q-difference equations of KdV type and "Chazy-type" second-degree difference equations

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Abstract. By imposing special compatible similarity constraints on a class of integrable partial q-difference equations of KdV-type we derive a hierarchy of second-degree ordinary q-difference equations. The lowest (non-trivial) member of this hierarchy is a second-order second-degree equation which can be considered as an analogue of equations in the class studied by Chazy. We present corresponding isomonodromic deformation problems and discuss the relation between this class of difference equations and other equations of Painlevé type.

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1. Introduction

The construction and study of discrete Painlevé equations has been a topic of research interest for almost two decades, [28, 13, 35, 21]. Reviews of the subject may be found in [14, 16]. The subject has culminated in the classification by H. Sakai of discrete as well continuous Painlevé equations based on the algebraic geometry of the corresponding rational surfaces associated with the spaces of initial conditions [37]. As a byproduct of the latter treatment, a "mistress" discrete Painlevé equation with elliptic dependence on the independent variable was discovered.

In the history of the Painlevé program, after the classification results for second-order first-degree equations, Painlevé's students, Chazy and Garnier, [6, 7, 15], investigated the classification of second-order second-degree equations and third-order first-degree equations. The classification of the second-degree class was completed by Cosgrove in recent years, [8, 9]. A partial classification for the third-order case was also obtained by the aforementioned authors. The work of Bureau, [3, 4], is also important in this respect. No classification results exist for the analogous discrete case and hardly any examples of second-order second-degree difference equations exist to date, with the notable exception of an (additive difference) equation given by Estévez and Clarkson [10].

A key result of this letter is a second-order second-degree equation, which can be considered as a q-analogue of an equation in the Chazy-Cosgrove class, together with its Lax pair (i.e., isomonodromic q-difference problem). This new equation contains four free parameters, which suggests that it could be a q-difference analogue of the second-order second-degree differential equation that is a counterpart of the sixth Painlevé equation. There are several forms of a second-order second-degree equation related to the sixth Painlevé equation that have appeared in the literature, notably one derived by Fokas and Ablowitz [12] and another appearing in the work of Okamoto [32]. Difference analogues of the Fokas-Ablowitz equation have been provided by Grammaticos and Ramani, [34], but these difference equations were all of first-degree. It has been argued by these authors that equations that are second-degree in the highest iterate cannot be viewed as "integrable", however, for the new equation we establish integrability through a Lax pair in the form of an isomonodromic deformation system. Furthermore we show that the equation arises as a similarity reduction from an integrable partial q-difference equation. Through the same procedure we also construct higher-order second-degree equations, which form a hierarchy associated with the new equation.

2. q-Difference Similarity Reduction

Lattice equations of KdV-type were introduced and studied over the last three decades [19, 29], see [27] for a review. These lattice equations can be formulated as partial difference equations on a lattice with step sizes that enter as parameters of the equation. Conventionally we think of these parameters as fixed constants. However, in agreement with the integrability of these equations, there exists the freedom to take the parameters as functions of the local lattice coordinate in each corresponding direction. In this paper we consider the case when the parameters depend exponentially with base q on the lattice coordinates.

We work in a space $\mathfrak{F}$ of functions f of arbitrarily many variables a_i ($i = 1, \dots, M$ for any M) on which we define the q-shift operations

$$ {}_q\mathcal{T}_i\, f(a_1, \dots, a_N) := f(a_1, \dots, q\, a_i, \dots, a_M) \ . $$

For $u, v, z \in \mathfrak{F}$, we consider the following systems of nonlinear partial q-difference equations:

$$ (u - {}_q\mathcal{T}_i\, {}_q\mathcal{T}_j u)\,({}_q\mathcal{T}_j u - {}_q\mathcal{T}_i u) = (a_i^2 - a_j^2) q^2 \ , \tag{2.1} $$

$$ a_j({}_q\mathcal{T}_j v)\, {}_q\mathcal{T}_i\, {}_q\mathcal{T}_j v + a_i({}_q\mathcal{T}_j v) v = a_i({}_q\mathcal{T}_i v)\, {}_q\mathcal{T}_j\, {}_q\mathcal{T}_i v + a_j({}_q\mathcal{T}_i v) v \tag{2.2} $$

and

$$a_i^2\,(z-{}_qT_iz)\,({}_qT_jz-{}_qT_i\,{}_qT_jz)$$
$$=a_j^2\,(z-{}_qT_jz)\,({}_qT_iz-{}_qT_i\,{}_qT_jz)\,, \tag{2.3}$$

where $i,j=1,\ldots,M$. Each of these systems, (2.1) to (2.3), represents a multi-dimensionally consistent family of partial difference equations, in the sense of [31, 1], which implies that they constitute holonomic systems of nonlinear partial q-difference equations. Another way to formulate this property is through an underlying linear system which takes the form

$${}_qT_i^{-1}\boldsymbol{\phi}=\boldsymbol{M}_i(k)\boldsymbol{\phi}, \tag{2.4}$$

where $\boldsymbol{\phi}=\boldsymbol{\phi}(k;\{a_j\})$ is a two-component vector-valued function and by consistency, ${}_qT_i^{-1}\,{}_qT_j^{-1}\boldsymbol{\phi}={}_qT_j^{-1}\,{}_qT_i^{-1}\boldsymbol{\phi}$, leads to the set of Lax equations (for each pair of indices i,j)

$$({}_qT_i^{-1}\boldsymbol{M}_j)\boldsymbol{M}_i=({}_qT_j^{-1}\boldsymbol{M}_i)\boldsymbol{M}_j\ . \tag{2.5}$$

We will consider three different cases, associated respectively with equations (2.1)–(2.3). To avoid proliferation of symbols we use the same symbol $\boldsymbol{M}_i(k)$ for each of the respective Lax matrices. For specific choices of the matrices $\boldsymbol{M}_i$ the Lax equations (2.5) lead to the nonlinear equations given above. In the case of the q-lattice KdV, (2.1), the Lax matrices $\boldsymbol{M}_i$ are given by

$$\boldsymbol{M}_i(k;\{a_j\})=\frac{1}{a_i-k}\begin{pmatrix} a_i-{}_qT_i^{-1}u & , & 1 \\ k^2-a_i^2+(a_i+u)(a_i-{}_qT_i^{-1}u) & , & a_i+u \end{pmatrix}. \tag{2.6}$$

In the case of the q-lattice mKdV, (2.2), the Lax matrices $\boldsymbol{M}_i$ are given by

$$\boldsymbol{M}_i(k;\{a_j\})=\frac{1}{a_i-k}\begin{pmatrix} a_i({}_qT_i^{-1}v)/v & , & k^2/v \\ {}_qT_i^{-1}v & , & a_i \end{pmatrix}\ . \tag{2.7}$$

Finally, in the case of the q-lattice SKdV, (2.3), the Lax matrices $\boldsymbol{M}_i$ are given by

$$\boldsymbol{M}_i(k;\{a_j\})=\frac{a_i}{a_i-k}\begin{pmatrix} 1 & , & (k^2/a_i^2)\left(z-{}_qT_i^{-1}z\right)^{-1} \\ z-{}_qT_i^{-1}z & , & 1 \end{pmatrix}\ . \tag{2.8}$$

These Lax matrices are straightforward generalizations of those with constant lattice parameters given in e.g. [30].

We mention that the solutions of the equations (2.1) to (2.3) are related through discrete Miura type relations, namely

$$a_i\left(z-{}_qT_i^{-1}z\right)=v\left({}_qT_i^{-1}v\right)\quad , \tag{2.9a}$$
$$s=\left(a_i-{}_qT_i^{-1}u\right)v-a_i\,{}_qT_i^{-1}v\quad , \tag{2.9b}$$
$${}_qT_i^{-1}s=a_iv-(a_i+u)\,{}_qT_i^{-1}v\ , \tag{2.9c}$$

where $s\in\mathfrak{F}$ is an auxiliary dependent variable. From these relations, the partial q-difference equations (2.1) to (2.3) can be derived by eliminating s.

Similarity reductions of lattice equations have been considered in [28, 25, 31, 26, 30] where it was shown that scaling invariance of the solution can be implemented through additional compatible constraints on the lattice equations. In the present case of (2.1) to (2.3) these constraints adopt the following form [11]

$$u(\{q^{-N}a_i\}) = q^{-N}\frac{1-\lambda(q^N-1)(-1)^{\sum_i {}^q\!\log a_i}}{1+\lambda(q^N-1)(-1)^{\sum_i {}^q\!\log a_i}}\,u(\{a_j\}), \tag{2.10a}$$

$$v(\{q^{-N}a_i\}) = \frac{1-\lambda(q^N-1)(-1)^{\sum_i {}^q\!\log a_i}}{1+\mu(q^N-1)}\,v(\{a_j\}), \tag{2.10b}$$

$$z(\{q^{-N}a_i\}) = q^{N}\frac{1-\mu(q^N-1)}{1+\mu(q^N-1)}\,z(\{a_j\}) \quad , \tag{2.10c}$$

where λ and μ are constant parameters of the reduction and $N \in \mathbb{N}$ represents a "periodicity freedom". The notation ${}^q\!\log x$ refers to the logarithm of x with base q.

In order to compute the corresponding isomonodromic deformation problems associated with the similarity reductions we have the following constraints on the vector function of the Lax pairs. In the case of (2.1) we have

$$\boldsymbol{\phi}(q^N k;\{a_j\}) = \tag{2.11}$$
$$\begin{pmatrix} \left(1+\lambda(q^N-1)(-1)^{\sum_i {}^q\!\log a_i}\right) & , & 0 \\ -2\lambda q\frac{q^N-1}{q-1}\left(\sum_i a_i\right)(-1)^{\sum_i {}^q\!\log a_i} & , & q^N\left(1-\lambda(q^N-1)(-1)^{\sum_i {}^q\!\log a_i}\right) \end{pmatrix}\boldsymbol{\phi}(k;\{q^{-N}a_j\}).$$

In the case of (2.2)

$$\boldsymbol{\phi}(q^N k;\{a_j\}) = \tag{2.12}$$
$$\begin{pmatrix} \left(1+\lambda(q^N-1)(-1)^{\sum_i {}^q\!\log a_i}\right) & 0 \\ 0 & q^{-N}(1+\mu(q^N-1)) \end{pmatrix}\boldsymbol{\phi}(k;\{q^{-N}a_j\}).$$

In the case of (2.3)

$$\boldsymbol{\phi}(q^N k;\{a_j\}) = \tag{2.13}$$
$$\begin{pmatrix} \left(1-\mu(q^N-1)\right) & 0 \\ 0 & q^{-N}(1+\mu(q^N-1)) \end{pmatrix}\boldsymbol{\phi}(k;\{q^{-N}a_j\}).$$

The similarity constraints, (2.11) to (2.13), in conjunction with the discrete linear equations (2.6) to (2.8) can be used to derive corresponding q-isomonodromic deformation problems. That is, (2.11) to (2.13) lead to q-difference equations in the spectral variable k, hence together with the lattice equation Lax pairs we obtain q-isomonodromic deformation problems for the corresponding reductions.

Remarks:

(i) The similarity constraints above were obtained through an approach based on Jackson-type integrals, the details of which will be presented elsewhere [11]. By construction, these constraints are compatible with the underlying lattice equations, which can be checked *a posteriori* by an explicit calculation, presented in the appendix.

(ii) In this approach, the dynamics in terms of the variables a_i appear through appropriately chosen q-analogues of exponential functions, whereas the relevant Jackson integrals exhibit an invariance through scaling by factors q^N.

(iii) The parameters λ and μ arise in this setting through boundary contributions in a manner analogous to the derivation in [30].

In the remainder of this letter our aim is to implement the similarity constraint to obtain explicit reductions to ordinary q-difference equations. For simplicity we consider only the reduction of the q-mKdV equation (2.2), leaving considerations of the q-KdV and q-SKdV to a future publication [11]. There are two possible scenarios to derive similarity reductions of the lattice equations using the constraint (2.10b).

"Periodic" similarity reduction: By fixing $M = 2$ and allowing N to vary, we select two lattice directions, say the variables a_1 and a_2, and consider similarity reductions with different values of N. This is a q-variant of the periodic staircase type reduction of partial difference equations on the two-dimensional lattice. For instance, with $N = 2$ the reduction is a second-order first-degree q-Painlevé equation. Increasing N leads to q-difference Painlevé type equations of increasing order. However, we will not pursue this route here but leave it to a subsequent publication [11]. These reductions are reminiscent of the work [18, 36, 17].

Multi-variable similarity reduction: The similarity constraints provide the mechanism to couple together two or more lattice directions. By considering the case $N = 1$ we implement the similarity constraints on an extended lattice of three or more dimensions in order to obtain coupled ordinary q-difference equations, in a way that is reminiscent of the approach of [31]. This is considered in the next section.

We have not considered the more general case of arbitrary $M, N \in \mathbb{N}$, which we will postpone to a future publication [11].

3. Multi-variable similarity reduction

In this section we consider explicitly the $M = 1$, 2 and 3 cases for $N = 1$.

For simplicity we shall in what follows denote the coefficient in (2.10b) as γ, i.e.

$$\gamma = \frac{1 - \lambda(q-1)(-1)^{\sum_i {}^{q}\log a_i}}{1 + \mu(q-1)} \quad \Rightarrow \quad v(\{q^{-1}a_i\}) = \gamma\, v(\{a_i\}) \ , \tag{3.1}$$

where γ alternates between two values, i.e., ${}_qT_i^2\gamma = \gamma$.

In contrast to the usual difference case which was explored in [31] where in the case of two variables we obtain a nontrivial OΔE as a reduction, in the q-difference case we have to consider at least three independent variables to obtain a nontrivial system of OΔEs as a reduction.

In [31] the compatibility between the similarity constraint and the lattice system was established and led to a system of higher order difference equations in the reduction, namely equations which were on the level of the first Garnier system. In contrast to the $q = 1$ work, the 3D similarity constraint here is somewhat simpler and leads to a second-order equation (which is of second-degree, and is a principal result of this letter).

Two-variable case

Let us now select among the collection of variables $\{a_j\}$ two specific ones which for simplicity we will call a and b. Denote the q-shifts in these variables by an over-tilde $\widetilde{\ }$ and an over-hat $\widehat{\ }$ respectively. Equation (2.2) may now be written

$$b\widehat{v}\widehat{\widetilde{v}} + a\widehat{v}v = a\widetilde{v}\widehat{\widetilde{v}} + bv\widetilde{v}, \tag{3.2}$$

where the over-tilde $\widetilde{\ }$ refers to the q-translation $a \mapsto qa$ and the over-hat $\widehat{\ }$ refers to the q-translation $b \mapsto qb$ (so if $v \equiv v(a,b)$, $\widetilde{v} \equiv v(qa,b)$, $\underset{\sim}{v} \equiv v(q^{-1}a,b)$, $\widehat{v} \equiv v(a,bq)$, $\ldots$).

Equation (3.1) gives the constraint $v = \gamma\widehat{\widetilde{v}}$ to impose on (3.2) (where $\widehat{\widetilde{\gamma}} = \widetilde{\widetilde{\gamma}} = \gamma$). This leads to the linear first-order (in that it is a two point) ordinary difference equation

$$\underset{\sim}{v} = C\,\tilde{v}, \tag{3.3}$$

where $C = \widetilde{\gamma}(a\gamma + b)/(a + b\gamma)$. In the appendix the consistency between the lattice equation (3.2) and the constraint (3.3) is shown by direct computation.

Three-variable case

Take three copies of the lattice mKdV equation with $a_1 = a$, $a_2 = b$, $a_3 = c$,

$$b\widehat{v}\widehat{\widetilde{v}} + a\widehat{v}v = a\widetilde{v}\widehat{\widetilde{v}} + bv\widetilde{v}, \tag{3.4a}$$

$$c\overline{v}\widetilde{\overline{v}} + a\overline{v}v = a\widetilde{v}\widetilde{\overline{v}} + cv\widetilde{v}, \tag{3.4b}$$

$$c\overline{v}\widehat{\overline{v}} + b\overline{v}v = b\widehat{v}\widehat{\overline{v}} + cv\widehat{v}, \tag{3.4c}$$

where $\overline{c} = qc$, together with the constraint

$$v(q^{-1}a, q^{-1}b, q^{-1}c) = \gamma v(a,b,c). \tag{3.5}$$

(The similarity constraint is shown by a direct computation to be compatible with the multidimensionally consistent system of mKdV lattice equations in the appendix.) We now proceed to derive the reduced system which leads to a (higher-degree) ordinary q-difference equation in terms of one selected independent variable, say the variable a. The remaining variables b and c will play the role of parameters in the reduced equation. Thus, we can derive the following system of two coupled OΔEs for $v(a,b,c)$ and $w(a,b,c) \equiv v(a,b,q^{-1}c)$:

$$\gamma v = w\frac{a\widetilde{\gamma}\widetilde{v} - b\,\underset{\sim}{w}}{a\,\underset{\sim}{w} - b\widetilde{\gamma}\widetilde{v}}\ , \tag{3.6a}$$

$$\underset{\sim}{w} = v\frac{aw - c\,\underset{\sim}{v}}{a\,\underset{\sim}{v} - cw}\ , \tag{3.6b}$$

where $\widetilde{\widetilde{\gamma}} = \gamma$. We consider the system (3.6a) and (3.6b) to constitute a q-Painlevé system with four free parameters.

The system (3.6a) and (3.6b) can be reduced to a second-order second-degree ordinary difference equation as follows. Introduce the variables

$$X = \frac{v}{w} \quad , \quad V = \frac{\widetilde{v}}{v} \quad , \quad W = \frac{\widetilde{w}}{w} \ , \tag{3.7}$$

then from (3.6a) we obtain

$$\widetilde{\gamma}\frac{\widetilde{v}}{\underset{\sim}{w}} = \widetilde{\gamma} V X \underset{\sim}{W} = \frac{a\gamma X + b}{b\gamma X + a} \ , \tag{3.8}$$

whereas from (3.6b) we get

$$W = \frac{VX}{\widetilde{X}} = \frac{a + q^{-1}cXV}{q^{-1}c/X + aV} \ , \tag{3.9}$$

using also the definitions (3.7). Thus, we obtain a quadratic equation for V in terms of X and $\widetilde{X}$ and hence also we have W in terms of X and $\widetilde{X}$. Inserting these into (3.8) we obtain a second-order algebraic equation for X. Alternatively, avoiding the emergence of square roots, the following second-order second-degree equation for X may be derived

$$\begin{aligned}
&\left[\widetilde{\gamma}^2 \widetilde{X}\, \underset{\sim}{X} - \left(\frac{a\gamma X + b}{b\gamma X + a}\right)^2\right]^2 \\
&\qquad = \widetilde{\gamma}\frac{c^2}{a^2}\frac{1}{X}\left(\frac{a\gamma X + b}{b\gamma X + a}\right)\left[\widetilde{\gamma}\widetilde{X}(1 - X\underset{\sim}{X}) + q^{-1}(1 - X\widetilde{X})\frac{a\gamma X + b}{b\gamma X + a}\right] \\
&\qquad\qquad \times \left[q^{-1}\widetilde{\gamma}\,\underset{\sim}{X}(1 - X\widetilde{X}) + (1 - X\underset{\sim}{X})\frac{a\gamma X + b}{b\gamma X + a}\right] \ .
\end{aligned} \tag{3.10}$$

We consider this second-degree equation to be one of the main results of this letter.

We now proceed to present the Lax pair for the q-Painlevé system (3.6a) and (3.6b) and the second-order second-degree equation (3.10). The Lax pair is formed by considering the compatibility of two paths on the lattice: along a 'period' then in the a direction and evolving in the a direction then along a 'period'. Using (2.12) the evolution along a period is converted into a dilation of the spectral parameter, k, by q. The result is the following isomonodromic q-difference system for the vector $\phi(k;a)$ which using the results of section 2 yields

$$\phi(k; q^{-1}a) = \boldsymbol{M}(k;a)\phi(k;a) \ , \tag{3.11a}$$

$$\phi(qk; a) \quad = \boldsymbol{L}(k;a)\phi(k;a) \ , \tag{3.11b}$$

where

$$\boldsymbol{M}(k;a) = \frac{1}{a-k}\begin{pmatrix} a\,\underset{\sim}{v}/v & k^2/v \\ \underset{\sim}{v} & a \end{pmatrix} \ , \tag{3.12a}$$

and

$$\boldsymbol{L}(k;a) = \frac{1}{a-k}\begin{pmatrix} a\gamma v/\widetilde{v} & k^2/\widetilde{v} \\ q^{-1}\gamma v & q^{-1}a \end{pmatrix}\begin{pmatrix} b\widetilde{\gamma}\widetilde{v}/w & k^2/w \\ \widetilde{\gamma}\widetilde{v} & b \end{pmatrix}\begin{pmatrix} cw/v & k^2/v \\ w & c \end{pmatrix} \tag{3.12b}$$

where we have suppressed the dependence on the variables b and c (which now play the role of parameters) and omitted the unnecessary prefactors $(b-k)^{-1}$ and $(c-k)^{-1}$, as well as an over factor $q^{-1}(1+\mu(q-1))$.

The consistency condition obtained from the two ways of expressing $\phi(qk;q^{-1}a)$ in terms of $\phi(k;a)$ is formed by the Lax equation

$$\boldsymbol{L}(k;q^{-1}a)\boldsymbol{M}(k;a)=\boldsymbol{M}(qk;a)\boldsymbol{L}(k;a)\ . \tag{3.13}$$

A gauge transformation can be obtained expressing the Lax matrices in terms of the variables introduced in (3.7). Setting

$$\mathcal{M}(k;a)=\frac{1}{a-k}\begin{pmatrix} a/\underset{\sim}{V} & k^2 \\ 1 & a\underset{\sim}{V}\end{pmatrix}, \tag{3.14a}$$

$$\mathcal{L}(k;a)=\frac{1}{a-k}\begin{pmatrix} \widetilde{\gamma}(ab\gamma X+k^2) & k^2(a\gamma X+b)/V \\ q^{-1}\widetilde{\gamma}V(a+b\gamma X) & q^{-1}(ab+k^2\gamma X)\end{pmatrix}\begin{pmatrix} c/X & k^2 \\ 1/X & c\end{pmatrix}, \tag{3.14b}$$

the Lax equations (3.13) (replacing $\boldsymbol{L}$ and $\boldsymbol{M}$ by $\mathcal{L}$ and $\mathcal{M}$ respectively) yield a set of relations equivalent to the following two equations:

$$\widetilde{\gamma}V\underset{\sim}{V}\underset{\sim}{X}=\frac{a\gamma X+b}{a+b\gamma X}\ , \tag{3.15}$$

$$aV^2+q^{-1}c\left(\frac{1}{X}-\widetilde{X}\right)V-a\frac{\widetilde{X}}{X}=0\ , \tag{3.16}$$

using also $\widetilde{\gamma}=\underset{\sim}{\gamma}$. This set follows directly from (3.8) and (3.9). Thus (3.14a) and (3.14b) form a q-isomonodromic Lax pair for the second-degree equation (3.10).

Four-variable case:

Suppose we have 4 variables a_i, $i=1,\dots,4$. Select $a=a_1$ to be the independent variable after reduction. Introduce the dependent variables $w_{j-2}={}_q T_j^{-1}v$, $j=3,4$. Then directly from the q-lattice mKdV equation (2.2) we have the set of equations

$$\underset{\sim}{w}_j=v\frac{aw_j-a_{j+2}\underset{\sim}{v}}{a\underset{\sim}{v}-a_{j+2}w_j}\quad ,\quad j=1,2\ , \tag{3.17}$$

where as before the tilde denotes a q-shift in the variable a. At the same time the multiply shifted object ${}_qT_3^{-1}\,{}_qT_4^{-1}\,\underset{\sim}{v}$ can be expressed in a unique way (due to the CAC property) in terms of $\underset{\sim}{v}$ and ${}_qT_j^{-1}v=w_{j-2}$, $j=3,4$, by iterating the relevant copies of the q-lattice mKdV equation in the variables a_j, $j\neq 2$, leading to an expression of the form ${}_qT_3^{-1}\,{}_qT_4^{-1}\,\underset{\sim}{v}=:F(\underset{\sim}{v},w_1,w_2)$, where F is easily obtained explicitly. Imposing the similarity constraint (3.1) we obtain $\widetilde{\gamma}\,{}_qT_2v=F(\underset{\sim}{v},w_1,w_2)$ and inserting this expression into the q-lattice mKdV (2.2) with $i=1,j=2$ we obtain

$$\left(a+\frac{a_2}{\gamma}\frac{a_3w_2-a_4w_1}{a_3w_1-a_4w_2}\right)\left(a_2\widetilde{\gamma}^{-1}F(\underset{\sim}{v},w_1,w_2)-a\widetilde{v}\right)=(a_2^2-a^2)v\widetilde{v}\ . \tag{3.18}$$

With the explicit form of $F(\underset{\sim}{v}, w_1, w_2)$ equation (3.18) reads

$$(a_2^2 - a^2)\gamma\widetilde{\gamma}\widetilde{v}(a_3w_1 - a_4w_2)\left[a(a_3^2 - a_4^2)\underset{\sim}{v} + a_3(a_4^2 - a^2)w_1 + a_4(a^2 - a_3^2)w_2\right]$$
$$= [(a_2a_3 - \gamma aa_4)w_2 + (\gamma aa_3 - a_2a_4)w_1]\ \Big[a(a_3^2 - a_4^2)(a_2w_1w_2 - \widetilde{\gamma}a\widetilde{v}\,\underset{\sim}{v})$$
$$+ \left(a_2a_4(a^2 - a_3^2)\underset{\sim}{v} - aa_3(a_4^2 - a^2)\widetilde{\gamma}\widetilde{v}\right) w_1 + \left(a_2a_3(a_4^2 - a^2)\underset{\sim}{v} - \widetilde{\gamma}aa_4(a^2 - a_3^2)\widetilde{v}\right) w_2\Big] , \tag{3.19a}$$

and this is supplemented by the two equations

$$avw_1 + a_3w_1\underset{\sim}{w}_1 = a_3v\,\underset{\sim}{v} + a\,\underset{\sim}{v}\,\underset{\sim}{w}_1 , \tag{3.19b}$$

$$avw_2 + a_4w_2\underset{\sim}{w}_2 = a_4v\,\underset{\sim}{v} + a\,\underset{\sim}{v}\,\underset{\sim}{w}_2 , \tag{3.19c}$$

which is equivalent to a five-point (fourth-order) q-difference equation in terms of v alone, containing five free parameters: a_2, a_3, a_4, λ and μ (inside γ and $\widetilde{\gamma}$). This would be an algebraic equation, so we proceed as follows in order to derive a higher-degree q-difference system. Introduce the variables

$$X_i = \frac{v}{w_i} \quad , \quad W_i = \frac{\widetilde{w}_i}{w_i} \quad , \quad i = 1, 2 , \tag{3.20}$$

while retaining the variable $V = \widetilde{v}/v$ as before. By definition we have

$$\frac{V}{W_i} = \frac{\widetilde{X}_i}{X_i} \quad , \quad i = 1, 2, \tag{3.21}$$

and from (3.19b), (3.19c) we obtain

$$W_i = \frac{qa + a_{i+2}VX_i}{qaV + a_{i+2}/X_i} = \frac{VX_i}{\widetilde{X}_i} \quad , \quad i = 1, 2 , \tag{3.22}$$

whilst from (3.19a) we get

$$(a_2^2 - a^2)\gamma\widetilde{\gamma}V\left(\frac{a_3}{X_1} - \frac{a_4}{X_2}\right)\left(a\frac{a_3^2 - a_4^2}{\underset{\sim}{V}} + a_3\frac{a_4^2 - a^2}{X_1} + a_4\frac{a^2 - a_3^2}{X_2}\right)$$
$$= \left(\frac{a_2a_3 - \gamma aa_4}{X_2} + \frac{\gamma aa_3 - a_2a_4}{X_1}\right)\ \Big[a(a_3^2 - a_4^2)(\frac{a_2}{X_1X_2} - \widetilde{\gamma}a\frac{V}{\underset{\sim}{V}})$$
$$+ \left(a_2a_4\frac{a^2 - a_3^2}{\underset{\sim}{V}} - aa_3(a_4^2 - a^2)\widetilde{\gamma}V\right)\frac{1}{X_1} + \left(a_2a_3\frac{a_4^2 - a^2}{\underset{\sim}{V}} - \widetilde{\gamma}aa_4(a^2 - a_3^2)V\right)\frac{1}{X_2}\Big] . \tag{3.23}$$

From (3.22) we obtain the set of quadratic equations for V

$$qa\frac{X_i}{\widetilde{X}_i}V^2 + a_{i+2}\left(\frac{1}{\widetilde{X}_i} - X_i\right)V - qa = 0 \quad , \quad i = 1, 2 , \tag{3.24}$$

from which by eliminating V we obtain

$$\left[a_3(1-X_1\widetilde{X}_1)X_2 - a_4(1-X_2\widetilde{X}_2)X_1\right]\left[a_3(1-X_1\widetilde{X}_1)\widetilde{X}_2 - a_4(1-X_2\widetilde{X}_2)\widetilde{X}_1\right]$$
$$= q^2a^2(X_1\widetilde{X}_2 - X_2\widetilde{X}_1)^2 \ . \tag{3.25}$$

Furthermore, solving V from the quadratic system as

$$V = qa\frac{X_2\widetilde{X}_1 - X_1\widetilde{X}_2}{a_3(1-X_1\widetilde{X}_1)X_2 - a_4(1-X_2\widetilde{X}_2)X_1} \ , \tag{3.26}$$

and inserting this into (3.23) we obtain a second-order equation in both X_1, X_2 coupled to the equation (3.25) which is first order in both X_1, X_2. It is this coupled system of two equations in X_1, X_2 which forms our higher order generalisation of (3.10). The system of (3.23) and (3.25) with (3.26) constitutes a third-order system with five parameters.

The derivation of the Lax pair follows the same approach as that for the three-variable case (with an extra factor in $\mathcal{L}$ due to the additional lattice direction). We omit details here, which we intend to publish in the future [11].

Beyond the four-variable case:

It is straight-forward to give the form of the full hierarchy, however due to lack of space we postpone this until a later publication [11].

4. Conclusion and discussion

In this letter we have presented the results of a scheme to derive partial q-difference equations of KdV type and consistent symmetries of the equations and demonstrated how it can be implemented. Lax matrices follow from this approach. A notable result is the derivation of the higher-degree equation (3.10), showing that the scheme presented here allows for the derivation of new results within the field of discrete integrable systems.

The first-, second- and third-order members of the $N = 1$ hierarchy have been presented. The scheme continues to give successively higher-order equations by considering successively higher dimensions of the original lattice equation. One may ask the natural question as to whether this gives an ‘interpolating’ hierarchy which, contrary to the usual cases, increases the order and number of parameters of the equations by one in each step, rather than a two step increase. A further natural question connected with this hierarchy is its relation to the q-Garnier systems of Sakai [38].

We will present full details of the scheme from which the lattice equations (2.1) to (2.3) and their associated constraints follow in a future publication [11]. There we will consider the most general case of symmetry reductions (arbitrary $N \in \mathbb{N}$) of all three lattice equations.

We also intend to return in a future publication to the question of limits and degeneracies of the equations presented in this paper. These include the $q \to 1$ continuum limit, the $q \to 1$ discrete limit and the $q \to 0$ crystal or ultradiscrete limit.

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5. Appendix

In this appendix we present the result of explicit calculations showing the consistency of the lattice equations and constraints.

Two-variable consistency

We shall check the consistency between the lattice equation (3.2) and the constraint $v = \gamma\widehat{\widetilde{v}}$ by direct computation. This computation is illustrated in the following diagram:

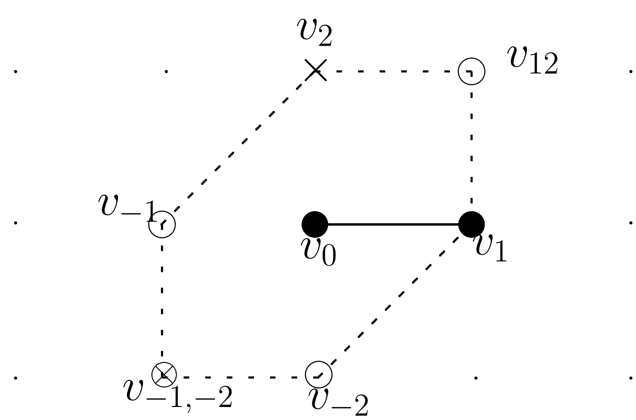

Fig 1. Consistency on the 2D lattice.

Assuming the values v_0, v_1 as indicated in Fig 1 are given, we compute successively v_{12},v_2 etc., where the subscripts refer to the shifts in the lattice variables a,b respectively, as is evident from Fig 1. Points other than v_0 and v_1 are computed using either the lattice equation (indicated by $\times$) or by using the similarity constraint (indicated by $\bigcirc$). The value $v_{-1,-2}$ is the first point which can be calculated in two different ways (hence indicated in the diagram by $\otimes$). Without making any particular assumptions on how γ depends on a and b, a straightforward calculation shows that the two ways of computing $v_{-1,-2}$ are indeed the same, for any choice of initial data v_0 and v_1, provided that γ obeys the relation

$$\left(\frac{a+b\gamma}{b+a\gamma}\right)^{\widehat{\widetilde{\ }}}\left(\frac{a+b\gamma}{b+a\gamma}\right)^{-1} = \frac{\widetilde{\gamma}}{\widehat{\gamma}}\ . \tag{5.1}$$

A particular solution of this relation is

$$\widehat{\widehat{\gamma}} = \gamma \quad \Leftrightarrow \quad \widehat{\gamma} = \widetilde{\gamma} \ , \tag{5.2}$$

and hence $\widetilde{\widetilde{\gamma}} = \gamma$ implying that γ is an alternating "constant" which is in accordance with the value given in (2.10b). The reduced equation in this case is (3.3), which can be readily integrated.

More generally, equation (5.1) can be resolved by setting

$$\frac{a+b\gamma}{b+a\gamma} = \frac{\widetilde{\nu}}{\widehat{\nu}} \quad , \quad \gamma = \frac{\widehat{\widehat{\nu}}}{\nu} \ , \tag{5.3}$$

leading to the consequence that ν has to solve the q-lattice mKdV (3.2). In principle we could take for ν any solution of the reduced equation (3.3) and use this to parametrise the reduced equation for v via the relations (5.3). In any event, we see that the two-variable case does not lead to interesting nonlinear equations.

Three-variable consistency

In this case the consistency diagram is as follows:

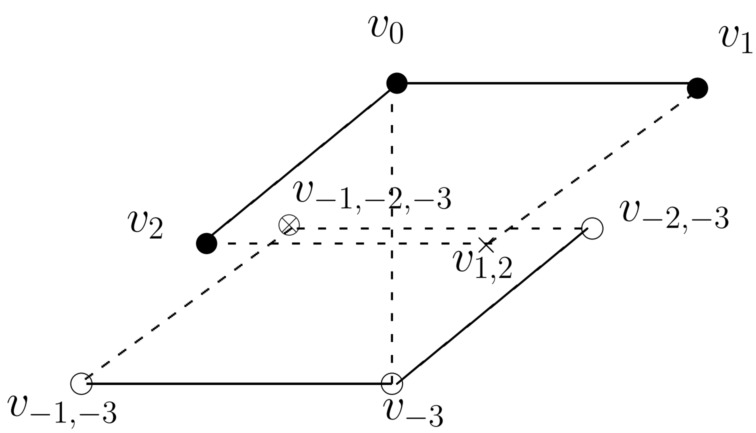

Fig 2. Consistency on the 3D lattice.

A similar notation as the previous case is used as is evident from Fig 2. The initial data v_0, v_1 and v_2 are given, and the indicated values on the vertices are computed either by using one of the lattice equations (3.4a) to (3.4c) or the similarity constraint (3.5) over the diagonal. Thus, $v_{1,2}$ is obtained from (3.4a) yielding

$$v_{1,2} = v_0\,\frac{av_2 - bv_1}{av_1 - bv_2} \ ,$$

whilst from the similarity constraint we obtain

$$v_{-1,-3} = \gamma_2 v_2 \quad , \quad v_{-3} = \gamma_{1,2} v_{1,2} \quad , \quad v_{-2,-3} = \gamma_1 v_1 \ ,$$

assuming that γ shifts along the lattice, indicated by the indices, and finally the value of $v_{-1,-2,-3}$ can be computed in two different ways, leading to

$$v_{-1,-2,-3} = \gamma_0 v_0 = \frac{av_{-2,-3} - bv_{-1,-3}}{av_{-1,-3} - bv_{-2,-3}} v_{-3} = \frac{a\gamma_1 v_1 - b\gamma_2 v_2}{a\gamma_2 v_2 - b\gamma_1 v_1}\gamma_{1,2} v_0 \frac{av_2 - bv_1}{av_1 - bv_2} \ ,$$

leading quadratic identity in v_1 and v_2. Assuming that the latter must hold identically, and thus setting all coefficients equal to zero, we obtain the following conditions on γ:

$$\gamma_{1,2,3} = \gamma_1 = \gamma_2 = \gamma_3 \quad ,$$

from which we conclude that γ is an alternating "constant", for instance

$$\gamma = \alpha\,\beta^{(-1)^{n+m+\cdots}} \qquad (\alpha, \beta \;\; \text{constants}) \tag{5.4}$$

and this leads to the conditions from which it is easily deduced that the form (3.1) of γ satisfies these conditions.

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