PD₄-complexes with strongly minimal models

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ABSTRACT

Let $X$ be a PD₄-complex with fundamental group $\pi$. We give conditions on the algebraic 2-type of $X$ under which the homotopy type of $X$ is determined by $\pi$, $w = w_1(X)$, the image of $[X]$ in $H_4(\pi; \mathbb{Z}^w)$ and the equivariant intersection pairing on $\pi_2(X)$. In particular, the homotopy type of an oriented Spin 4-manifold with fundamental group a PD₂-group $\pi$ is determined by $\pi$ and this pairing.

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In this paper we shall provide further evidence for the view that the homotopy type of a PD₄-complex should be largely determined by its algebraic 2-type and Poincaré duality [Pl]. This idea is made precise in [HK], where it is shown that the homotopy type of an oriented PD₄-complex $X$ with given Postnikov 2-stage $f : X \to P$ is determined by the image of $[X]$ in $H_4(P; \mathbb{Z})$, provided $H_2(X; \mathbb{Q}) \neq 0$. It is then shown that if $\pi = \pi_1(X)$ is finite and of cohomological period dividing 4 then $f_*[X]$ is in turn determined by the equivariant intersection form on the $\mathbb{Z}[\pi]$-module $\pi_2(X)$, which is the principal manifestation of Poincaré duality in this dimension. Our main result extends the latter result to certain cases with infinite fundamental group, under further hypotheses on the algebraic 2-type.

A PD₄-complex $Z$ with fundamental group $\pi$ is a model for a PD₄-complex $X$ if there is a 2-connected degree-1 map $f : X \to Z$, and is strongly minimal if $\text{Hom}_{\mathbb{Z}[\pi]}(\pi_2(Z), \mathbb{Z}[\pi]) = 0$. In §1 we show that there is a strongly minimal model for $X$ if and only if $\text{Hom}_{\mathbb{Z}[\pi]}(\pi_2(X), \mathbb{Z}[\pi])$ is a projective $\mathbb{Z}[\pi]$-module.
and $H^3(\pi; \mathbb{Z}[\pi]) = 0$, and that under the latter assumption duality defines a nonsingular $w$-hermitean pairing $\lambda_X$ on $\text{Hom}_{\mathbb{Z}[\pi]}(\pi_2(X), \mathbb{Z}[\pi])$, where $w = w_1(X)$. In §2 we show that if $\pi$ has finitely many ends $X$ has a model $Z$ with $\pi_2(Z) = 0$ if and only if either $\pi = 1$ or $\pi = \mathbb{Z}/2\mathbb{Z}$ and $w^4 \neq 0$ or $\pi$ has two ends and every finite subgroup of $\pi$ has cohomological period dividing 4 or if $\pi$ is a $PD_4$-group, $w = w_1(\pi)$ and $\pi_2(X)$ is a projective $\mathbb{Z}[\pi]$-module.

The main result is in §3, where we show that if $X$ has a strongly minimal model $Z$, $\pi$ has no 2-torsion and $H^3(\pi; \pi_2(X)) = 0$ the homotopy type of $X$ is determined by $Z$ and $\lambda_X$. Several applications of these results are given in §4. If $c.d.\pi \leq 2$ then $X$ always has a strongly minimal model, and the other conditions clearly hold, while every nonsingular $w$-hermitean pairing is realized by some $PD_4$-complex. The model $Z$ is determined by $\pi$ and $w$ if $\pi$ is free, and by $\pi$, $w$ and $w_2(X)$ if $\pi$ is a $PD_2$-group [Hi1,2]. If $\pi$ is a $PD_4$-group then $Z = K(\pi, 1)$ is the unique strongly minimal model, but it is only a model for $X$ if $w = w_1(\pi)$ and $\pi_2(X)$ is projective. The minimal model is not unique if $\pi \cong (\mathbb{Z}/n\mathbb{Z}) \rtimes \mathbb{Z}$.

§1. Intersection pairings

Let $w : \pi \to \{\pm 1\}$ be a homomorphism, and define an involution on $\mathbb{Z}[\pi]$ by $\bar{g} = w(g)g^{-1}$, for all $g \in \pi$. Let $Z$ and $Z^w$ be the augmentation and $w$-twisted augmentation modules, and $\varepsilon : Z[\pi] \to Z$ and $\varepsilon_w : Z[\pi] \to Z^w$ be the augmentation and the $w$-twisted augmentation, defined by $\varepsilon(g) = 1$ and $\varepsilon_w(g) = w(g)$, for all $g \in \pi$, respectively. If $R$ is a right $\mathbb{Z}[\pi]$-module let $\overline{R}$ be the corresponding left $\mathbb{Z}[\pi]$-module with the conjugate structure given by $g.r = r.\bar{g}$, for all $g \in \mathbb{Z}[\pi]$ and $r \in R$. If $L$ is a left $\mathbb{Z}[\pi]$-module let $L^\dagger = \text{Hom}_{\mathbb{Z}[\pi]}(L, \mathbb{Z}[\pi])$ be the conjugate dual module, and let $E^iL = \text{Ext}_{\mathbb{Z}[\pi]}^i(L, \mathbb{Z}[\pi])$, for $i \geq 0$. Thus $L^\dagger = E^0L$, while $E^i\mathbb{Z} = \overline{H^i(\pi; \mathbb{Z}[\pi])}$, for $i \geq 0$. If $L$ is free, stably free or projective then so is $L^\dagger$.

If $S$ is a topological space with fundamental group $\pi$ let $\tilde{S}$ be its universal covering space, $c_S : S \to K(\pi, 1)$ the classifying map, and $f_S : S \to P_2(S)$ be the second stage of the Postnikov tower.

Let $X$ be a $PD_4$-complex with fundamental group $\pi$ and $w_1(X) = w$, and let $C_* = C_*(X; \mathbb{Z}[\pi])$ be the equivariant cellular chain complex of $\tilde{X}$. This is a complex of left $\mathbb{Z}[\pi]$-modules, and is equivariantly chain homo-
topology equivalent to a finitely generated complex of projective modules, since $X$ is finitely dominated. Let $\Pi = \pi_2(X) \cong H_2(\tilde{X};\mathbb{Z}[\pi]) = H_2(C)$ and $H = H^2(\tilde{X};\mathbb{Z}[\pi]) = H^2(C)$, where the dual cochain complex is given by $C^q = \text{Hom}_{\mathbb{Z}[\pi]}(C_q,\mathbb{Z}[\pi])$, for all $q \geq 0$. Let $ev : H \to \Pi^!$ be the evaluation homomorphism, given by $ev([x])([z]) = [x] \cap [z] = c(z)$ for all 2-cycles $z \in C_2$ and 2-cocycles $c \in C^2$. This homomorphism sits in an exact sequence $0 \to E^2\mathbb{Z} \to H \xrightarrow{ev} \Pi^! \to E^3\mathbb{Z} \to 0$, by Lemma 3.3 of [Hi].

A choice of generator $[X]$ for $H_4(X;\mathbb{Z}^\omega) \cong \mathbb{Z}$ determines a Poincaré duality isomorphism $D : H \to \Pi$ by $D(u) = u \cap [X]$, for all $u \in H$. In particular, if there is a strongly minimal $PD_4$-complex $Z$ with fundamental group $\pi$ then $\pi_2(Z) \cong E^2\mathbb{Z}$ and $E^3\mathbb{Z} = 0$. The cohomology intersection pairing $\lambda : H \times H \to \mathbb{Z}[\pi]$ is defined by $\lambda(u,v) = ev(v)(D(u))$, for all $u,v \in H$. This pairing is $\pi$-hermitian: $\lambda(gu,v) = g\lambda(u,v)h$ and $\lambda(v,u) = \overline{\lambda(u,v)}$ for all $u,v \in H$ and $g,h \in \pi$. It is equivalent under Poincaré duality to the equivariant intersection pairing on $\pi_2(X)$. (See Chapter 4 of [Ra].) Since $\lambda(e,v) = 0$ for all $e \in E \cong E^2\mathbb{Z}$ and $v \in H$ the pairing $\lambda$ induces a pairing $\lambda_X : H/E \times H/E \to \mathbb{Z}[\pi]$. The adjoint homomorphism $\lambda_X : H/E \to (H/E)^!$ is given by $\lambda_X([v])([u]) = \lambda(u,v) = ev(v)(D(u))$, for all $u,v \in H$. It is a monomorphism, and $\lambda_X$ is nonsingular if $\lambda_X$ is an isomorphism.

If $Y$ is a second $PD_4$-complex we write $\lambda_X \cong \lambda_Y$ if there is an isomorphism $\theta : \pi \cong \pi_1(Y)$ such that $w_1(X) = w_1(Y)\theta$ and a $\mathbb{Z}[\pi]$-module isomorphism $\Theta : \pi_2(X) \cong \theta^*\pi_2(Y)$ inducing an isometry of cohomology intersection pairings. If $f : X \to Y$ is a 2-connected degree-1 map the “surgery kernel” $K_2(f) = \text{Ker}(\pi_2(f))$ and “surgery cokernel” $K^2(f) = \text{Cok}(H^2(f;\mathbb{Z}[\pi]))$ are finitely generated and projective, and are stably free if $X$ and $Y$ are finite complexes, by Lemma 2.2 of [Wl]. Moreover cap product with $[X]$ induces an isomorphism from $K^2(f)$ to $K_2(f)$. The pairing $\lambda_f = \lambda|_{K^2(f) \times K^2(f)}$ is nonsingular, by Theorem 5.2 of [Wl]. Thus if $f : X \to Z$ is a strongly minimal model $\lambda_f \cong \lambda_X$.

**Theorem 1.** Let $X$ be a $PD_4$-complex with fundamental group $\pi$, and let $E = E^2\mathbb{Z}$. Suppose that $E^3\mathbb{Z} = 0$. Then

(i) $H/E \cong \Pi^!$ and $\lambda_X$ is nonsingular;

(ii) if $\Pi^!$ is a projective $\mathbb{Z}[\pi]$-module then $E^3 = 0$;
(iii) if $\Pi$ is a projective $\mathbb{Z}[\pi]$-module then $E = 0$.

**Proof.** Since $E^3\mathbb{Z} = 0$ the Poincaré duality isomorphism $D$ induces an isomorphism $\gamma : H/E = \Pi \rightarrow M = \Pi / D(E)$, where $E = E^2\mathbb{Z}$. Let $p : \Pi \rightarrow M$ be the canonical epimorphism. Since $ev$ is an epimorphism $\eta(D(e)) = 0$ for all $\eta \in \Pi$ and $e \in E$. Hence $p^\dagger : M^\dagger \rightarrow \Pi^\dagger$ is an isomorphism and $\lambda_X p^\dagger = \gamma^\dagger$.

Since $\gamma$ and $p$ are isomorphisms so is $\lambda_X$.

If $\Pi^\dagger$ is projective then $\Pi \cong M \oplus D(E)$. Since $ev$ is an epimorphism and $ev(u)(D(e)) = ev(e)(D(u)) = 0$ for all $u \in H$ and $e \in E$ it follows that $D(E)^\dagger = 0$, and so $E^\dagger = 0$.

If $\Pi$ is projective and $E^3\mathbb{Z} = 0$ then $E$ is projective, since $\Pi \cong H \cong E \oplus \Pi^\dagger$, and so $E \cong E^\dagger = 0$. □

If $X$ is a finite $PD_4$-complex and $\pi_1(X)$ is a $PD_3$-group then $\Pi$ is stably isomorphic to the augmentation ideal, by Theorem 3.13 of [Hi], and so $\Pi^\dagger$ is stably free. However $E^3\mathbb{Z} \cong \mathbb{Z} \neq 0$. Note also that if $\lambda_X$ is nonsingular and $E^2\mathbb{Z} = 0$ then $E^3\mathbb{Z} = 0$.

**Corollary.** The $PD_4$-complex $X$ has a strongly minimal model if and only if $\Pi^\dagger$ is a finitely generated projective $\mathbb{Z}[\pi]$-module and $(E^3\mathbb{Z})^\dagger = E^3\mathbb{Z} = 0$.

**Proof.** If $f : X \rightarrow Z$ is a 2-connected degree-1 map then $K_2(f)$ is a finitely generated projective direct summand of $\Pi$, and if $Z$ is strongly minimal $(E^2\mathbb{Z})^\dagger = E^3\mathbb{Z} = 0$. Therefore $\Pi^\dagger = K_2(f)^\dagger$ is also projective, and $E^\dagger = 0$, by the theorem, and so the conditions are necessary. If they hold $\Pi \cong E \oplus \Pi^\dagger$, and the argument of Theorem 1 of [Hi2] gives a 2-connected degree-1 map $f : X \rightarrow Z$ with $\pi_2(Z) \cong E$. □

§2. $PD_4$-complexes with $\pi_2 = 0$

Let $Z$ be a $PD_4$-complex with fundamental group $\pi$. Then $Z$ is strongly minimal if and only if $\pi_2(Z) \cong E^2\mathbb{Z}$, and $(E^2\mathbb{Z})^\dagger = E^3\mathbb{Z} = 0$. In particular, $Z$ is strongly minimal if $\pi_2(Z) = 0$. Poincaré duality then gives $\pi_3(Z) \cong E^1\mathbb{Z}$, while $E^2\mathbb{Z} = E^3\mathbb{Z} = 0$, and the homotopy type of $Z$ is determined by $\pi$, $w$ and the first nontrivial $k$-invariant $\kappa \in H^4(\pi; \pi_3(Z))$. If moreover $E^1\mathbb{Z}$ is finitely generated it is 0 or $\mathbb{Z}$, and so $\tilde{Z}$ is homotopy equivalent to $S^4$ or $S^3$ or is contractible. Hence $\pi = 1$ or $Z/2Z$, has two ends or is a $PD_4$-group. (Taking
connected sums of such complexes gives examples with $E^1Z$ of infinite rank.) A $PD_4$-complex $X$ with fundamental group $\pi$ has a model $Z$ with $\pi_2(Z) = 0$ if and only if $E^3Z = 0$ and $\Pi$ is a finitely generated projective $\mathbb{Z}[\pi]$-module, by the argument of Theorem 1 of [Hi2].

If $\pi$ is finite and $\pi_2(Z) = 0$ then $Z \simeq S^4$ or $RP^4$. (See Lemma 12.1 of [Hi].) Every orientable $PD_n$-complex admits a degree-1 map to $S^n$.

**Lemma 2.** Let $X$ be a $PD_4$-complex with $\pi_1(X) = Z/2Z$ and let $w = w_1(X)$. Then $RP^4$ is a model for $X$ if and only if $w^4 \neq 0$.

**Proof.** The condition is clearly necessary. Conversely, we may assume that $X = X_0 \cup e^4$ is obtained by attaching a single 4-cell to a 3-complex $X_0$, by Lemma 2.9 of [Wi]. The map $c_X : X \to RP^\infty = K(Z/2Z, 1)$ factors through a map $f : X \to RP^4$, and $w = f^*w_1(RP^4)$, since $w \neq 0$. The degree of $f$ is well-defined up to sign, and is odd since $w^4 \neq 0$. We may arrange that $f$ is a degree-1 map, after modifying $f$ on a disc, if necessary. \(\square\)

The two $RP^2$-bundles over $S^2$ provide contrasting examples. If $X = S^2 \times RP^2$ then $w^3 = 0$ and $\Pi \cong \mathbb{Z} \oplus \mathbb{Z}^w$, which is not projective. On the other hand, if $X$ is the nontrivial bundle space then $w^4 \neq 0$ and $\Pi \cong Z[Z/2Z]$.

If $\pi$ has two ends and $\pi_2(Z) = 0$ then $\pi$ is an extension of $Z$ or $D = Z/2Z \times Z/2Z$ by a finite normal subgroup and $Z \simeq S^3$. Hence finite subgroups of $\pi$ have cohomological period dividing 4. (See Chapter 11 of [Hi].) We shall show that any $PD_4$-complex $X$ with $\pi_1(X) \cong \pi$ has a strongly minimal model. It is convenient to use the following notation. If $R$ is a noetherian ring and $M$ is a finitely generated $R$-module let $\Omega^1M = \text{Ker}(\phi)$, where $\phi : R^m \to M$ is any epimorphism, and define $\Omega^kM$ for $k > 1$ by iteration, so that $\Omega^{n+1}M = \Omega^1\Omega^nM$. These modules are finitely generated, and are well-defined up to stabilization by direct sums with a free module, by Schanuel’s Lemma.

**Lemma 3.** Let $X$ be a $PD_4$-complex such that $\pi = \pi_1(X)$ has two ends. Then $X$ has a strongly minimal model $Z$ with $\pi_2(Z) = 0$ if and only if every finite subgroup of $\pi$ has cohomological period dividing 4.

**Proof.** If $\pi_2(Z) = 0$ then $Z \simeq S^3$, by Theorem 11.1 of [Hi], and so the condition is necessary. Conversely, since $\pi$ is virtually $Z$ the condition imples
that the Farrell cohomology of $\pi$ has period dividing 4 [Fa]. The chain complex $C_s$ for $\tilde{X}$ gives rise to three exact sequences:

$$0 \to Z_2 \to C_2 \to C_1 \to C_0 \to \mathbb{Z} \to 0.$$ 

$$0 \to Z_3 \to C_3 \to Z_2 \to \Pi \to 0$$

and

$$0 \to C_4 \to Z_4 \to E^1\mathbb{Z} \to 0.$$ 

Elementary manipulations show that $1\mathbb{Z}$ is stably isomorphic to $5\mathbb{Z}$, and so to $1\mathbb{Z}$, by periodicity. Therefore $\text{Ext}^q_{\mathbb{Z}[\pi]}(\Omega^1\mathbb{Z} \oplus \Omega^2\Pi, N) \cong \text{Ext}^q_{\mathbb{Z}[\pi]}(\Omega^1\mathbb{Z}, N)$, for all $q > 1$, and any $\mathbb{Z}[\pi]$-module $N$. If $N$ is finitely generated so is $\text{Ext}^q_{\mathbb{Z}[\pi]}(\Omega^1\mathbb{Z}, N)$, and so $\text{Ext}^{q+1}_j(\Pi, N) = \text{Ext}^q_{\mathbb{Z}[\pi]}(\Omega^2\Pi, N) = 0$, for all $q > 1$. Since $\Pi$ is finitely generated $\text{Ext}^q_{\mathbb{Z}[\pi]}(\Pi, -)$ commutes with direct limits and so is 0, for all $r > 3$. Therefore $\Pi$ has finite projective dimension. Now it follows easily from the UCSS and Poincaré duality that $\text{Ext}^s_{\mathbb{Z}[\pi]}(\Omega^2\Pi, \mathbb{Z}[\pi]) = 0$ for all $s \geq 1$. Hence $\Pi$ is projective, and so we may construct a 2-connected degree-1 map $f : X \to Z$ to a PD$_4$-complex $Z$ with $\pi_2(Z) = 0$, as in Theorem 1 of [Hi2].

It can be shown that if $\pi$ has one end and $\Pi$ is projective then $c.d.\pi = 4$ and $E^4\mathbb{Z} \cong \mathbb{Z}$. Does $\Pi$ projective imply $E^3\mathbb{Z} = 0$? If so then $\pi$ is a PD$_4$-group.

**Lemma 4.** Let $\pi$ be a PD$_4$-group. Then $K(\pi, 1)$ is the unique strongly minimal PD$_4$-complex with fundamental group $\pi$, and a PD$_4$-complex $X$ with $\pi_1(X) \cong \pi$ has a strongly minimal model if and only if $w_1(X) = w_1(\pi)$ and the $\mathbb{Z}[\pi]$-module $\Pi = \pi_2(X)$ is projective.

**Proof.** The PD$_4$-complex $K(\pi, 1)$ is clearly strongly minimal. If $f : X \to K(\pi, 1)$ is a 2-connected degree-1 map then $f = c_X$ and $\Pi = \text{Ker}(\pi_2(f))$ is projective [Wl]. Conversely, a strongly minimal PD$_4$-complex $Z$ with fundamental group $\pi$ is aspherical, since $E^i\mathbb{Z} = 0$ for $i \leq 3$. If $\Pi$ is projective the Universal Coefficient spectral sequence collapses to give an isomorphism $c^*_X : H^4(\pi; \mathbb{Z}[\pi]) \cong H^4(X; \mathbb{Z}[\pi])$. Poincaré duality for $X$ and $\pi$ now show that $c^*_X$ is a degree-1 map. □
The sufficiency of the condition also follows from the construction of Theorem 1 of [Hi2], for if $I$ is projective this gives a 2-connected degree-1 map $f : X \to Z$ to a $PD_4$-complex $Z$ with $\pi_2(Z) = 0$, which must be aspherical since $E^1Z = 0$.

Let $K$ be the 2-complex determined by a finite presentation of an orientable $PD_4$-group and let $X = \partial N$, where $N$ is a regular neighbourhood of an embedding of $K$ in $R^5$. Then $c_X$ factors through $N \simeq K$, and so has degree 0. Hence $X$ has no strongly minimal model, and $\pi_2(X)$ is not projective.

§3. The main result

If $A$ is an abelian group the universal quadratic functor of Whitehead $\gamma_A : A \mapsto \Gamma_W(A)$ determines a map $s$ from the symmetric product $A \circ A$ to $\Gamma_W(A)$ by $s(a \circ b) = \gamma(a + b) - \gamma(a) - \gamma(b)$, and there is an exact sequence

$$A \circ A \xrightarrow{s} \Gamma_W(A) \to A/2A \to 0,$$

where the right-hand map is induced by the projection of $A$ onto $A/2A$ (which is quadratic) [Ba]. If $A$ and $B$ are abelian groups the inclusions into $A \oplus B$ induce a canonical splitting $\Gamma_W(A \oplus B) \cong \Gamma_W(A) \oplus \Gamma_W(B) \oplus (A \oplus B)$. Since $\Gamma(Z) \cong Z$ it follows by a finite induction that if $A \cong Z^r$ then $\Gamma_W(Z^r)$ is again finitely generated and free, and that $s$ is injective. The latter conditions hold for $A$ any free abelian group, since every finitely generated subgroup of such a group lies in a finitely generated direct summand.

If $M$ is a finitely generated $Z[\pi]$-module let $Her_w(M^1)$ be the group of $w$-hermitean pairings on $M^1$. Let $ev(m)(n, n') = w(m)n'(m)$ for all $m \in M$ and $n, n' \in M^1$. Then $ev(m)(n, n')$ is $Z$-quadratic in $m$ and $w$-hermitean in $n$ and $n'$ and $ev(gm) = w(g)ev(m)$ for all $g \in \pi$ and $m \in M$. Hence $ev$ determines a homomorphism $B_M : Z^w \otimes_{Z[\pi]} \Gamma_W(M) \to Her_w(M^1)$.

Lemma 5. Let $\pi$ be a group, $w : \pi \to Z^r$ a homomorphism and $M$ a finitely generated projective $Z[\pi]$-module. If $\text{Ker}(w)$ has no element of order 2 then $B = B_M$ is surjective, while if there is no element $g \in \pi$ of order 2 such that $w(g) = -1$ then $B$ is injective.

Proof. Let $P$ be a projective complement to $M$, so that $M \oplus P \cong Z[\pi]^r$ for some $r \geq 0$. The inclusion of $M$ into the direct sum induces a split.
monomorphism from $\Gamma_W(M)$ to $\Gamma_W(\mathbb{Z}[\pi]')$ which is clearly compatible with $B$ and $B_{\mathbb{Z}[\pi]}'$. We may extend an hermitian form $h$ on $M \dagger$ to a form $\tilde{h}$ on $M \dagger \oplus P \dagger$ by setting $\tilde{h}(n,p) = \tilde{h}(p',p) = 0$ for all $n \in M \dagger$ and $p,p' \in P \dagger$. If $\tilde{h} = B(\theta)$ then $h = B(\theta_M)$, where $\theta_M$ is the image of $\theta$ under the homomorphism induced by the projection from $M \oplus P$ onto $M$. In this way we may easily reduce to the case when $M$ is a free module with basis $e_1, \ldots, e_r$. Let $e_1^*, \ldots, e_r^*$ be the dual basis for $M \dagger$, defined by $e_i^*(e_i) = 1$ and $e_i^*(e_j) = 0$ if $i \neq j$.

In particular, as $M$ is a free abelian group there is a short exact sequence

$$0 \rightarrow M \otimes \mathbb{Z} \rightarrow \Gamma_W(M) \rightarrow M/2M \rightarrow 0,$$

and $\Gamma_W(M)$ is free as an abelian group. This is a sequence of $\mathbb{Z}[\pi]$-modules and homomorphisms, if we define the action on $M \otimes \mathbb{Z}$ by $g(m \otimes n) = gm \otimes gn$, for all $g \in \pi$ and $m,n \in M$.

The sequence

$$0 \rightarrow \mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} M \otimes \mathbb{Z} \rightarrow \mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} \Gamma_W(M) \rightarrow \mathbb{F}_2 \otimes_{\mathbb{Z}[\pi]} M \rightarrow 0$$

is also exact, since $\text{Tor}_{\mathbb{Z}[\pi]}^2(\mathbb{Z}^w, M/2M) = \text{Ker}(2: \mathbb{Z}^w \otimes_{\mathbb{F}_2} M \rightarrow \mathbb{Z}^w \otimes_{\mathbb{F}_2} M) = 0$.

Let $\eta_M : M \rightarrow \mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} \Gamma_W(M)$ be the composite of $\gamma_M$ with the reduction from $\Gamma_W(M)$ to $\mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} \Gamma_W(M)$. Then the composite of $\eta_M$ with the projection to $\mathbb{F}_2 \otimes_{\mathbb{Z}[\pi]} M$ is the canonical epimorphism.

Since $m \circ gn = g(g^{-1}m \circ n) = \tilde{g}m \circ n$ in $\mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} (M \otimes \mathbb{Z})$, the typical element of $\mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} (M \otimes \mathbb{Z})$ may be expressed in the form $\mu = \sum_{i \leq j} (r_{ij}e_i) \circ e_j$.

For such an element $B(\mu)(e_i^*, e_j^*) = r_{kl}$, if $k < l$, and $= r_{kk} + \hat{r}_{kk}$, if $k = l$. In particular, $B(\mu)$ is even: if $\varepsilon : \mathbb{Z}[\pi] \rightarrow \mathbb{F}_2$ is the composite of the augmentation with reduction mod (2) then $\varepsilon_2(B(\mu)(n,n)) = 0$ for all $n \in M \dagger$.

If $m \in M$ has nontrivial image in $\mathbb{F}_2 \otimes_{\mathbb{Z}[\pi]} M$ then $\varepsilon_2(e_i^*(m)) \neq 0$ for some $i \leq r$. Hence $B(\eta_M(m))$ is not even, and it follows easily that $\text{Ker}(B) \leq \mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} (M \circ M)$. Suppose that $B(\mu) = 0$, for some $\mu = \sum_{i \leq j} (r_{ij}e_i) \circ e_j$.

Then $r_{kl} = 0$, if $k \neq l$, and $r_{ii} + \hat{r}_{ii} = 0$, for all $i$. Since $\pi$ has no orientation reversing element of order 2 we have $r_{ii} = \sum_{g \in F(i)} a_{ij} (g - \hat{g})$, where $F(i)$ is a finite subset of $\pi$, for $1 \leq i \leq r$. Since $(g - \hat{g})e_i \circ e_i = 0$ it follows easily that $\mu = \sum (r_{ii}e_i) \circ e_i = 0$. Hence $B$ is injective.

To show that $B$ is surjective it shall suffice to assume that $M$ has rank 1 or 2, since $h$ is determined by the values $h_{ij} = h(e_i^*, e_j^*)$. Let $\varepsilon_w[m, m']$
be the image of $m \circ m'$ in $\mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} \Gamma_W(M)$. Then $B(\varepsilon_w[m, m'])(n, n') = n(m)n'(m') + n(m')n(m)$, for all $m, m' \in M$ and $n, n' \in M^1$. Suppose first that $M$ has rank 1. Since $h_{11} = \tilde{h}_{11}$ and $\text{Ker}(w)$ has no element of order 2 we may write $h_{11} = 2b + \delta + \sum_{g \in F}(g + \tilde{g})$, where $b = \tilde{b}$, $\delta = 1$ or 0 and $F$ is a finite subset of $\pi$. Let $\mu = \varepsilon_w[(b + \delta + \sum_{g \in F}g)e_1, e_1] + \delta \eta_M(e_1)$. Then $B(\mu)(d_{ij}^1, d_{ij}^1) = h_{11}$. If $M$ has rank 2 and $h_{11} = h_{12} = 0$, let $\mu = \varepsilon_w[h_{12}e_1, e_2]$. Then $B(\mu)(d_{ij}^1, d_{ij}^1) = h_{ij}$. In each case $B(\mu) = h$, since each side of the equation is a $w$-hermitian form on $M^1$. \[ \Box \]

In particular, if $\pi$ has no 2-torsion then $B_M$ is an isomorphism, for any projective $\mathbb{Z}[\pi]$-module $M$.

**Lemma 6.** Let $M$ be a finitely generated projective $\mathbb{Z}[\pi]$-module and $\theta : M \to E$ be a $\mathbb{Z}[\pi]$-module homomorphism. Let $\alpha_\theta(m, e) = (m, e + \theta(m))$ for all $(m, e) \in \Pi = M \oplus E$. Then $\alpha_\theta$ is an automorphism of $\Pi$ and $\Gamma_W(\alpha_\theta)(\gamma) - \gamma = (B(\gamma) \otimes 1)(\theta) \mod \Gamma_W(E)$ for all $\gamma \in \mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} \Gamma_W(M)$.

**Proof.** The homomorphism $\alpha_\theta$ is clearly an automorphism of $\Pi$ which restricts to the identity on the summands $E$ and $M$, and $\Gamma_W(\alpha_\theta)(\gamma_M(m)) = \gamma_M(m) + \gamma_M(\theta(m)) + m \otimes \theta(m)$, for all $m \in M$. (See Chapter 1 of [Ba].)

Let $\beta_m = B(1 \otimes \gamma_M(m))$, for $m \in M$. Let $M^*$ be the right $\mathbb{Z}[\pi]$-module $\text{Hom}_{\mathbb{Z}[\pi]}(M, \mathbb{Z}[\pi])$ (so that $M^! = M^{op}$). Since $M$ is finitely generated and projective the functions $d : M \to M^{op}$ and $t : M^* \otimes_{\mathbb{Z}[\pi]} E \to \text{Hom}_{\mathbb{Z}[\pi]}(M, E)$ given by $d(m)(\mu) = \mu(m)$ and $t(\mu \otimes e)(m) = \mu(m)e$ are isomorphisms (of left $\mathbb{Z}[\pi]$-modules and abelian groups, respectively), for all $m \in M$, $\mu \in M^{op}$ and $e \in E$. Now $\beta_m(\mu) = \mu(m)d(m)$, which is $d(m)\mu(m)$ in $M^*$. Since $t$ is surjective we have $\check{\theta} = t(\Sigma \mu_i \otimes e_i)$, for some $\mu_i \in M^*$ and $e_i \in E$. Then $B_m \otimes 1)(t^{-1}(\check{\theta})) = \Sigma \beta_m(\mu_i) \otimes e_i = \Sigma d(m)\mu_i(m) \otimes e_i = d(m) \otimes \theta(m) = (d \otimes 1)(m \otimes \theta(m))$.

Since $\Gamma_W(\alpha_\theta)(\gamma_M(m)) = \gamma_M(m) \equiv \beta_m \otimes 1)(\theta) \mod \Gamma_W(E)$, for all $m \in M$, and each side is $Z$-quadratic in $m$, we have $\Gamma_W(\alpha_\theta)(\gamma) - \gamma \equiv (B(\gamma) \otimes 1)(\theta) \mod \Gamma_W(E)$, for all $\gamma \in \mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} \Gamma_W(M)$. \[ \Box \]

If $M$ is a $\mathbb{Z}[\pi]$-module let $L_\pi(M, 2)$ be the “generalized Eilenberg-Mac Lane space” with fundamental group $\pi$ and universal covering space $\simeq K(M, 2)$, and such that $c_{L_\pi(M, 2)}$ admits a section. The group of based self homotopy
Therefore assume that \( f = \text{composition} \) since \( H \) and then adjoining higher dimensional cells to kill the higher homotopy. Let \( \text{jective and } L \) Lemmas 4 and 5 of [Hi2]. Now \( H \) 2.2 of [Wl]. Then by adjoining 3-cells to \( L \) also determines a projection \( E \) where \( E = \text{projection onto the second factor} \). We may assume that \( \lambda_f = \lambda_g \). 

\textbf{Proof.} The conditions are clearly necessary. Suppose that they hold. Then \( M = K_2(gx) \cong K_2(gy) \) is projective and \( \pi_2(X) \cong \pi_2(Y) \cong \Pi = M \oplus E \), where \( E = \pi_2(Z) \) and the isomorphisms are chosen so that \( \pi_2(gx) \) and \( \pi_2(gy) \) correspond to projection onto the second factor. We may assume that \( M \neq 0 \), for otherwise \( gx \) and \( gy \) are homotopy equivalences.

We have \( P_2(X) \cong P_2(Y) \cong P = L_\pi(\Pi, 2) \), since \( H^3(\pi; \Pi) = 0 \). After composing \( f_Y \) with a self homotopy equivalence of \( P \), if necessary, we may assume that \( f_Z = gf_X \) and \( f_Zg_Y = gf_Y \) for some common 2-connected map \( g : P \to P_2(Z) \). The map \( g \) is a fibration with fibre \( K(M, 2) \), and the inclusion of \( E \) into \( \Pi = M \oplus E \) determines a section \( s \) for \( g \). The splitting \( \Pi = M \oplus E \) also determines a projection \( q : P \to L = L_\pi(M, 2) \). We may construct \( L \) by adjoining 3-cells to \( X \) to kill the kernel of projection from \( \Pi \) onto \( M \) and then adjoining higher dimensional cells to kill the higher homotopy. Let \( j : X \to L \) be the inclusion, and let \( X_0 \) and \( L_0 \) be the 3-skeletons of \( X \) and \( L \), respectively. Let \( \sigma : K(\pi, 1) \to L \) be a section for \( c_L \).

Since \( \hat{L} \cong K(M, 2) \) we have \( H_4(L; \mathbb{Z}[\pi]) \cong \Gamma_W(M) \), and since \( M \) is projective and \( c_L \sigma = id_{K(\pi, 1)} \) the Cartan-Leray spectral sequence for \( c_L \) gives \( H_4(L; \mathbb{Z}^w) = H_4(\pi; \mathbb{Z}^w) \oplus (\mathbb{Z}^w \otimes \mathbb{Z}[\pi]) \Gamma_W(M) \). Then \( [X] = j_*[X] - \sigma_c X_\pi[X] \) is an element of \( \mathbb{Z}^w \otimes \mathbb{Z}[\pi] \Gamma_W(M) \).

We may identify \( M^! \) with a direct summand of \( H^2(X; \mathbb{Z}[\pi]) \), by Lemma 2.2 of [Wl]. Then \( B_M([\mathbb{Z}][u, v]) = ev(u)(u \cap [X]) \) for all \( u, v \in M^! \), by Lemmas 4 and 5 of [Hi2]. Now \( u \cap \sigma_c X_\pi[X] = \sigma_c(\sigma^* u \cap c_X_\pi[X]) = 0 \), since \( H_2(\pi; \mathbb{Z}[\pi]) = 0 \), and so \( B_M([\mathbb{Z}][u, v]) = \lambda_f(u, v) \) for all \( u, v \in M^! \). Therefore \( f_{X_\pi}[X] \) and \( f_{Y_\pi}[Y] \) have image \( \lambda_f \) and \( \lambda_g \) in \( Her_w(M^!) \). After composing \( f_Y \) with a self homotopy equivalence of \( P \), if necessary, we may assume that \( \lambda_f = \lambda_g \).
Since $P_2(Z)$ is a retract of $P$ comparison of the Cartan-Leray spectral sequences for $c_P$ and $c_{P_2(Z)}$ shows that $\text{Cok}(H_4(s; \mathbb{Z}^w))$ is isomorphic to $H_0(\pi_2; H_4(K(\Pi, 2))/H_0(\pi; H_4(K(E, 2))) \cong \mathbb{Z}^w \otimes g_{\pi} \Gamma_{W}(\Pi)/\Gamma_{W}(E))$. Since $\pi$ has no orientation reversing element of order 2 the homomorphism $B_M$ is injective, by Lemma 5, and therefore since $\lambda_f = \lambda_g$ the images of $f_{X*}[X]$ and $f_{Y*}[Y]$ in $\mathbb{Z}^w \otimes g_{\pi} \Gamma_{W}(\Pi)/\Gamma_{W}(E))$ differ by an element of the subgroup $\mathbb{Z}^w \otimes g_{\pi} (M \otimes E)$. Using the nonsingularity of $\gamma = \lambda_f = \lambda_g$ and Lemma 6 we may choose a homomorphism $\theta : M \to E$ and hence a self homotopy equivalence $P(\theta)$ of $P$ such that $gP(\theta) = g$ and $P(\theta)*f_{Y*}[Y] = f_{X*}[X]$ mod $\Gamma_{W}(E)$. Since $g_{X*}[X] = g_{Y*}[Y]$ in $H_4(Z; \mathbb{Z}^w)$ and hence $(gf_{X})_*[X] = (gf_{Y})_*[Y]$ in $H_4(P_2(Z); \mathbb{Z}^w)$ it follows that $P(\theta)*f_{Y*}[Y] = f_{X*}[X]$ in $H_4(P, \mathbb{Z}^w)$.

There is then a map $h : X \to Y$ with $f_Y h = f_X$, by the argument of Lemma 1.3 of [HK]. Let $X^+$ and $Y^+$ be the orientable covering spaces corresponding to Ker$(w)$. Then $h$ lifts to a map $h^+ : X^+ \to Y^+$. Since $f_X$ and $f_Y$ are 3-connected $\pi_1(h^+)$, $\pi_2(h^+)$ and $H_3(h^+; \mathbb{Z})$ are isomorphisms. Since $M$ is projective and nonzero $\mathbb{Z} \otimes \text{Ker}(w)$ is a nontrivial torsion-free direct summand of $H_2(X^+; \mathbb{Z})$, and so $h^+$ has degree 1, by Poincaré duality with coefficients $\mathbb{Z}$. Hence $h^+$ is a homotopy equivalence, and therefore so is $h$. □

The condition on 2-torsion is in general necessary, for the intersection pairing is no longer a complete invariant when $w : \pi \to \mathbb{Z}^+$ is an isomorphism ([HKT] - see below). The hypothesis $H^3(\pi; \pi_2(X)) = 0$ is used to identify $P_2(X)$ and $P_2(Z)$ with $P$, and in the appeal to [R]; it is not clear that it is essential. Note that if $\pi_2(X)$ is a direct summand of $\mathbb{Z}[\pi]^+$ then $H^3(\pi; \pi_2(X))$ is a direct summand of $(E^3\mathbb{Z})^r$.

Corollary A. If $X$ has a strongly minimal model $Z$, $\pi$ has no 2-torsion and $H^3(\pi; \pi_2(X)) = 0$ the homotopy type of $X$ is determined by $Z$ and $\lambda_X$. □

Corollary B. [HRS] If $g : X \to Z$ is a 2-connected degree-1 map of $PD_3$-complexes with fundamental group $\pi$ such that $w_4(X)$ is trivial on elements of order 2 and $H^3(\pi; \pi_2(X)) = 0$ then $X$ is homotopy equivalent to $M \# Z$ with $M$ 1-connected if and only if $\lambda_g$ is extended from a nonsingular pairing over $\mathbb{Z}$. □
The result of [HRS] assumes that $X$ is orientable, $\pi$ is infinite and either $E^2\mathbb{Z} = 0$ or $\pi$ acts trivially on $\pi_2(\mathbb{Z})$. (Since $\pi$ is infinite the latter condition implies that $Z$ is strongly minimal.)

**Corollary C.** Let $\pi$ be a finitely presentable group with no 2-torsion and such that $E^2\mathbb{Z} = E^3\mathbb{Z} = 0$. Then two $PD_4$-complexes $X$ and $Y$ with fundamental group $\pi$, $w_1(X) = w_1(Y) = w$ and $\Pi = \pi_2(X) \cong \pi_2(Y)$ a nonzero projective $\mathbb{Z}[\pi]$-module are homotopy equivalent if and only if $c_{\pi^*}[X] = c_{\pi^*}[Y]$ in $H_4(\pi; \mathbb{Z}^w)$ and $\lambda_X \cong \lambda_Y$.

**Proof.** The hypotheses imply that $X$ and $Y$ have strongly minimal models $Z_X$ and $Z_Y$ with $\pi_2(Z_X) = \pi_2(Z_Y) = 0$, and hence $P_2(Z_X) \simeq P_2(Z_Y) \cong K(\pi, 1)$. Moreover $H^3(\pi; \Pi) = 0$, since $E^3\mathbb{Z} = 0$, and so the result follows by the argument of Theorem 7. (Note that it is not clear a priori that $X$ and $Y$ have a common minimal model $Z$.) \(\square\)

§4. Realization of pairings

In Theorem 6 of [Hi2] it is shown that if $\pi$ has one end and $c.d.\pi = 2$ every nonsingular $w$-hermitean pairing on a finitely generated projective $\mathbb{Z}[\pi]$-module is the intersection pairing $\lambda_X$ of some $PD_4$-complex $X$ with fundamental group $\pi$ and $w_1(X) = w$. However the argument on lines 5 and 6 of page 54 of [Hi2] purporting to show that the attaching map $\phi$ of the top cell of $X$ defines an element $[\phi]$ in $\mathbb{Z}^w \otimes \Gamma_W(M)$ is inadequate. (The error was in assuming that the natural homomorphism from $\mathbb{Z}^w \otimes_\mathbb{Z}[\pi] H_3(X; \mathbb{Z}[\pi])$ to $H_3(X; \mathbb{Z}^w)$ is a monomorphism).

Here we shall provide an argument which assumes only that $c.d.\pi \leq 2$ and does not require that $\pi$ have one end. Let $M$ be a projective $\mathbb{Z}[\pi]$-module, let $L_0$ be the 3-skeleton of $L = L_\pi(M, 2)$ and let $Z_i$ be the submodule of $i$-cycles in $C_i = C_i(L_0; \mathbb{Z}[\pi])$. If $c.d.\pi \leq 2$ then $Z_2$ is a direct summand of $C_2$, by Schanuel’s Lemma. (Compare the first exact sequence of Lemma 3 above.) Hence $H_3(L_0; \mathbb{Z}[\pi])$ is a direct summand of $C_3$ and so is also projective. Moreover $\mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} H_3(L_0; \mathbb{Z}[\pi]) \cong H_3(L_0; \mathbb{Z}^w)$, since $M = H_2(L; \mathbb{Z}[\pi])$ is projective and $H_3(\pi; \mathbb{Z}^w) = H_4(\pi; \mathbb{Z}^w) = 0$. Hence there is an exact sequence

$$0 \to \mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} \Gamma_W(M) \to \mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} \pi_3(L_0) \to H_3(L_0; \mathbb{Z}^w) \to 0.$$
Therefore if $\phi \in \pi_3(X_o)$ has image 0 in $H_3(X_o; \mathbb{Z}^w)$ and $j : X_o \to L_o$ is the inclusion (as described in [Hi2], or in Theorem 7 above) $[\phi] = (\varepsilon_w \otimes 1)(j\phi)$ is in $\mathbb{Z}^w \otimes \mathbb{Z}[\varepsilon] \Gamma_W(M)$.

**Theorem 8.** Let $Z$ be a strongly minimal $PD_4$-complex with fundamental group $\pi$ such that $c.d.\pi \leq 2$ and let $w = w_1(Z)$. Then every nonsingular $w$-hermitean pairing on a finitely generated projective $\mathbb{Z}[\pi]$-module is realized by some $PD_4$-complex with minimal model $Z$.

**Proof.** Let $N$ be a finitely generated projective $\mathbb{Z}[\pi]$-module and $\Lambda$ be a nonsingular $w$-hermitean pairing on $N^\dagger$. Suppose $N \oplus F_1 \cong F_2$, where $F_1$ and $F_2$ are free $\mathbb{Z}[\pi]$-modules with countable bases $I$ and $J$, respectively. (These may be assumed finite if $N$ is stably free.) We may assume $Z = Z_o \cup_\varepsilon e^4$ is obtained by attaching a single 4-cell to a 3-complex $Z_o$, by Lemma 2.9 of [Wl]. Construct a 3-complex $X_o$ with $\pi_2(X_o) \cong \pi_2(Z_o) \oplus N$ by attaching $J$ 3-cells to $Z_o \vee (\vee^1 S^2)$, along sums of translates under $\pi$ of the 2-spheres in $\vee^1 S^2$, as in Theorem 1 of [Hi2]. Let $i : Z_o \to X_o$ be the natural inclusion. Collapsing $\vee^1 S^2$ gives $X_o/\vee^1 S^2 \cong Z_o \vee (\vee^1 S^3)$, and so there is a retraction $q : X_o \to Z_o$. Let $j : X_o \to L = L_x(N; 2)$ be the map corresponding to the projection of $\pi_2(X_o)$ onto $N$. Then $\pi_2(ji) = 0$ and so $ji$ factors through $K(\pi, 1)$. The map $B_N : \mathbb{Z}^w \otimes \mathbb{Z}[\varepsilon] \Gamma_W(N) \to Her_w(N^\dagger)$ is an isomorphism, by Lemma 5. Therefore we may choose $\psi \in \Gamma_W(M) \leq \pi_3(X_o)$ so that $B_N([\psi]) = \Lambda$. Let $\phi = \psi - iq\psi + i\theta$. Then $j\phi = j\psi$ and $q\phi = \theta$, and so $X = Z_o \cup_\psi D^4$ is a $PD_4$-complex with $\lambda_X \cong \Lambda$, by part (ii) of Theorem 6 of [Hi2].

Every pair $(\pi, w)$ with $\pi$ finitely presentable is realized by a closed 4-manifold, and hence by a strongly minimal $PD_4$-complex, if also $c.d.\pi \leq 2$.

**§5. Applications**

We shall first return briefly to the cases considered in §2.

It is well known that the (oriented) homotopy type of a 1-connected $PD_4$-complex is determined by its intersection pairing and that every such pairing is realized by some 1-connected topological 4-manifold [FQ]. The argument for Theorem 6 breaks down when $\pi = \mathbb{Z}/2\mathbb{Z}$ and $w$ is nontrivial, for then the homomorphism $B : \mathbb{Z}^w \otimes \mathbb{Z}[\varepsilon] \Gamma_W(M) \to Her_w(M^\dagger)$ is no longer injective.
However nonorientable topological 4-manifolds with fundamental group \( Z/2Z \) are classified up to homeomorphism in [HKT], and it is shown there that the homotopy types are determined by the Euler characteristic, \( w^4 \), the “\( w_2 \)-type” and an Arf invariant (for \( w_2 \)-type III). The authors remark that their methods show that \( X \) together with a quadratic enhancement \( q : \Pi \to Z/4Z \) due to [KKR] is also a complete invariant for the homotopy type of such a manifold.

If \( \pi \) has two ends, no 2-torsion and its finite subgroups have cohomological period dividing 4 then \( \pi \cong (Z/nZ) \rtimes \mathbb{Z} \) for some odd \( n \). Any generator of \( H^4(\pi; \mathbb{Z}^w) \cong Z/nZ \) can occur as the \( k \)-invariant of a \( PD_4 \)-complex \( Z \) with \( \pi_1(Z) \cong \pi \) and \( \pi_2(Z) = 0 \). (See Theorem 11.1 of [Hi].) However Theorem 7 implies that, for instance, if \( L \) and \( L' \) are any two lens spaces with fundamental group \( Z/nZ \) then \( (L \times S^1)\# CP^2 \cong (L' \times S^1)\# CP^2 \). Thus \( (L \times S^1)\# CP^2 \) has both \( L \times S^1 \) and \( L' \times S^1 \) as minimal models.

If \( \pi \) is a \( PD_4^+ \)-group then \( c_X \) has degree 1 if and only if \( k_1(X) = 0 \) [CH]. (It is assumed there that \( \pi \) and \( X \) are orientable, but the argument needs only that \( w_1(X) = w_1(\pi) \).) Thus Theorem 7 gives an alternative proof of the main result of [CH], namely that a \( PD_4 \)-complex \( X \) with fundamental group \( \pi \) a \( PD_4 \)-group and \( w_1(X) = w_1(\pi) \) is homotopy equivalent to \( M\# K(\pi, 1) \) with \( M \) 1-connected if and only if \( k_1(X) = 0 \) and \( \lambda_X \) is extended from a nonsingular pairing over \( Z \).

We consider finally two other cases. If \( r > 1 \) the free group \( F(r) \) has infinitely many ends. The manifolds \( \#^r(S^1 \times S^3) \) and \( (S^1 \times S^3)\#(\#^{r-1}(S^1 \times S^3)) \) have \( \pi_2 = 0 \), and every strongly minimal \( PD_4 \)-complex with free fundamental group is homotopy equivalent to one of these. In [Hi1] we showed that the homotopy type of a \( PD_4 \)-complex \( X \) with \( \pi \cong F(r) \) is determined by \( r, w \) and \( \lambda_X \), and that every nonsingular \( \pi \)-hermitean pairing on a finitely generated free \( \mathbb{Z}[F(r)] \)-module is realized by some such \( PD_4 \)-complex.

Suppose now that \( \pi \) is finitely presentable and \( c.d.\pi = 2 \). Then \( X \) has a strongly minimal model \( Z \) with \( \pi_2(Z) = E^2\mathbb{Z} \neq 0 \) [Hi2].

**Theorem 9.** Let \( \pi \) be a finitely presentable group with \( c.d.\pi \leq 2 \). Then two \( PD_4 \)-complexes \( X \) and \( Y \) with fundamental group \( \pi \), \( w_1(X) = w_1(Y) = w \) and \( \pi_2(X)^\dagger \cong \pi_2(Y)^\dagger \) a nonzero projective \( \mathbb{Z}[\pi] \)-module are homotopy equivalent if and only if \( \lambda_X \cong \lambda_Y \) and there are 2-connected degree-1 maps \( g_X : X \to Z \).
and \( g_Y : Y \to Z \) to the same strongly minimal \( PD_4 \)-complex \( Z \). Moreover every nonsingular \( w \)-hermitean pairing on a finitely generated projective \( \mathbb{Z}[\pi] \)-module is the cohomology intersection pairing of some such \( PD_4 \)-complex.

**Proof.** The first assertion follows from Theorem 7, while the second assertion follows from Theorem 8. \( \square \)

**Corollary.** Let \( \pi \) be a \( PD_2 \)-group. Then two \( PD_4 \)-complexes \( X \) and \( Y \) with fundamental group \( \pi \) and orientation character \( w : \pi \to \mathbb{Z}^\times \) and with second Wu class \( v_2(X) = v_2(Y) = 0 \) are homotopy equivalent if and only if \( \lambda_X \cong \lambda_Y \).

**Proof.** The minimal model \( Z \) for \( X \) is the total space of an \( S^2 \)-bundle over \( K(\pi, 1) \), by Theorem 7 of [Hi2], and \( v_2(Z) = 0 \) if \( v_2(X) = 0 \). \( \square \)

The situation is more complicated if \( v_2(X) \neq 0 \), but it remains true that the minimal model \( Z \) is determined by the cohomology of \( X \), as in Theorem 7 of [Hi2]. The strongly minimal \( PD_4 \)-complexes with fundamental group \( \pi \cong F(r) \times Z \) are mapping tori of self homeomorphisms of \( \#^r(S^1 \times S^2) \) or \( (S^1 \times S^2) \# (\#^{r-1}(S^1 \times S^2)) \), by Theorem 4.5 of [Hi]. What can one say about the other strongly minimal \( PD_4 \)-complexes with \( \pi \) of cohomological dimension 2? In particular, what can one say when \( \pi \) is solvable?
References


