) but \( \text{Rev}(\sigma_b) = \sigma_b \) is a \( \sigma_a \)-chain to \( \sigma_a \neq \tau_a \). Also \( \tau_a \sigma_b \) is a simple \( \sigma_b \)-chain to \( \sigma_a \), but the target of the compound \( \sigma_a \)-chain \( \text{Rev}(\tau_a \sigma_b) = \sigma_b \tau_a \) is \( \sigma_a \), not equal to the source of \( \tau_a \sigma_b \).
Lemmas 2.4, 2.5 and 2.6 below are restatements of Lemmas 3, 5 and 4(2) respectively of [Cor1].

**Lemma 2.4.** If $C$ is an $a$-chain to $b$ and $W$ is a common multiple of $a$ and $C$ then $W$ is also a common multiple of $a$ and $Cb$. In particular $a$ does not divide $C$.

**Lemma 2.5.** If $W$ is a word in $(S \cup T)^*$, $a \in S \cup T$, such that $a$ does not divide $W$ but $\text{lcm}(a, W)$ exists, then $W$ either is empty or there is an $a$-chain $C$ such that $W \sim C$.

**Lemma 2.6.** If $C$ is an $a$-chain such that $a$ divides $Cb$ then $b$ is the target of $C$.

**Corollary 2.1.** Let $a \in S$ and $W$ a non-empty word in $(S \cup T)^*$. Then $a \not\sim W$ if and only if $W \sim C$, for some $a$-chain $C$.

**Proof:** Clearly $a$ divides $\Delta$ so by Lemma 2.1 we deduce that $W \Delta \sim \Delta R(W)$ is a common multiple of $a$ and $W$, showing that $\text{lcm}(a, W)$ exists. Thus if $a$ does not divide $W$, Lemma 2.5 yields the existence of an $a$-chain $C$ such that $W \sim C$. On the other hand, if $W \sim C$, where $C$ is an $a$-chain, then $a \not\sim C \sim W$, by Lemma 2.4. □

The following two results are also proved in [Cor2, Lemma 6.5].

**Lemma 2.7.** If $C$ is an $a$-chain to $b$, for some generator $a$ in $S$, then $b$ also lies in $S$.

**Proof:** Write $C = C_1 \ldots C_k$ where each $C_i$ is simple and suppose $d$ is the target of $C_1$. Then $(aP, C_1dQ) \in \mathbb{R}_i^\Sigma$, for some generator $d$ and words $P, Q$. If $d \in T$ inspection of the defining relations, $\mathbb{R}$, shows that $Q = 1$ (since $C_1 \neq 1$) and $a \in T$. Hence $d$ must lie in $S$. If $k = 1$ then $b = d$ and we are done. Otherwise, $C_2 \ldots C_k$ is a $d$-chain to $b$ and $d \in S$, so by induction $b$ must be a letter from $S$ and we are done. □

**Lemma 2.8.** If $C$ is a $\sigma_a$-chain to $\sigma_b$ then $C$ is not right divisible by $\sigma_b$.

**Proof:** Write $C = C_1 \ldots C_k$ where each $C_i$ is simple. Since $\sigma_a$ clearly lies in $S$, Lemma 2.7 tells us that the source of $C_k$ must also be an element of $S$ and so is $\sigma_c$, for some $c$ in $I$. Hence there exist words $P, Q$ in $SG^*_1\Sigma$ such that $(C_k \sigma_aP, \sigma_cQ) \in \mathbb{R}_i^\Sigma$. Inspection of $\mathbb{R}$ immediately shows that $C_k$ is not right divisible by its target $\sigma_b$. If $k = 1$ then $C = C_k \not\sim \sigma_b$ and we are done. So suppose $k \geq 2$ and put $C' = C_1 \ldots C_{k-1}$. Then since the source of $C_k$, $\sigma_c$, is the target of $C_{k-1}$, $C'$ is a $\sigma_a$-chain to $\sigma_c$ and so, by induction, $C' \not\sim \sigma_c$. Thus

$$\sigma_c \not\sim \text{Rev}(C').$$

(2.1)

Now put $W = \text{Rev}(C) = \text{Rev}(C_k)\text{Rev}(C')$. We show that $W$ is not divisible by $\sigma_b$ so that $C \not\sim \sigma_b$ as required. Recall that $C_k$ is a simple $\sigma_c$-chain to $\sigma_b$. If $C_k$ is over $S$ then $\text{Rev}(C_k)$ is a $\sigma_b$-chain to $\sigma_c$; so if $W$ were divisible by $\sigma_b$ then $W$ would be a common multiple of $\text{Rev}(C_k)$ and $\sigma_b$ showing that $\sigma_c$ divides $\text{Rev}(C')$, by Lemma 2.4 and cancellation, contradicting (2.1). Suppose then that $C_k$ is not over $S$. Then $C_k = \tau_c$ or $C_k = \tau_d(\sigma_c\sigma_d)^q$ for some non-negative integer $q \leq m_{cd} - 2$.

If $C_k = \tau_c$ then, noting $C_k$ is a $\sigma_c$ chain to $\sigma_b$, we deduce that $b = c$; so if $\sigma_b = \sigma_c$
were to divide \( W = \tau_c \Rev(C') \), reduction would show that \( \Rev(C') \) is divisible by \( \sigma_c \) contradicting (2.1). So assume that
\[
C_k = \tau_d \langle \sigma_c \sigma_d \rangle^q \quad (\exists q) \quad 0 \leq q \leq m_{cd} - 2.
\]
Thus \( C_k \) is a \( \sigma_c \)-chain to \( \sigma_b \) where \( b = c \) if \( q \) is even and \( b = d \) if \( q \) is odd. Hence,
\[
W = \Rev(C_k) \Rev(C') \sim \Rev(\langle \sigma_c \sigma_d \rangle^q \tau_d \Rev(C'))
\]
\[
= \begin{cases} 
\langle \sigma_d \sigma_c \rangle^q \tau_d \Rev(C') & \text{if } q \text{ is even,} \\
\langle \sigma_c \sigma_d \rangle^q \tau_d \Rev(C') & \text{if } q \text{ is odd.} 
\end{cases}
\]
Suppose that \( \sigma_b \) divides \( W \). Then, recalling \( q \leq m_{cd} - 2 \), reduction and cancellation yield a word \( R \) such that
\[
\tau_d \Rev(C') \sim \langle \sigma_d \sigma_c \rangle^{m_{cd} - q} R = \sigma_d \sigma_e \langle \sigma_d \sigma_c \rangle^{m_{cd} - (q+2)} R.
\]
Another application of reduction tells us that \( \tau_d \) divides \( \sigma_c \langle \sigma_d \sigma_c \rangle^{m_{cd} - (q+2)} R \) so that, yet again by reduction,
\[
\sigma_c \langle \sigma_d \sigma_c \rangle^{m_{cd} - (q+2)} R \sim \langle \sigma_c \sigma_d \rangle^{m_{cd} - 1} R',
\]
for some word \( R' \). Thus
\[
\tau_d \Rev(C') \sim \sigma_d \sigma_c \langle \sigma_d \sigma_c \rangle^{m_{cd} - (q+2)} R \sim \sigma_d \langle \sigma_c \sigma_d \rangle^{m_{cd} - 1} R' \sim \langle \sigma_c \sigma_d \rangle^{m_{cd} - R'}.
\]
A final application of reduction now shows that \( \sigma_c \not\prec \Rev(C') \) which contradicts (2.1). This implies that \( \sigma_b \) does not divide \( W = \Rev(C) \) so \( C \not\sim \sigma_b \) as required. The result now follows by induction.

**Remark 2.** The condition in Lemma 2.8 above for the source of \( C \) to be an element of \( S \) is necessary. Let \( m_{ab} = 3 \) and put \( U = \sigma_a^2 \sigma_b \sigma_a \). Then \( U \) is clearly right divisible by \( \sigma_a \). Furthermore, \( U \sim \langle \sigma_a \sigma_b \rangle \langle \sigma_a \rangle \langle \sigma_b \rangle \), the latter being a compound \( \tau_b \)-chain with target \( \sigma_a \).

### 2.3 The structure of \( \Delta \)

For the remainder of this section we resume our supposition that \( M \) is of finite type. The following result provides an important property of the fundamental word. It was originally discovered by Garside for the positive braid monoid in [Gar, Theorem 8] and was generalised by Brieskorn and Saito in [BS, Lemma 5.3] who showed that it holds for positive Artin monoids of finite type.

**Proposition 2.1.** Let \( X \) and \( Y \) be in \((S \cup T)^*\). If \( \Delta \not\prec XY \) then for every \( i \in I \) either \( X \not\succ \sigma_i \) or \( \sigma_i \not\prec Y \).

**Proof:** Suppose there exists an \( i \in I \) such that \( \sigma_i \) does not right divide \( X \) nor does it divide \( Y \). We show \( \Delta \not\prec XY \) by induction on \( \ell(Y) \geq 0 \). The result certainly is true if \( \ell(X) = 0 \) whilst if \( \ell(Y) = 0 \) the claim holds by Lemma 2.1. So suppose that both
X and Y are non-empty. Since X \not \subset \sigma_i, we infer from Lemma 2.1 that X cannot be right divisible and so divisible by \Delta, the lowest common multiple of the set S. Hence \sigma_j \not \subset X, for some j \in I. Noting X is non-empty, Corollary 2.1 may be applied yielding a \sigma_j-chain, C, such that X \sim C. By Lemma 2.7 we deduce that the target of C is \sigma_k, for some k \in I, so C \not \subset \sigma_k, by Lemma 2.8. Hence

\[ C \text{ is a } \sigma_j\text{-chain to } \sigma_k, \quad C \sim X \not \subset \sigma_i \text{ and } \sigma_k. \quad (2.2) \]

If \sigma_k does not divide Y then by another application of Corollary 2.1 we infer the existence of a \sigma_k-chain \( C' \) such that \( Y \sim C' \) so that \( CC' \) is a \( \sigma_j \)-chain, by (2.2); this implies that \( \sigma_j \) and so \( \Delta \) cannot divide \( CC' \sim XY \), by Lemma 2.4. So suppose that \( \sigma_k \prec Y \). Then \( i \neq k \), since \( \sigma_i \not \subset Y \). Thus there exists a largest integer \( q \) and word \( Y' \) such that

\[ Y \sim \langle \sigma_k \sigma_i \rangle^q Y' \quad \text{and} \quad \sigma_d \not \subset Y' \quad (2.3) \]

where \( d = k \) if \( q \) is even and \( d = i \) if \( q \) is odd. Since \( \langle \sigma_k \sigma_i \rangle^{mi_k} \sim \langle \sigma_i \sigma_k \rangle^{mi_k} \) and \( \sigma_i \not \subset Y \), we have \( 1 \leq q \leq m_{ik} - 1 \). Put \( X' = X \langle \sigma_k \sigma_i \rangle^q \). Then

\[ XY \sim X \langle \sigma_k \sigma_i \rangle^q Y' = X'Y' \]

and \( \ell(Y') < \ell(Y) \). If \( \sigma_d \) were to right divide \( X' = X \langle \sigma_k \sigma_i \rangle^q \) reduction and reversal would yield a word \( X'' \) such that \( X \langle \sigma_k \sigma_i \rangle^q \sim X'' \langle \sigma_k \sigma_i \rangle^{mi_k} \) so, noting \( q < m_{ik} \), \( \sigma_i \) or \( \sigma_k \) would right divide \( X \), by cancellation, contradicting (2.2). Hence \( X' \not \subset \sigma_d \), and by (2.3) \( \sigma_d \not \subset Y' \). Since \( \ell(Y') < \ell(Y) \) we deduce by an inductive hypothesis that \( \Delta \not \subset X'Y' \sim XY \) as required. The result now follows by induction.

**Corollary 2.2.** Let \( a \) be in \( S \), and \( S' = S \setminus \{a\} \). Then \( \text{lcm}(S') \prec \Delta_a \), so that \( \text{lcm}(\Delta_a, a) = \Delta \).

**Proof:** By the previous proposition applied to \( \Delta \sim a\Delta_a \) we immediately obtain that \( \text{lcm}(S') \prec \Delta_a \). The last statement of the corollary follows since \( \Delta = \text{lcm}(S) \) and \( \Delta_a \prec \Delta \) by Lemma 2.3(2). \( \square \)

**Corollary 2.3.** Let \( a \) be a letter in \( S \), \( U \) a word over \( S \cup T \), \( r \) any integer \( \geq 2 \) and suppose \( \Delta \) divides \( a^rU \). Then for any \( m \) such that \( 1 \leq m \leq r - 1 \), \( \Delta \) also divides \( a^{r-m}U \).

**Proof:** Put \( U' = a^{r-m}U \). Then \( a^rU = a^mU' \) is divisible by \( \Delta \) so that \( b \prec U' \) for every \( b \) in \( S \) such that \( b \neq a \), by Proposition 2.1. But \( a \) also divides \( U' = a^{r-m}U \), since \( 1 \leq m \leq r - 1 \), whence \( \text{lcm}(S) = \Delta \prec U' \) as required. \( \square \)

**Lemma 2.9.** Let \( I \) and \( J \) be words in \( (S \cup T)^* \) and \( a \) an element in \( S \). Then the following are equivalent:

1. \( \Delta \prec I\Delta_a\Delta_a'J \)
2. \( \Delta \prec I\Delta_a \) or \( \Delta \prec \Delta_a'J \),
(3) \( I \succ a \) or \( a \prec J \).

In particular \((\Delta_a \Delta_{a'})^m\) is not divisible by \( \Delta \) for any \( a \) in \( S \) and any positive integer \( m \).

**Proof**: Suppose \( \Delta \prec I \Delta_a \Delta_{a'} \). By Proposition 2.1, \( I \succ a \) or \( a \prec \Delta_a \Delta_{a'} \). If \( I \succ a \) then \( I \sim I_1a \) for some word \( I_1 \) in \((S \cup T)^*\), whence \( I \Delta_a \sim I_1a \Delta_a \sim I_1 \Delta \), so that \( I \Delta_a \) is right divisible, and so, by Lemma 2.1, is divisible by \( \Delta \). If \( a \prec \Delta_a \Delta_{a'} \) then \( \Delta \prec \Delta_a \Delta_{a'} \) since \( \text{lc}(\Delta_a, a) = \Delta \); by Lemma 2.3(2) and cancellation, we then obtain that \( a' \prec \Delta_{a'} \) showing that \( \Delta \prec \Delta_{a'} \). Thus (2) follows from (1).

Now assume that \( \Delta \) divides \( I \Delta_a \) or \( \Delta_{a'} \). If \( \Delta \prec I \Delta_a \), then by Lemma 2.1, \( I \Delta_a \) is right divisible by \( \Delta \), whence there is a word \( I_1 \) in \((S \cup T)^*\) such that \( I \Delta_a \sim I_1 \Delta \sim I_1 a \Delta_a \); cancellativity then yields that \( I \succ a \). That \( \Delta \prec \Delta_{a'} \) implies \( a \prec J \) follows immediately by Lemma 2.3(2) and cancellativity. Thus (3) follows from (2). That (3) implies (1) follows from Lemmas 2.3(2) and 2.1. Notice that (3) implies \( \ell(I) \) or \( \ell(J) \geq 1 \), thus proving the last statement of the lemma, by a simple induction on \( m \). \( \square \)

# 3 Birman’s conjecture

Except when explicitly stated, assume throughout this section that \( M \) if of any type. The map \( \eta \) from \( SB_{n+1} \) to the group algebra \( \mathbb{Z}B_{n+1} \) induced by

\[
\sigma_i \mapsto \sigma_i, \quad \sigma_i^{-1} \mapsto \sigma_i^{-1}, \quad \tau_i \mapsto \sigma_i - \sigma_i^{-1}
\]

is easily verified to be a monoid homomorphism; \( \eta \) is sometimes referred to as the Vassiliev homomorphism [Vas] or desingularisation map [Par]. In [Bir, Remark 1] Birman conjectured that \( \eta \) is faithful, so that the singular braid monoid embeds into the group algebra of the braid group. In 1996 Rolfsen et al. showed that the above map is injective on singular braids with up to two singularities (where a singularity is denoted by a \( \tau_i \)); the following year this result was extended by Zhu, who showed that it holds for up to three singular points. Dasbach and Gemein [DG], simultaneously but independently of Jár [Jár], discovered that the conjecture holds for the singular braid monoid on three strings. Paris [Par] proved the truth of the conjecture in its entirety whilst East demonstrated that it holds for all singular Artin monoids of type \( I_2(p) \). Furthermore, in a recent preprint, Godelle and Paris proved in [GP] that the conjecture is true for right-angled singular Artin monoids. In effect Birman’s conjecture generalises to arbitrary Artin types since the Vassiliev homomorphism, \( \eta \), from any singular Artin monoid to the group algebra of the corresponding Artin group is well defined. This fact was observed by Corran in [Cor1, Remark 25]. Thus we may conjecture the following:

**Conjecture 1.** The Vassiliev homomorphism \( \eta : SG_M \rightarrow \mathbb{Z}G_M \) is faithful, so that the singular Artin monoid embeds into the group algebra of the Artin group.

We write \( I = J \) if \( I \) and \( J \) are equal elements of \( \text{Im}(\eta) \), in which case the context of the equality signs should be made clear.
Analogously the map, also denoted by $\eta$, from $SG^+_M$ to $\mathbb{Z}G_M$, induced by $\sigma_i \mapsto \sigma_i$, $\tau_i \mapsto \sigma_i - \sigma_i^{-1}$ is a monoid homomorphism, again referred to as the \textit{Vassiliev homomorphism}. 

\textbf{Conjecture 2.} The Vassiliev homomorphism $: SG^+_M \rightarrow \mathbb{Z}G_M$ is injective.

In effect, for finite type $M$, Conjecture 2 implies Conjecture 1, as the following result demonstrates:

\textbf{Observation 1.} Whenever $M$ is of finite type, Conjecture 2 implies Conjecture 1.

\textit{Proof:} Suppose Conjecture 2 holds and that $\eta(U) = \eta(V)$ for some words $U$ and $V$ in $SG^+_M$ where, without causing confusion, we denote the equivalence class of a word by the word itself. By Theorem 2.1, there are integers $p(U)$ and $p(V)$, and words $\overline{U}$ and $\overline{V}$ in $SG^+_M$ such that $U \approx \Delta^{p(U)} \overline{U}$ and $V \approx \Delta^{p(V)} \overline{V}$. Then there exist positive integers $k_1$, $k_2$ and $k$ such that

$$\Delta^k U \approx \Delta^{k_1} \overline{U} \quad \text{and} \quad \Delta^k V \approx \Delta^{k_2} \overline{V}. \quad (3.1)$$

Hence (recalling $\Delta$ is over $S$),

$$\eta(\Delta^{k_1} \overline{U}) = \eta(\Delta^k U) = \eta(\Delta^k V) = \eta(\Delta^{k_2} \overline{V}).$$

But $k_1$ and $k_2$ are positive integers, $\overline{U}$ and $\overline{V}$ are over $S \cup T$ so that $\Delta^{k_1} \overline{U}$ and $\Delta^{k_2} \overline{V}$ also represent elements of $SG^+_M$ and their images under $\eta$ are the same, in either interpretation of $\eta$. Hence, since Conjecture 2 holds, $\Delta^{k_1} \overline{U} \sim \Delta^{k_2} \overline{V}$. By (3.1) it follows that $\Delta^k U \approx \Delta^k V$. The result now follows by cancellativity. \hfill $\square$

Observation 1 thus shows that, when $M$ is of finite type, it is sufficient to prove Birman’s conjecture in the positive singular Artin monoid. Many properties hold only in $SG^+_M$ and not in $SG_M$, the most obvious of which is preservation of word length, and so allowing for inductive arguments. In what follows we make some elementary observations about the Vassiliev homomorphism and its relationship with $SG^+_M$. 

Define monoid homomorphisms $\epsilon$ and $\mathcal{N}$ from $SG_M$ to $(\mathbb{Z}, +)$ by

$$\epsilon : \sigma_i^{\pm 1} \mapsto \pm 1, \quad \tau_i \mapsto 0, \quad \mathcal{N} : \sigma_i^{\pm 1} \mapsto 0, \quad \tau_i \mapsto 1$$

and a map $+$ from $(S \cup T)^*$ to $S^*$ by $\alpha_i \mapsto \sigma_i$, where $\alpha = \sigma$ or $\tau$. So $\epsilon(A)$ is the exponent sum of sigmas in any word $A$ in $SG_M$, $\mathcal{N}(A)$ counts the number of singularities (taus) of any word in $SG_M$ and $+$ turns every letter from $T$ into a corresponding one in $S$. Notice that $+$ induces a homomorphism $SG^+_M \rightarrow G^+_M$ and that for any word $A$ in $SG^+_M$, $\epsilon(A^+) = \epsilon(A) + \mathcal{N}(A)$. In [Jár, Lemma 1] Járai showed that we can replace $\eta$ with a simpler homomorphism $\psi$ introduced below and that the group algebra $\mathbb{Z}B_{n+1}$ contains no zero divisors. Lemma 3.1 below is obtained by replacing $SB_{n+1}$ with $SG_M$ in the proof of [Jár, Lemma 1].

\textbf{Lemma 3.1.} Define the homomorphism $\psi : SG_M \rightarrow \mathbb{Z}G_M$, by $\psi(\tau_i) = \sigma_i + \sigma_i^{-1}$ and $\psi(\sigma_i^{\pm 1}) = \sigma_i^{\pm 1}$. Then for any words $C$, $A$ and $B$ in $SG_M$, 

\[ \]
(1) \( \eta(A) = \eta(B) \) if and only if \( \psi(A) = \psi(B) \) and \\
(2) \( \psi(CA) = \psi(CB) \iff \psi(A) = \psi(B) \iff \psi(AC) = \psi(BC) \).

We define the homomorphism \( \psi : SG^+_M \to \mathbb{Z}G_M \) as in Lemma 3.1. From the above definitions and result we deduce the following:

**Lemma 3.2.** Let \( U \) and \( V \) be in \( SG^+_M \) and suppose \( \psi(U) = \psi(V) \). Then \( \ell(U) = \ell(V) \) and \( \mathcal{N}(U) = \mathcal{N}(V) \).

**Proof:** Observe that \( U^+ \sim V^+ \) since they both represent the unique monomial of maximal exponent sum of \( \psi(U) = \psi(V) \). But
\[
\ell(U) = \epsilon(U) + \mathcal{N}(U) = \ell(U^+) = \ell(V^+) = \epsilon(V) + \mathcal{N}(V) = \ell(V).
\] (3.2)

Notice that for every word \( A \) in \( SG_M \) there is a unique monomial, represented by \( A^- \), obtained by replacing each \( \tau_i \) by \( \sigma_i^{-1} \), in the support of \( \psi(A) \) with minimal exponent sum \( \epsilon(A) - \mathcal{N}(A) \). Then since \( \psi(U) = \psi(V) \), it follows that \( U^- \approx V^- \), so
\[
\epsilon(U) - \mathcal{N}(U) = \epsilon(U^-) = \epsilon(V^-) = \epsilon(V) - \mathcal{N}(V).
\] (3.3)

By (3.2) and (3.3) it follows that \( \mathcal{N}(U) = \mathcal{N}(V) \). \( \square \)

## 4 Common divisors and the Vassiliev homomorphism

For the remainder of this paper we resume our assumption that \( M \) is of finite type. In this section we provide a criterion (expressed in Corollary 4.2 below) for determining when two elements of \( SG^+_M \), with the same image under \( \psi \), have a non-trivial common divisor.

### 4.1 The positive form

Let \( W \) be a word over \( S \cup S^{-1} \cup T \). Then there are words \( W_i \) over \( S \cup T \) and generators \( \sigma_{a_i}^{-1} \in S^{-1} \) such that
\[
W = W_0 \sigma_{a_1}^{-1} W_1 \sigma_{a_2}^{-1} W_2 \ldots W_{k-1} \sigma_{a_k}^{-1} W_k.
\]

According to [Cor1, p.278] define maps \( \theta_1 \) and \( \theta_2 \) by
\[
\theta_1(W) = W_0 \zeta_{a_1} W_1 \zeta_{a_2} W_2 \ldots W_{k-1} \zeta_{a_k} W_k \quad \text{and} \quad \theta_2(W) = k.
\]

So \( \theta_1 \) turns \( W \) into a word over \( S \cup T \) by replacing each letter \( \sigma_{a}^{-1} \) from \( S^{-1} \) with a corresponding \( \zeta_{a} \), whilst \( \theta_2 \) counts the number of occurrences of letters from \( S^{-1} \) in \( W \). Furthermore, \( \theta_1 \) acts as the identity on \( S \cup T \) and for any words \( X \) and \( Y \), \( \theta_1(XY) = \theta_1(X)\theta_1(Y) \). Since \( \zeta_{\alpha} \sim \alpha \zeta \) for any generator \( \alpha \) in \( S \cup T \), by centrality, it can be shown [Cor1, p. 278] that \( \theta_1(W) \approx \zeta^{\theta_2(W)} W \).

For every \( W \) over \( S \cup S^{-1} \cup T \), let \( q(W) \) be the largest integer such that the word
\((\theta_1(W)/\Delta^{q(W)})\) is defined. Observe that, for any word \(W\), and any \(a\) in \(S\), \(q(a^{-1}) = 1\) and \(q(W) \geq \theta_2(W) \geq 0\). This follows from the fact that, by Lemmas 2.1 and 2.3(1), \(\theta_1(a^{-1}) = \zeta_a \sim \Delta_a \sim \Delta \Delta_a\)' for every generator \(a\) in \(S \cup T\). Moreover, \(\Delta^{q(X)+q(Y)} < \theta_1(X)\theta_1(Y) = \theta_1(XY)\), by Lemma 2.1 again, giving \(q(XY) \geq q(X) + q(Y) \geq 0\), for any words \(X\) and \(Y\) over \(S \cup S^{-1} \cup T\). Since \(q(W) \geq \theta_2(W) \geq 0\), the word \((\theta_1(W)/\Delta^{\theta_2(W)})\) is always defined and we shall denote it by \(N(W)\). So \(N(W)\) is also a word over \(S \cup T\) and \(N\) fixes elements of \((S \cup T)^*\). Notice that for any words \(X\) and \(Y\) over \(S \cup S^{-1} \cup T\),

\[ \Delta^{\theta_2(X)}N(XY) \sim \theta_1(XY) = \theta_1(X)\theta_1(Y) \sim \Delta^{\theta_2(X)}N(X)\Delta^{\theta_2(Y)}N(Y). \]

Thus, since \(\theta_2(XY) = \theta_2(X) + \theta_2(Y)\), Lemmas 2.1 and 2.2(2) yield

\[ N(XY) \sim \begin{cases} N(X)N(Y) & \text{if } \theta_2(Y) \text{ is even,} \\ \mathcal{R}(N(X))N(Y) & \text{if } \theta_2(Y) \text{ is odd,} \end{cases} \tag{4.1} \]

by cancellation. Since \(N(\Delta) = \Delta\) and, by Lemma 2.2(1), \(\mathcal{R}(\Delta) \sim \Delta\) we obtain immediately that \(N(\Delta Y) \sim \Delta N(Y)\) for any \(Y\) over \(S \cup S^{-1} \cup T\).

We call a word \(W\) in \((S \cup S^{-1} \cup T)^*\) minimal if \(q(W) = \theta_2(W)\). We call a word \(W\) over \(S \cup T\) prime if it is not divisible by \(\Delta\). Recalling \(q(W)\) is the largest integer such that \(\Delta^{q(W)}\) divides \(\theta_1(W)\) we see that \(W\) is minimal gives \(N(W) = (\theta_1(W)/\Delta^{q(W)})\) is prime; whereas \(\Delta \neq N(W) = (\theta_1(W)/\Delta^{\theta_2(W)})\) implies that \(q(W) = \theta_2(W)\). Thus \(W\) is minimal if and only if \(N(W)\) is prime.

**Lemma 4.1.** Let \(X\) and \(Y\) be words over \(S \cup S^{-1} \cup T\) such that \(X \approx Y\), and \(a, b\) distinct elements in \(S\). Then:

1. \(N(X) \sim N(Y)\) if and only if \(\theta_2(X) = \theta_2(Y)\);
2. If \(X = a^{-1}X_1, Y = b^{-1}Y_1\) and \(N(X) \sim N(Y)\), then \(X\) and \(Y\) are not minimal.

**Proof:** (1) If \(\theta_2(X) = \theta_2(Y)\) then, since \(X \approx Y\),

\[ \theta_1(X) \approx \zeta^{\theta_2(X)}X \approx \zeta^{\theta_2(Y)}Y \approx \theta_1(Y) \]

giving, by Theorem 1.1, \(\theta_1(X) \sim \theta_1(Y)\), so that

\[ N(X) = (\theta_1(X)/\Delta^{\theta_2(X)}) \sim (\theta_1(Y)/\Delta^{\theta_2(Y)}) = N(Y). \]

Now suppose \(N(X) \sim N(Y)\) and suppose further, without loss of generality, that \(\theta_2(X) - \theta_2(Y) \geq 0\). Then, since \((\theta_1(X)/\Delta^{\theta_2(X)}) = N(X)\), multiplying through on the left by \(\Delta^{\theta_2(X)}\) yields

\[ \theta_1(X) \sim \Delta^{\theta_2(X)}N(X) \sim \Delta^{\theta_2(X)}N(Y) \]

\[ = \Delta^{\theta_2(X)-\theta_2(Y)} \Delta^{\theta_2(Y)}(\theta_1(Y)/\Delta^{\theta_2(Y)}) \]

\[ \approx \Delta^{\theta_2(X)-\theta_2(Y)} \theta_1(Y) \]

\[ \approx \Delta^{\theta_2(X)-\theta_2(Y)} \zeta^{\theta_2(Y)}Y \]

\[ \approx \Delta^{\theta_2(X)+\theta_2(Y)}X \quad \text{(since } X \approx Y, \zeta = \Delta^2). \]
Thus $\Delta^2 \theta_2(X) X = \zeta^2 \theta_2(X) X \approx \theta_1(X) \approx \Delta \theta_2(X) + \theta_2(Y) X$ which gives $\Delta^2 \theta_2(X) \sim \Delta \theta_2(X) + \theta_2(Y)$, by cancellation and Theorem 1.1, so that $\theta_2(X) = \theta_2(Y)$.

(2) Suppose $X = a^{-1} X_1$, $Y = b^{-1} Y_1$ and $N(X) \sim N(Y)$. By (1), $\theta_2(X) = \theta_2(Y)$ yielding

$$1 + \theta_2(X_1) = \theta_2(a^{-1} X_1) = \theta_2(b^{-1} Y_1) = 1 + \theta_2(Y_1)$$

so that $\theta_2(X_1)$ is even if and only if $\theta_2(Y_1)$ is even. Combined with (4.1), this implies,

$$N(a^{-1}) N(X_1) \sim N(X) \sim N(Y) \sim N(b^{-1}) N(Y_1)$$

if $\theta_2(X_1)$ is even, and

$$\mathcal{R}(N(a^{-1})) N(X_1) \sim N(X) \sim N(Y) \sim \mathcal{R}(N(b^{-1})) N(Y_1)$$

if $\theta_2(X_1)$ is odd. Let $L = \text{lcm}(N(a^{-1}), N(b^{-1}))$ which exists by the preceding equivalence. Then $L \sim \text{lcm}(\Delta_\sigma, \Delta_\phi)$ (see Lemma 4.2(1) below) and, since $a \neq b$, it follows by Corollary 2.2 that $\Delta \prec L$. Since $\mathcal{R}(\Delta) \sim \Delta$ this implies that $\Delta$ also divides $\mathcal{R}(L)$. Thus $\Delta \prec N(X) \sim N(Y)$, showing that $X$ and $Y$ are not minimal.

Lemma 4.2. For any integers $r, s \geq 1$ and generator $a$ in $S$,

(1) the word $a^{-r}$ is minimal and

$$N(a^{-r}) \sim \begin{cases} (\Delta_a \Delta_{a'})^m & \text{if } r = 2m, \\ (\Delta_{a'} \Delta_a)^m \Delta_{a'} & \text{if } r = 2m + 1. \end{cases}$$

(2) the word $(a^{-r} a^s)$ is not minimal.

Proof: (1) The claim certainly holds for $r = 1 = \theta_2(a^{-1})$, since, by Lemmas 2.1 and 2.3(1), we obtain $\theta_1(a^{-1}) = \zeta_a \sim \Delta_a \Delta \sim \Delta \Delta_{a'}$, so that $N(a^{-1}) \sim \Delta_{a'}$, which is clearly prime. For $r = 2 = \theta_2(a^{-2})$ we deduce by (4.1) that

$$N(a^{-2}) \sim \mathcal{R}(N(a^{-1})) N(a^{-1}) \sim \Delta_a \Delta_{a'}$$

by Lemma 2.3(1) again. Thus, $N(a^{-2}) \sim \Delta_a \Delta_{a'}$, which is prime by Lemma 2.9. So suppose that $r$ is any integer $\geq 3$ and that the claim holds for all $l < r$. If $r = 2m$, then

$$\theta_1(a^{-r}) = \zeta_a^r = \zeta_a^{r-1} \zeta_a \sim \Delta_{a'}^{2m-1} (\Delta_a' \Delta_a)^{m-1} \Delta_a' \Delta_a$$

$$\sim \Delta_{a'}^{2m-1} (\Delta_a' \Delta_a)^{m-1} \Delta_{a'} \Delta_a$$

$$= \Delta_{a'}^{2m} (\Delta_a \Delta_{a'}).$$

Note that (4.2) was obtained inductively (since $2m - 1 < r$) and (4.3) holds by Lemmas 2.1 and 2.3(1). Thus $N(a^{-r}) \sim (\Delta_a \Delta_{a'})^m$ and, by Lemma 2.9, it is prime. For $r = 2m + 1$, (4.1) yields

$$N(a^{-r}) = N(a^{-1} a^{-2m}) = N(a^{-1}) N(a^{-2m})$$

$$\sim \Delta_{a'} (\Delta_a \Delta_{a'})^m$$
which, by Lemma 2.9, is also prime. The result for all \( r \) now follows by induction.

(2) Since \( \theta_2(a^s) = 0 \), (4.1) gives \( N(a^{-r} a^s) \sim N(a^{-r}) N(a^s) \sim N(a^{-r}) a^s \) from which it follows, by the first part of this lemma, that

\[
N(a^{-r} a^s) \sim \begin{cases} 
(\Delta_a \Delta_{a'})^m a^s & \text{if } r = 2m, \\
(\Delta_{a'} \Delta_a)^m \Delta_{a'} a^s & \text{if } r = 2m + 1.
\end{cases}
\]

But by Lemma 2.3(2) we know that \( \Delta \sim \Delta_{a'} a \). Hence, by Lemma 2.1, \( \Delta \) divides \( N(a^{-r} a^s) \). Thus \( N(a^{-r} a^s) \) is not prime and so the word \( (a^{-r} a^s) \) cannot be minimal. \( \square \)

### 4.2 Minimal words and the support of \( \psi \)

Let \( U \) be any word over \( S \cup T \) also regarded as an element of \( SG_M^+ \). A summand of \( \psi(U) \) is any word over \( S \cup S^{-1} \) obtained by replacing any given instance of \( \tau \) by \( \sigma \) or \( \sigma^{-1} \). The support of \( \psi(U) \) is the set of summands of \( \psi(U) \). Let \( \mathcal{M}_U \) denote the summands of \( \psi(U) \) that are minimal. For example, in type \( A_2 \), \( \psi(\tau_1 \tau_2 \sigma_3 \tau_1) \) has summands \( \sigma_1 \sigma_2^2 \sigma_1, \sigma_1 \sigma_2 \sigma_1 \), \( \sigma_1 \sigma_2 \sigma_2 \sigma_1 \), \( \sigma_1 \sigma_2^{-1} \sigma_2 \sigma_1 \), \( \sigma_1^{-1} \sigma_2 \sigma_1 \), \( \sigma_1^{-1} \sigma_2^{-1} \sigma_2 \sigma_1 \), and \( \sigma_1^{-1} \sigma_2^{-1} \sigma_2 \sigma_1 \) and a routine calculation shows

\[
\mathcal{M}_{\tau_1 \tau_2 \sigma_3 \tau_1} = \{ \sigma_1 \sigma_2^2 \sigma_1, \sigma_1 \sigma_2 \sigma_1, \sigma_1^{-1} \sigma_2 \sigma_1, \sigma_1^{-1} \sigma_2 \sigma_1 \}.
\]

**Lemma 4.3.** Let \( U, V \) be words in \( (S \cup T)^* \) also regarded as elements of \( SG_M^+ \) such that \( \psi(U) = \psi(V) \). For every summand \( F \) of \( \psi(U) \) there is a corresponding summand \( G \) of \( \psi(V) \) such that \( F \approx G \), \( \theta_2(F) = \theta_2(G) \), \( N(F) \sim N(G) \) and \( F \in \mathcal{M}_U \) if and only if \( G \in \mathcal{M}_V \).

**Proof:** Let \( F \) be any element in the support of \( \psi(U) \). Then, since \( F \) represents an element of the Artin group \( G_M \), and \( \psi(U) = \psi(V) \) there is an element \( G \) in the support of \( \psi(V) \) such that \( F \approx G \). Thus, since \( F \) and \( G \) are equivalent monomials in \( G_M \), their exponent sums must be the same; that is \( \epsilon(F) = \epsilon(G) \). But \( \epsilon(A) = \ell(A) - 2\theta(A) \), for any word \( A \) in \( (S \cup S^{-1})^* \), whence \( \ell(F) - 2\theta_2(F) = \ell(G) - 2\theta_2(G) \). By Lemma 3.2, \( \ell(F) = \ell(U) = \ell(V) = \ell(G) \). Hence \( \theta_2(F) = \theta_2(G) \) and \( F \approx G \) so that \( N(F) \sim N(G) \), by Lemma 4.1(1).

Now let \( F \) be an element of \( \mathcal{M}_U \). Then \( F \) is a minimal word in the support of \( \psi(U) \).

So, by the previous argument, there is an element \( G \) in the support of \( \psi(V) \) such that \( F \approx G \), \( \theta_2(F) = \theta_2(G) \) and \( N(F) \sim N(G) \). But \( F \) is minimal so that \( N(F) \sim N(G) \) is prime. Thus \( G \) is a minimal word in the support of \( \psi(V) \); that is \( G \in \mathcal{M}_V \). \( \square \)

**Lemma 4.4.** Let \( U \) be a word over \( S \cup T \) that is divisible by \( \Delta \). Then \( \mathcal{M}_U = \emptyset \).

**Proof:** Since \( \Delta \) divides \( U \) there exists a word \( U_1 \) over \( S \cup T \) such that \( U \sim \Delta U_1 \). Thus \( \psi(U) = \psi(\Delta U_1) = \Delta \psi(U_1) \). Now let \( X \) be any summand of \( \psi(U) \). Then by Lemma 4.3
there is a word $Y$ in the support of $\psi(\Delta U_1)$ such that $X \approx Y$ and $N(X) \sim N(Y)$. But $Y = \Delta Y_1$ for some word $Y_1$ in the support of $\psi(U_1)$. Hence

$$N(X) \sim N(Y) = N(\Delta Y_1) \sim \Delta N(Y_1).$$

Thus $\Delta$ divides $N(X)$ so that $X$ is not minimal and therefore not an element of $\mathcal{M}_U$. □

Corollary 4.2 below motivates the next proposition.

**Proposition 4.1.** Let $C$ be a $\sigma_a$-chain to $\sigma_b$. Then there exists a word $Z \in \mathcal{M}_C$ such that $N(Z) \sim C'$ where $C'$ is a $\sigma_a [\sigma_a']$-chain to $\sigma_b$ if $\theta_2(Z)$ is even [odd].

**Proof:** By Lemma 2.4, $\sigma_a$ does not divide $C$. Write $C = C_1 \ldots C_k$ where each $C_i$ is simple. Lemma 2.7 tells us that the target of $C_1$ must lie in $S$. So suppose that $\sigma_c$ is the target of $C_1$. Then there exist words $P, Q$ such that $(\sigma_a P, C_1 \sigma_a Q) \in \mathcal{R}^C$. If $C_1$ is over $S$ we have that $\sigma_a \neq C_1 = C_1^+$ is a $\sigma_a$-chain to $\sigma_c$, $C_1^+ \in \mathcal{M}_{C_1}$ and $\theta_2(C_1^+) = 0$ is even. Otherwise inspection of the relations $\mathcal{R}$ shows that

$$C_1 = \tau_j (\sigma_a \sigma_j)^q \quad (\exists q) \quad 0 \leq q \leq m_{ja} - 2$$

or $C_1 = \tau_a$. Suppose first $C_1 = \tau_j (\sigma_a \sigma_j)^q$. Then

$$\sigma_a \neq N(C_1^+) = C_1^+ = \sigma_j (\sigma_a \sigma_j)^q = (\sigma_j \sigma_a)^q + 1$$

since $q + 1 < m_{ja}$. Clearly $C_1^+$ is a summand of $\psi(C_1)$ which is prime, since it is not divisible by $\sigma_a$, and so must lie in $\mathcal{M}_{C_1}$. Moreover, $\theta_2(C_1^+) = 0$ is even and $C_1^+$ is a simple $\sigma_a$-chain to $\sigma_c$ (by definition). So assume that $C_1 = \tau_a$. Then the target of $C_1$ is $\sigma_c = \sigma_a$. Put $X = \sigma_a^{-1}$ which is clearly a summand of $\psi(C_1)$ and note that $\theta_2(X) = 1$ is odd. Then $X$ is minimal and $N(X) \sim \Delta_{\sigma_c}$ by Lemma 4.2(1). Corollary 2.2 tells us that $\sigma_c'$ does not divide $N(X)$, whence $N(X) \sim D$, for some $\sigma_c'$-chain $D$, by Corollary 2.1. Since

$$\sigma_c' \prec \Delta \sim \Delta_{\sigma_c} \sigma_a \sim D \sigma_a$$

by Lemma 2.3(2), the target of $D$ must be $\sigma_a = \sigma_c$, by Lemma 2.6. Hence, in all cases, there exists a word $X$ in $\mathcal{M}_{C_1}$ such that $N(X) \sim C_1'$ where

$$C_1'$$

is a $\sigma_a [\sigma_a']$-chain to $\sigma_c$ if $\theta_2(X)$ is even [odd].

(4.4)

Now if $k = 1$ then $C = C_1$, $\sigma_b = \sigma_c$ and we are done. Otherwise $C_2 \ldots C_k$ is a $\sigma_c$-chain to $\sigma_b$, yielding, by induction, a word $Y$ in $\mathcal{M}_{C_2 \ldots C_k}$ such that $N(Y) \sim C_2'$ where

$$C_2'$$

is a $\sigma_c [\sigma_c']$-chain to $\sigma_b$ if $\theta_2(Y)$ is even [odd].

(4.5)

Put $Z = XY$, noting that it is a summand of $\psi(C) = \psi(C_1 C_2 \ldots C_k)$. Then (4.1) gives

$$N(Z) \sim \begin{cases} C_1' \ C_2' \quad & \text{if } \theta_2(Y) \text{ is even}, \\ \mathcal{R}(C_1') \ C_2' \quad & \text{if } \theta_2(Y) \text{ is odd}. \end{cases}$$
Case 1: \( \theta_2(Y) \) is even.

Put \( C' = C'_1C'_2 \) so that \( N(Z) \sim C' \). Then \( C'_2 \) is a \( \sigma_c \)-chain to \( \sigma_b \) by (4.5). If \( \theta_2(Z) \) is even then \( \theta_2(X) \) is also even so \( C'_1 \) is a \( \sigma_a \)-chain to \( \sigma_c \) by (4.4) whence \( C' \) is a \( \sigma_a \)-chain to \( \sigma_b \) and \( C' \sim N(Z) \). On the other hand if \( \theta_2(Z) \) is odd then \( \theta_2(X) \) is odd and \( C'_1 \) is a \( \sigma'_a \)-chain to \( \sigma_c \), again by (4.4), which shows that \( C' \) is a \( \sigma'_a \)-chain to \( \sigma_b \) as required.

Case 2: \( \theta_2(Y) \) is odd.

Put \( C' = \mathcal{R}(C'_1)C'_2 \) so that \( N(Z) \sim C' \). Then \( C'_2 \) is a \( \sigma'_c \)-chain to \( \sigma_b \) by (4.5). Recalling \( \mathcal{R} \) is an involutorial automorphism of \( (S \cup T)^* \) which preserves the relations \( \mathcal{R}, \) (4.4) gives

\[
\mathcal{R}(C'_1) \text{ is a } \sigma'_a [\sigma_a] \text{-chain to } \sigma'_b \text{ if } \theta_2(X) \text{ is even [odd].} \tag{4.6}
\]

Thus if \( \theta_2(Z) \) is even then \( \theta_2(X) \) is odd so that \( \mathcal{R}(C'_1) \) is a \( \sigma_a \)-chain to \( \sigma'_b \), by (4.6), showing that \( C' \) is a \( \sigma_a \)-chain to \( \sigma_b \). On the other hand if \( \theta_2(Z) \) is odd, then \( \theta_2(X) \) is even so that \( \mathcal{R}(C'_1) \) is a \( \sigma'_a \)-chain to \( \sigma'_c \), again by (4.6), whence \( C' \) is a \( \sigma'_a \)-chain to \( \sigma_b \).

Cases 1 and 2 both show that \( N(Z) \sim C' \) where

\[
C'_1 \text{ is a } \sigma_a [\sigma'_a] \text{-chain to } \sigma_b \text{ if } \theta_2(Z) \text{ is even [odd],}
\]

which, by Corollary 2.1, is prime. Since \( Z \) is a minimal element in the support of \( \psi(C) \) it must (by definition) lie in \( \mathcal{M}_C \) and our proof is complete.

**Corollary 4.1.** Let \( U \) be a non-empty word in \( (S \cup T)^* \) and \( a \) any generator in \( S \). Suppose that \( a \not\in \mathcal{U} \). Then \( \mathcal{M}_U \) contains an element \( Z \) such that \( a [a'] \not\in N(Z) \) if \( \theta_2(Z) \) is even [odd]. In particular \( \mathcal{M}_U \neq \emptyset \) whenever \( U \) is prime.

**Proof:** Since \( a \in S, U \sim C \) for some \( a \)-chain \( C \), by Corollary 2.1. Proposition 4.1 now yields a word \( Z \) in \( \mathcal{M}_C \) such that \( N(Z) \sim C' \) where \( C' \) is an \( a [a'] \)-chain if \( \theta_2(Z) \) is even [odd]. By Corollary 2.1 again, we infer that \( a [a'] \not\in N(Z) \) if \( \theta_2(Z) \) is even [odd]. Certainly \( \psi(U) = \psi(C) \) where \( U \) and \( C \) are regarded as (the same) elements of \( SG_M^+ \). By Lemma 4.3 there exists \( Z' \in \mathcal{M}_U \) such that \( N(Z') \sim N(Z) \). Hence

\[
a [a'] \not\in N(Z') \text{ if } \theta_2(Z') \text{ is even [odd]}
\]

and the result is proved.

**Corollary 4.2.** Let \( U, V \) be words in \( (S \cup T)^* \) also regarded as elements of \( SG_M^+ \) such that \( \psi(U) = \psi(V) \) and \( C \) any word over \( S \). Then \( C \) divides \( U \) if and only if \( C \) divides \( V \).

**Proof:** We first prove the “only if” part of the statement. Suppose first that \( U \sim aU_1 \), for some generator \( a \) in \( S \) and word \( U_1 \) over \( S \cup T \). Put \( F = \Delta_{a'} U, G = \Delta_{a'} V \). Then \( \Delta < \Delta_{a'} U_1 \sim F \), by Lemma 2.3(2), and \( \psi(F) = \Delta_{a'} \psi(U) = \Delta_{a'} \psi(V) = \psi(G) \) giving a one-one correspondence between the sets \( \mathcal{M}_F \) and \( \mathcal{M}_G \) by Lemma 4.3. Since
Δ divides F we deduce, by Lemma 4.4, that $\mathcal{M}_F = \emptyset$; whilst if G is prime Corollary 4.1 yields $\mathcal{M}_G \neq \emptyset$ contradicting the existence of the bijection between $\mathcal{M}_F$ and $\mathcal{M}_G$. Hence Δ also divides $G = \Delta_a V$ showing that $a \prec V$ by Lemma 2.3(2) and cancellation. This proves the result for $\ell(C) = 1$ and, noting that it holds trivially for $\ell(C) = 0$, starts an induction. So assume that C divides U and $\ell(C) \geq 2$. Then there exists a letter a in S and non-empty word C₁ over S such that $C = C_1 a$ and $U \sim C_1 a U_1$ for some word $U_1$ over $S \cup T$. By induction we infer the existence of a word $V_1$ over $S \cup T$, such that $V \sim C_1 V_1$. Thus, $\psi(C_1 a U_1) = \psi(U) = \psi(V) = \psi(C_1 V_1)$ which gives $\psi(a U_1) = \psi(V_1)$, by Lemma 3.1(2), and shows that a divides $V_1$. Hence $C = C_1 a$ also divides $V \sim C_1 V_1$ as required and the result for any $\ell(C)$ follows by induction. Swapping the roles of U and V in the preceding argument proves the converse of the result. □

5 The Intermediate Lemma

In this section we prove that the Intermediate Property – discovered by Corran in [Cor1, Intermediate Lemma] and expressed in Lemma 5.1 below – is preserved under the Vasiliev homomorphism. As a corollary we deduce that $\eta$ is injective for a class of monoids which include singular Artin monoids of type $I_2(p)$.

Lemma 5.1. Let U, V be words in $(S \cup T)^*$ such that $\tau_i U \sim \tau_j V$. Then $i = j$ or $m_{ij} = 2$.

The proofs of Lemmas 5.2 and 5.3 below, although technical, are straightforward and lead to Proposition 5.1 below.

Lemma 5.2. Let F be a minimal word in $(S \cup S^{-1} \cup T)^*$, $q$ any integer $\geq 1$. If $\theta_2(F)$ is even [odd] then

1. $\sigma_s^q F$ is minimal whenever $\sigma_s [\sigma_s'] \not\prec N(F)$ and

2. $\sigma_s^{-q} F$ is minimal whenever $\sigma_s [\sigma_s'] \not\prec N(F)$.

Proof: Supposing F is minimal gives $N(F)$ is prime.

Case 1: $\theta_2(F)$ is even.

Suppose first that $\sigma_s$ divides $N(F)$ so that $N(F) \sim \sigma_s F_1$ for some word $F_1$ over $S \cup T$. Put $F' = \sigma_s^q F$. Then, since $\theta_2(F)$ is even, (4.1) yields

$N(F') = N(\sigma_s^q F) \sim N(\sigma_s^q) N(F) \sim \sigma_s^q N(F)$.

Now $\Delta \not\prec N(F) \sim \sigma_s F_1$ so by Corollary 2.3 it follows that

$\Delta \not\prec \sigma_s^{q+1} F_1 = \sigma_s^q \sigma_s F_1 \sim \sigma_s^q N(F) \sim N(F')$.

Hence $N(F')$ is prime showing that $F' = \sigma_s^q F$ is minimal. Now assume that $\sigma_s \not\prec N(F)$ and put $F' = \sigma_s^{-q} F$. Then since $\theta_2(F)$ is even we infer, by (4.1) again,

$N(F') = N(\sigma_s^{-q} F) \sim N(\sigma_s^{-q}) N(F)$.
which yields
\[ N(F') \sim N(\sigma_s^{-q}) N(F) \sim \begin{cases} (\Delta_{\sigma_s, \Delta_{\sigma_s'}})^m N(F) & \text{if } q = 2m, \\ (\Delta_{\sigma_s, \Delta_{\sigma_s'}})^m \Delta_{\sigma_s'} N(F) & \text{if } q = 2m + 1, \end{cases} \]
by Lemma 4.2(1). Since \( \sigma_s \neq N(F) \), Lemma 2.9 shows that \( N(F') \) is prime so \( F' = \sigma_s^{-q}F \) is minimal as required.

**Case 2:** \( \theta_2(F) \) is odd.

Assume \( F' = \sigma_s^{\pm q}F \). Then, recalling \( \mathcal{R} \) is an involution, (4.1) gives
\[ N(F') \sim \mathcal{R}(N(\sigma_s^{\pm q})) N(F), \quad \text{so } \mathcal{R}(N(F')) \sim N(\sigma_s^{\pm q}) \mathcal{R}(N(F)). \]

Since \( \mathcal{R}(\sigma_s) = \sigma_s' \prec N(F) \) if and only if \( \sigma_s \prec \mathcal{R}(N(F)) \) and \( \mathcal{R}(N(F')) \) is prime if and only if \( N(F') \) is prime, the argument proceeds exactly as that of each alternative in the previous case. \( \Box \)

**Lemma 5.3.** Let \( F = \sigma_sF_1 \) be a minimal word in \((S \cup S^{-1} \cup T)^*\) such that \( \Delta \) divides \( \mathcal{R}(\sigma_s F_1) \). Then \( \sigma_s^{-1}F_1 \) is minimal or \( m_{rs} = 2 \).

**Proof:** Suppose \( m_{rs} \neq 2 \). We show that \( \sigma_s^{-1}F_1 \) is minimal.

**Case 1:** \( \theta_2(F_1) \) is even.

By (4.1) we obtain that \( N(F) = N(\sigma_s F_1) \sim \sigma_s N(F_1) \). Since \( N(F) \) is prime, by assumption, Lemma 2.1 thus shows that \( N(F_1) \) is also prime. Now \( \Delta \) divides \( N(\sigma_r F) = N(\sigma_r \sigma_s F_1) \) giving, again by (4.1),
\[ \Delta \prec N(\sigma_r \sigma_s F_1) \sim N(\sigma_r \sigma_s) N(F_1) \sim \sigma_r \sigma_s N(F_1). \]

Noting \( m_{rs} \neq 2 \) it is clear that \( \sigma_s \) is the only generator which right divides \( \sigma_r \sigma_s \), whence an application of Proposition 2.1 to (5.1) yields \( \sigma_j \prec N(F_1) \) for every \( j \neq s \). Since \( N(F_1) \) is prime this implies that \( \sigma_s \) does not divide \( N(F_1) \). Hence \( \sigma_s^{-1}F_1 \) is minimal, by Lemma 5.2(2), as required.

**Case 2:** \( \theta_2(F_1) \) is odd.

By (4.1) we obtain that \( N(F) = N(\sigma_s F_1) \sim \sigma_s' N(F_1) \). Since \( N(F) \) is prime, by assumption, Lemma 2.1 thus shows that \( N(F_1) \) is also prime. Now \( \Delta \) divides \( N(\sigma_r F) = N(\sigma_r \sigma_s F_1) \) giving, again by (4.1),
\[ \Delta \prec N(\sigma_r \sigma_s F_1) \sim \mathcal{R}(N(\sigma_r \sigma_s)) N(F_1) \sim \sigma_r' \sigma_s' N(F_1). \]

Recalling \( m_{rs'} = m_{rs} \neq 2 \) it is clear that \( \sigma_{s'} = \sigma_{s'}' \) is the only generator which right divides \( \sigma_r' \sigma_{s'} \), whence an application of Proposition 2.1 to (5.2) yields \( \sigma_j' = \sigma_j' \) divides \( N(F_1) \) for every \( j' \neq s' \). Since \( N(F_1) \) is prime this implies that \( \sigma_{s'} \) does not divide \( N(F_1) \). Hence \( \sigma_{s'}^{-1}F_1 \) is minimal, by Lemma 5.2(2), as required. \( \Box \)
Proposition 5.1. Let $U = \tau_i U_1$, $V = \tau_j V_1$ be words in $(S \cup T)^*$ also regarded as elements of $S G_M^+$ such that $\psi(U) = \psi(V)$. Then $i = j$ or $m_{ij} = 2$.

**Proof:** Suppose $U = \tau_i U_1$ and $V = \tau_j V_1$ provide a counterexample. That is $\psi(\tau_i U_1) = \psi(\tau_j V_1)$ but $m_{ij} \geq 3$. Suppose further that this counterexample is minimal with respect to $\ell(U)$ which, by Lemma 3.2 is equal to $\ell(V)$. Then $\ell(U) \geq 2$ since $\ell(U) = \ell(V) = 1$ gives $\sigma_i + \sigma_i^{-1} = \psi(U) = \psi(V) = \sigma_j + \sigma_j^{-1}$ showing $i = j$. We first show that $V = \tau_j V_1$ is not divisible by $\sigma_j$. Suppose, by way of contradiction, that it is. Reduction yields a word $P$ such that $V_1 \sim \sigma_j P$ and, recalling $\psi(U) = \psi(V)$, Corollary 4.2 implies that $\sigma_j$ also divides $U = \tau_i U_1$ yielding, by reduction again, a word $Q$ such that $U_1 \sim \langle \sigma_j \sigma_i \rangle^{m_{ij}-1} Q$. Put $C = \langle \sigma_j \sigma_i \rangle^{m_{ij}-1}$. Then

$$U = \tau_i U_1 \sim \tau_i C Q \sim C \tau_i Q \quad (5.3)$$

where $\tau_d$ is the target of $C$. Noting $C$ is over $S$, another application of Corollary 4.2 shows that

$$C = \sigma_j \langle \sigma_i \sigma_j \rangle^{m_{ij}-2} \sim V \sim \sigma_j \tau_j P \sim \sigma_j \tau_j P.$$

Recalling $m_{ij} \geq 3$ we deduce, by cancellation, $\sigma_i \not< \tau_j P$ so that $P \sim \langle \sigma_i \sigma_j \rangle^{m_{ij}-1} P'$, for some word $P'$ over $S \cup T$. Thus

$$V \sim \sigma_j \tau_j P \sim \sigma_j \tau_j \langle \sigma_i \sigma_j \rangle^{m_{ij}-1} P' \sim \sigma_j \langle \sigma_i \sigma_j \rangle^{m_{ij}-1} \tau_c P' \sim \langle \sigma_j \sigma_i \rangle^{m_{ij}-1} \sigma_d \tau_c P' \sim C \sigma_d \tau_c P',$$

where, since $\{c, d\} = \{i, j\}$, $m_{cd} \geq 3$. (5.3) combined with the preceding equivalence gives,

$$\psi(C \sigma_d \tau_c P') = \psi(V) = \psi(U) = \psi(C \tau_d Q),$$

so that $\psi(\sigma_d \tau_c P') = \psi(\tau_d Q)$, by Lemma 3.1(2). Another application of Corollary 4.2 shows that $\sigma_d$ divides $\tau_d Q$ yielding a word $Q'$ such that $Q \sim \sigma_d Q'$. Hence

$$\psi(\sigma_d \tau_c P') = \psi(\tau_d Q) = \psi(\sigma_d \tau_d Q') = \psi(\sigma_d \tau_d Q')$$

so that $\psi(\tau_d P') = \psi(\tau_d Q')$, again by Lemma 3.1(2), and $m_{cd} \geq 3$. Since $\ell(V) > \ell(\tau_d P')$ this contradicts the minimality of $\ell(U) = \ell(V)$. Thus $\sigma_j \not< V$ and so, by a final application of Corollary 4.2, we deduce that $\sigma_j \not< U$. This shows that the latter word is prime giving $\mathcal{M}_U \neq \emptyset$, by Corollary 4.1. So let $X$ be an element of $\mathcal{M}_U$ such that

$$\sigma_j [\sigma_j'] \not< N(X) \text{ if } \theta_2(X) \text{ is even [odd]}, \quad (5.4)$$

the existence of which is guaranteed also by Corollary 4.1. Assume further that $\theta_2(X) = k$ is maximal, that is, if $G$ is any other word in $\mathcal{M}_U$ such that

$$\sigma_j [\sigma_j'] \not< N(G) \text{ if } \theta_2(G) \text{ is even [odd]}$$
then $\theta_2(G) \leq k$. Now since $U = \tau_1 U_1$, we deduce that $X = \sigma_i^{-1} X_1$ or $X = \sigma_i X_1$ for some summand $X_1$ of $\psi(U_1)$. We consider both cases separately and show that each implies a contradiction.

**Case I: $X = \sigma_i^{-1} X_1$.**

Then $\theta_2(X) = 1 + \theta_2(X_1)$ so $\theta_2(X)$ is even precisely when $\theta_2(X_1)$ is odd. Thus

$$N(X) = N(\sigma_i^{-1} X_1) \sim \begin{cases} N(\sigma_i^{-1} X_1) & \text{if } \theta_2(X_1) \text{ is even,} \\ R(N(\sigma_i^{-1})) X_1 & \text{if } \theta_2(X_1) \text{ is odd,} \end{cases}$$

by (4.1), whence

$$N(X) \sim \begin{cases} \Delta \sigma_i N(X_1) & \text{if } \theta_2(X) \text{ is odd,} \\ \Delta \sigma_i N(X_1) & \text{if } \theta_2(X) \text{ is even,} \end{cases}$$

by Lemmas 4.2(1) and 2.3(1). This implies, by Corollary 2.2, that $\sigma_j [\sigma'_j] < N(X)$ if $\theta_2(X)$ is even [odd], which clearly contradicts (5.4).

**Case II: $X = \sigma_i X_1$.**

Since $\psi(U) = \psi(V)$ and $X \in \mathcal{M}_U$, Lemma 4.3 yields the existence of a word $Y$ in $\mathcal{M}_V$ such that

$$N(\sigma_i X_1) = N(X) \sim N(Y) \quad \text{and} \quad \theta_2(X) = \theta_2(Y) = k. \quad (5.5)$$

Noting $V = \tau_j V_1$, we deduce that $Y = \sigma_j^{\pm 1} Y_1$, for some word $Y_1$ in the support of $\psi(V_1)$. If $Y = \sigma_j Y_1$ then $\theta_2(X) = \theta_2(Y) = \theta_2(Y_1)$, by (5.5) which gives

$$N(X) \sim N(\sigma_j X_1) \sim \begin{cases} \sigma_j N(Y_1) & \text{if } \theta_2(X) \text{ is even,} \\ \sigma'_j N(Y_1) & \text{if } \theta_2(X) \text{ is odd,} \end{cases}$$

by (4.1), and contradicts (5.4). Hence $Y = \sigma_j^{-1} Y_1$. Observe that the word $\sigma_j Y = \sigma_j \sigma_j^{-1} Y_1$ is not minimal, by Lemma 4.2(2), so

$$\Delta \sim N(\sigma_j Y) \sim \begin{cases} \sigma_j N(X) & \text{if } \theta_2(X) \text{ is even,} \\ \sigma'_j N(X) & \text{if } \theta_2(X) \text{ is odd,} \end{cases}$$

by (4.1) and (5.5). Recalling $m_{ij} \geq 3$ and the word $X = \sigma_i X_1$ is minimal, we thus deduce, by Lemma 5.3 that the word $\sigma_i^{-1} X_1$ is also minimal. Since $U = \tau_1 U_1$, and $X_1$ is a summand of $\psi(U_1)$ this implies that $\sigma_i^{-1} X_1 \in \mathcal{M}_U$. Put $F_1 = \sigma_i^{-1} X_1$. Then

$$\theta_2(F_1) = 1 + \theta_2(X_1) = 1 + \theta_2(\sigma_i X_1) = 1 + \theta_2(X) = 1 + k.$$
and

\[ N(F_1) \sim \begin{cases} \Delta_{\sigma'_i} N(X_1) & \text{if } \theta_2(X_1) \text{ is even}, \\ \Delta_{\sigma_i} N(X_1) & \text{if } \theta_2(X_1) \text{ is odd}, \end{cases} \]

by (4.1), Lemmas 4.2(1) and 2.3(1). Since \( F_1 \) is an element of \( \mathcal{M}_U \), Lemma 4.3 yields the existence of a word \( G_1 \) in \( \mathcal{M}_V \) such that \( F_1 \approx G_1 \),

\[ N(F_1) \sim N(G_1) \quad \text{and} \quad \theta_2(G_1) = \theta_2(F_1) = 1 + k. \] (5.6)

Noting \( V = \tau_j V_1 \) we deduce that \( G_1 = \sigma_j^\pm Y_2 \) for some summand \( Y_2 \) of \( \psi(V_1) \). Assuming \( G_1 = \sigma_j^{-1} Y_2 \) contradicts that \( F_1 \) is minimal, by (5.6) and Lemma 4.1(2). Thus \( G_1 = \sigma_j Y_2 \) so

\[ N(\sigma_j^{-1} X_1) = N(F_1) \sim N(G_1) = N(\sigma_j Y_2) \] (5.7)

and

\[ 1 + k = \theta_2(F_1) = \theta_2(G_1) = \theta_2(\sigma_j Y_2) \] (5.8)

by (5.6). Observe that, By Lemma 4.2(2), the word \( \sigma_i F_1 = \sigma_i \sigma_j^{-1} X_1 \) is not minimal so

\[ \Delta \prec N(\sigma_i F_1) \sim \begin{cases} \sigma_i N(G_1) & \text{if } \theta_2(G_1) \text{ is even}, \\ \sigma'_i N(G_1) & \text{if } \theta_2(G_1) \text{ is odd}, \end{cases} \]

\[ \sim N(\sigma_i G_1) = N(\sigma_i \sigma_j Y_2), \]

by (4.1), (5.7) and (5.8). Noting \( m_{ij} \geq 3 \) and the word \( G_1 = \sigma_j Y_2 \) is an element of \( \mathcal{M}_V \), we deduce, by Lemma 5.3, that the word \( \sigma_j^{-1} Y_2 \) is also minimal. Since \( V = \tau_j V_1 \) and \( Y_2 \) is a summand of \( \psi(V_1) \), this implies that \( \sigma_j^{-1} Y_2 \) must lie in \( \mathcal{M}_V \). Put \( G_2 = \sigma_j^{-1} Y_2 \). Then (5.8) gives

\[ \theta_2(G_2) = \theta_2(\sigma_j^{-1} Y_2) = 1 + \theta_2(Y_2) = 2 + k \] (5.9)

and, since \( G_2 \) is minimal we obtain

\[ \Delta \not\prec N(G_2) = N(\sigma_j^{-1} Y_2) \sim \begin{cases} \Delta_{\sigma'_i} N(Y_2) & \text{if } \theta_2(Y_2) \text{ is even}, \\ \Delta_{\sigma_j} N(Y_2) & \text{if } \theta_2(Y_2) \text{ is odd}, \end{cases} \]

by (4.1), Lemmas 4.2(1) and 2.3(1). This implies, by Corollary 2.2, that

\[ \sigma_j [\sigma'_i] \not\prec N(G_2) \quad \text{if } \theta_2(G_2) \text{ is even [odd]} \] (5.10)

since \( \theta_2(G_2) \) is even if and only if \( \theta_2(Y_2) \) is odd by (5.9). Recalling \( G_2 \in \mathcal{M}_V \), a final application of Lemma 4.3 yields the existence of a word \( F_2 \) in \( \mathcal{M}_U \) such that

\[ N(F_2) \sim N(G_2) \quad \text{and} \quad \theta_2(F_2) = \theta_2(G_2). \]

Thus \( \theta_2(F_2) = 2 + k > k = \theta_2(X) \), by (5.9), and

\[ \sigma_j [\sigma'_i] \not\prec N(F_2) \quad \text{if } \theta_2(F_2) \text{ is even [odd]}, \]
Thus I putting Lemma 3.1(2), for some words the proposition. Recall also that the singular braid monoid on \( n + 1 \) strings, \( SB_{n+1} \), is the singular Artin monoid of type \( A_n \); the special case obtained when \( I = \{1, 2, \ldots, n\} \), \( m_{ij} = 3 \) when \(|i - j| = 1\) and \( m_{ij} = 2 \) whenever \(|i - j| \geq 2\). The singular Artin monoid of type \( I_2(p) \) is the special case when \( I = \{1, 2\} \) and \( m_{12} = p \geq 3 \). Thus if \( p = 3 \), types \( A_2 \) and \( I_2(3) \) coincide; the singular braid monoid on three strings, \( SB_3 \), is also the singular Artin monoid of type \( I_2(3) \). Both types \( A_n \) and \( I_2(p) \) are finite (see, for example, [Hum, Chapter 2]).

For any \( i, j \in I \) such that \( m_{ij} \geq 3 \), let \( T_{ij} \) denote the monoid generated by \( S \cup S^{-1} \cup \{\tau_i, \tau_j\} \) subject to the same defining relations as \( SG_M \). Let \( T_{ij}^+ \) denote the set of equivalence classes of words in \((S \cup \{\tau_i, \tau_j\})^* \) under \( \sim \). Then \( T_{ij} \) and \( T_{ij}^+ \) are both submonoids of \( SG_M \) and \( SG_M^+ \) respectively.

**Proposition 5.2.** The restriction of \( \eta \) from \( T_{ij} \) to the group algebra \( \mathbb{Z}G_M \) is injective. In particular the Vassiliev homomorphism from \( \eta : SG_{I_2(p)} \longrightarrow \mathbb{Z}G_I \) is faithful.

**Proof:** By Lemma 3.1(1) it suffices to prove the result for \( \psi \). We first prove the result for the positive monoid \( T_{ij}^+ \). Suppose that \( U, V \) in \((S \cup \{\tau_i, \tau_j\})^* \) provide a counterexample. That is, assume that \( U \not\prec V \) but \( \psi(U) = \psi(V) \). Suppose further that this counterexample is minimal with respect to \( \ell(U) \) which, by Lemma 3.2, is the same as \( \ell(V) \). Clearly \( \ell(U) \geq 2 \). If \( U \sim CV', V \sim CV' \) for some non-empty word \( C \) then \( \psi(U') = \psi(V') \), by Lemma 3.1(2), \( U' \not\prec V' \), \( \ell(U') < \ell(U) \) and hence the minimality of \( \ell(U) \) is contradicted. Thus \( U, V \) have no common divisor from which we infer, by Corollary 4.2, that \( U \) and \( V \) are not divisible by any generator from \( S \). This tells us that \( U = \tau_r U_1 \) and \( V = \tau_s V_1 \) for some words \( U_1 \) and \( V_1 \) in \( T_{ij}^+ \) and generators \( \tau_r, \tau_s \in \{\tau_i, \tau_j\} \). Noting \( m_{ij} \geq 3 \) we deduce by Proposition 5.1 that \( r = s \); this shows that \( \tau_r \) is a common divisor of \( U \) and \( V \) contradicting that \( \gcd(U, V) = 1 \). The result thus holds for \( T_{ij}^+ \).

Observe that \( \zeta^{-\theta_2(W)} \theta_1(W) \approx W \) for any word \( W \) in \( T_{ij} \) and \( \theta_1(W) \) is an element of \( T_{ij}^+ \). The result for \( T_{ij} \) thus follows by an argument identical to that of Observation 1. Putting \( I = \{1, 2\} \), \( m_{12} = p \geq 3 \) gives \( SG_{I_2(p)} = T_{12} \) and proves the second statement of the proposition.

**References**


