Cofull Embeddings in Coset Monoids

James East

School of Mathematics and Statistics, University of Sydney, Sydney, NSW 2006, Australia

July 13, 2005

Abstract

Easdown, East and FitzGerald (2004) gave a sufficient condition for a (factorizable inverse) monoid to embed as a cofull submonoid of the coset monoid of its group of units. We show that this condition is also necessary. This yields a simple description of the class of finite monoids which embed in the coset monoids of their group of units. We apply our results to give a short proof of the result of McAlister (1980) that the symmetric inverse semigroup on a finite set $X$ does not embed in the coset monoid of the symmetric group on $X$. We also present examples which show that the word “cofull” may not be removed.

Keywords: Factorizable inverse monoid, coset monoid, symmetric inverse semigroup.

MSC: Primary 20M18; Secondary 20M30.

1 Factorizable Inverse Monoids and Coset Monoids

If $M$ is a monoid then we denote by $G_M$ the group of units of $M$. We say that a submonoid of $M$ is cofull if it contains $G_M$. If $N$ is another monoid and $\psi : M \rightarrow N$ an embedding, then we say that $\psi$ is cofull if $M\psi$ is a cofull submonoid of $N$.

If $M$ is an inverse monoid then we denote by $E_M$ the semilattice of idempotents of $M$. An inverse monoid $M$ is factorizable if $M = E_M G_M$. The study of factorizable inverse monoids (henceforth FIMs) was initiated in [2]; for related studies see [4, 5, 6, 10] and references therein.

Let $G$ be a group and denote by $S(G)$ the join semilattice of all subgroups of $G$. The join $H \vee K$ of two subgroups $H, K \in S(G)$ is defined to be $\langle H \cup K \rangle$, the smallest subgroup of $G$ containing $HK$. Now let

$$C(G) = \{ Hg \mid H \in S(G), \ g \in G \}$$

*jamese@maths.usyd.edu.au
be the set of all cosets of all subgroups of $G$. An associative product $*$ is defined on $\mathcal{C}(G)$, for $H, K \in \mathcal{S}(G)$ and $g, l \in G$, by

$$(Hg) * (Kl) = (H \lor gKg^{-1})gl,$$

the smallest coset of $G$ containing $HgKl$. The set $\mathcal{C}(G)$ is a FIM under $*$ with identity $\{1\}$, known as the coset monoid of $G$. The coset monoid was introduced in [8, 9]; see also [7]. Following is a collection of some elementary properties of coset monoids; these, along with other properties, are stated in [7].

**Lemma 1** Let $G$ be a group. Then

(i) $E_{\mathcal{C}(G)} = \mathcal{S}(G)$;
(ii) $G_{\mathcal{C}(G)} = \{ \{g\} \mid g \in G \} \cong G$; and
(iii) the subgroups of $\mathcal{C}(G)$ are precisely the sections of $G$ (a section of $G$ is a quotient of a subgroup of $G$).

The following was proved in [7].

**Theorem 2 (McAlister)** Let $X$ be a set. Then

(i) the symmetric inverse semigroup $\mathcal{I}_X$ embeds in the coset monoid $\mathcal{C}(\mathcal{G}_Y)$ of the symmetric group $\mathcal{G}_Y$ on a set $Y$ with $|Y| = |X| + 1$; and
(ii) $\mathcal{I}_X$ does not embed in $\mathcal{C}(\mathcal{G}_X)$ if $X$ is finite and nonempty.

It follows by Theorem 2(i) and the Wagner-Preston Theorem that any inverse monoid (indeed any inverse semigroup) embeds in the coset monoid of some group. An interesting question that arises is “given an inverse monoid $M$, what is the minimum cardinality of a group $G$ for which an embedding $M \to \mathcal{C}(G)$ exists?” Now suppose that $\Psi : M \to \mathcal{C}(G)$ is an embedding of an inverse monoid $M$ in the coset monoid of a group $G$. By Lemma 1(iii), the image of $G_M$ under $\Psi$ is a section of $G$, showing that the cardinality of $G$ is bounded below by the cardinality of $G_M$. Thus, another natural question arises: “Which inverse monoids $M$ embed in $\mathcal{C}(G_M)$?” The goal of this article is to give necessary and sufficient conditions for an inverse monoid $M$ to embed as a cofull submonoid of $\mathcal{C}(G_M)$.

Let $M$ be an inverse monoid and write $E = E_M$ and $G = G_M$. For $e \in E$ let

$$G_e = \{ g \in G \mid eg = e \}.$$ 

It is easy to check that each $G_e$ is a subgroup of $G$, and that $G_e \lor G_f \subseteq G_{ef}$ for each $e, f \in E$. Define a map

$$\psi_M : E \to \mathcal{S}(G) : e \mapsto G_e \quad \text{for each } e \in E.$$ 

Let $\mathcal{C}$ denote the class of factorizable inverse monoids $M$ for which $\psi_M$ is a semilattice embedding. The following was proved in [5].

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Theorem 3 (Easdown, East, FitzGerald) A monoid $M$ embeds as a cofull submonoid of $\mathcal{C}(G_M)$ if $M \in \mathcal{C}$.

This theorem was proved by showing that if $M \in \mathcal{C}$, then the map

$$M \to \mathcal{C}(G) : eg \mapsto G_e g$$

for each $e \in E$ and $g \in G$

is a cofull embedding. Our main goal is to show that the condition $M \in \mathcal{C}$ is also necessary for a monoid to embed as a cofull submonoid of $\mathcal{C}(G_M)$. In addition we show that the word “cofull” may be removed within the class of finite (but not infinite) inverse monoids.

2 Cofull Embeddings

Our goal in this section is to show that a monoid $M$ embeds as a cofull submonoid of $\mathcal{C}(G_M)$ if and only if $M \in \mathcal{C}$.

Lemma 4 Any cofull submonoid of a factorizable inverse monoid is a factorizable inverse monoid.

Proof Suppose that $N$ is a cofull submonoid of a FIM $M$, and choose $m \in N$. Then $m = eg$ for some $e \in E_M$ and $g \in G_M$. Since $N$ is cofull, we have $g^{-1} \in N$ and so $m^{-1} = g^{-1}e = g^{-1}(eg)g^{-1} = g^{-1}mg^{-1} \in N$, showing that $N$ is inverse. We also have $e = mg^{-1} \in N$ so that $N$ is factorizable.

Theorem 5 A monoid $M$ embeds as a cofull submonoid of $\mathcal{C}(G_M)$ if and only if $M \in \mathcal{C}$.

Proof The “if” part of the theorem is true by Theorem 3. To show the converse, it suffices to show that $N \in \mathcal{C}$ for every cofull submonoid $N$ of $\mathcal{C}(G_M)$. Write $G = G_M$, and $\mathcal{G} = G_{\mathcal{C}(G)} = \{g \mid g \in G\}$. Now $N$ is a FIM by Lemma 4, so it remains only to show that

$$\psi_N : E_N \to \mathcal{S}(G_N)$$

is an embedding. Now $E_N = \mathcal{S}(G) \cap N$, and $G_N = \mathcal{G}$ since $N$ is cofull. Further, if $H \in E_N$, then $H\psi_N = \mathcal{P} = \{\{h\} \mid h \in H\}$. It follows that $\psi_N$ is an embedding since $H \lor K = H \lor K$ for any subgroups $H, K \in \mathcal{S}(G)$.

As a corollary, we have the following.

Theorem 6 A finite monoid $M$ embeds in $\mathcal{C}(G_M)$ if and only if $M \in \mathcal{C}$.

Proof Write $G = G_M$. Any embedding $\Psi : M \to \mathcal{C}(G)$ gives rise to an embedding $\overline{\Psi} : G \to G_{\mathcal{C}(G)} \cong G$ since the image of $G$ under $\Psi$, being finite, cannot be contained in a proper section of $G$. Since $G$ is finite, $\overline{\Psi}$ is an isomorphism whence $\Psi$ is cofull, and we are done by Theorem 5.

We now apply Theorem 6 to provide an alternative proof of Theorem 2(ii).
Corollary 7 (McAlister) Let $X$ be a finite nonempty set. Then $\mathcal{I}_X$ does not embed in $\mathcal{C}(G_X)$.

**Proof** Put $G = G_X = G_{I_X}$. For $A \subseteq X$ denote by $id_A$ the identity map on $A$ so that $E_{I_X} = \{id_A \mid A \subseteq X\}$. Then for each $A \subseteq X$,

$$G_{id_A} = \{ \pi \in G \mid a\pi = a \ (\forall a \in A) \}$$

is the pointwise stabilizer of $A$, which we will denote by $\text{Stab}(A)$. Now if $x \in X$, then $\text{Stab}(X) = \text{Stab}(X \setminus \{x\}) = \{id_X\} \iff \psi_{I_X}$ is not injective. We are now done by Theorem 6.

We remark that if $X$ is any set with $|X| \geq 2$, then the map $\psi_{I_X}$ is not a semilattice homomorphism since if $x, y \in X$ with $x \neq y$, then, writing $A = X \setminus \{x\}$ and $B = X \setminus \{y\}$, we have

$$G_{id_A} \lor G_{id_B} = \text{Stab}(A) \lor \text{Stab}(B) = \text{Stab}(A) = \text{Stab}(B) = \{id_X\}$$

while the transposition which interchanges $x$ and $y$ is in

$$\text{Stab}(X \setminus \{x, y\}) = \text{Stab}(A \cap B) = G_{id_A \lor id_B} = G_{id_A \lor id_B}.$$

### 3 Other Embeddings

In this final section we consider examples of FIMs $M$ which embed in $\mathcal{C}(G_M)$ but do not belong to $\mathcal{C}$. These FIMs are necessarily infinite, and the embeddings are not cofull.

**Example 8** Let $X$ be an infinite set. Then the symmetric inverse semigroup $I_X \notin \mathcal{C}$ by the comments after the proof of Corollary 7. On the other hand, $I_X$ does embed in the coset monoid of the symmetric group $G_X = G_{I_X}$ by Theorem 2(i).

Now $I_X$ (indeed $E_{I_X}$) is uncountable for any infinite set $X$. Our second example is a countable FIM $M$ for which $|E_M| = 3$ and $\text{rank}(G_M) = 1$. Here for a group $G$ we have written $\text{rank}(G)$ for the minimal cardinality of a set which generates $G$ (as a group).

**Example 9** Let $G = \langle x \rangle$ be the infinite cyclic group generated by $x$, and let $G^y$ be the semigroup obtained by adjoining a zero $y$ to $G$. Let $M = (G^y)^z$ be the semigroup obtained by adjoining a new zero $z$ to $G^y$. It is easy to check that $M$ is a FIM with $G_M = G$ and $E_M = \{1, y, z\}$. We also have $G_y = G_z = G$ so that $M \notin \mathcal{C}$. Now define

$$\Psi : M \to \mathcal{C}(G) : \begin{cases} x & \mapsto \{x^2\} \\ y & \mapsto \{x^2\} \\ z & \mapsto G. \end{cases}$$

Then one may easily check that $\Psi$ is an embedding.
Our final example is also a countable FIM $M$ although in this case we have $|E_M| = 2$ and $\text{rank}(G_M) = 2$.

**Example 10** Let $G = \langle x, y \rangle$ be the free group freely generated by $\{x, y\}$. Define a homomorphism

$$\varphi : G \to G : x \mapsto x^2, \ y \mapsto y^2,$$

and put $K = \langle x^2, y^2 \rangle$, the image of $\varphi$. Let $B = G/N$ where $N$ is the normal closure in $G$ of $\{xyxy^{-1}x^{-1}y^{-1}\}$. So $B$ has presentation $\langle x, y \mid xyx = yxy \rangle$ and is isomorphic to the braid group on 3 strings; see [1]. It is well known that $Nx^2$ and $Ny^2$ generate a free subgroup of $B$ of rank 2; see for example [3]. It follows that $N \cap K = \{1\}$.

Now let $E = \{0, 1\}$ which we consider as a semilattice under multiplication, and put $M = E \times G$. So $M$ is a FIM with $E_M = (E, 1) \cong E$ and $G_M = (1, G) \cong G$, and $M \not\in \mathcal{C}$ since $G_{(1,1)} = G_{(0,1)} = \{(1,1)\}$. Define

$$\Psi : M \to C(G) : \begin{cases} (1, g) &\mapsto \{g\varphi\} \quad \text{for each } g \in G \\ (0, g) &\mapsto N(g\varphi) \quad \text{for each } g \in G. \end{cases}$$

Then $\Psi$ is a homomorphism since $N$ is normal in $G$ and $\varphi$ is a homomorphism. To show that $\Psi$ is injective, suppose that $e_1, e_2 \in E$ and $g_1, g_2 \in G$ such that

$$(e_1, g_1)\Psi = (e_2, g_2)\Psi.$$

Then we clearly must have $e_1 = e_2$. Suppose first that $e_1 = e_2 = 1$. Then

$$\{g_1\varphi\} = (e_1, g_1)\Psi = (e_2, g_2)\Psi = \{g_2\varphi\}.$$

It then follows that $g_1 = g_2$ since $\varphi$ is injective and so $(e_1, g_1) = (e_2, g_2)$. Finally, suppose that $e_1 = e_2 = 0$. Then

$$N(g_1\varphi) = (e_1, g_1)\Psi = (e_2, g_2)\Psi = N(g_2\varphi)$$

from which it follows that $(g_1g_2^{-1})\varphi = (g_1\varphi)(g_2\varphi)^{-1} \in N$. But then $(g_1g_2^{-1})\varphi = 1$ since $N \cap K = \{1\}$, and so $g_1g_2^{-1} = 1$ since $\varphi$ is injective, whence $g_1 = g_2$ and $(e_1, g_1) = (e_2, g_2)$. This completes the proof that $\Psi$ is injective.

While the monoids $M$ considered in Examples 9 and 10 had different values of $|E_M|$ and $\text{rank}(G_M)$, they shared the property that $|E_M| + \text{rank}(G_M) = 4$. It turns out that 4 is the minimum value of $|E_M| + \text{rank}(G_M)$ for any FIM $M \not\in \mathcal{C}$ which embeds in $C(G_M)$.

**Proposition 11** Suppose that $M \not\in \mathcal{C}$ is a FIM for which there exists an embedding $\Psi : M \to C(G_M)$. Then $|E_M| + \text{rank}(G_M) \geq 4$.

**Proof** Write $E = E_M$ and $G = G_M$ and suppose that $|E| + \text{rank}(G) \leq 3$. Since $M \not\in \mathcal{C}$, we have $|E| \geq 2$, and since $\Psi$ is injective, we have $\text{rank}(G) \geq 1$. It then follows that $|E| = 2$ and $\text{rank}(G) = 1$. Write $E = \{1, e\}$ where 1 is the identity of $M$. Since $M \not\in \mathcal{C}$ we must
have $G_e = \{1\}$. Since $M$ is infinite, $G$ must be an infinite cyclic group generated by $x$ say, and since $\Psi$ is an embedding, we have $x\Psi = \{x^i\}$ and $e\Psi = \langle x^i \rangle$ for some $i, j \in \mathbb{Z} \setminus \{0\}$. But then $ex^j \neq e$ since $G_e = \{1\}$, yet

\[(ex^j)\Psi = (e\Psi) * (x^i\Psi) = \langle x^i \rangle * \{x^{ij}\} = \langle x^i \rangle x^{ij} = \langle x^i \rangle = e\Psi\]

contradicting the injectivity of $\Psi$. This completes the proof. \qed

References


