DENSITY OF COMMENSURATORS FOR UNIFORM LATTICES OF RIGHT-ANGLED BUILDINGS

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Abstract. Let $G$ be the automorphism group of a regular right-angled building $X$. The “standard uniform lattice” $\Gamma_0 \leq G$ is a canonical graph product of finite groups, which acts discretely on $X$ with quotient a chamber. We prove that the commensurator of $\Gamma_0$ is dense in $G$. For this, we develop a technique of “unfoldings” of complexes of groups. We use unfoldings to construct a sequence of uniform lattices $\Gamma_n \leq G$, each commensurable to $\Gamma_0$, and then apply the theory of group actions on complexes of groups to the sequence $\Gamma_n$. As further applications of unfoldings, we determine exactly when the group $G$ is nondiscrete, and we prove that $G$ acts strongly transitively on $X$.

Introduction

Two subgroups $\Gamma_0$ and $\Gamma_1$ of a group $G$ are commensurable if the intersection $\Gamma_0 \cap \Gamma_1$ has finite index in both $\Gamma_0$ and $\Gamma_1$. The commensurator of $\Gamma \leq G$ in $G$ is the group

$$\text{Comm}_G(\Gamma) := \{g \in G \mid g\Gamma g^{-1} \text{ and } \Gamma \text{ are commensurable}\}.$$ 

Note that $\text{Comm}_G(\Gamma)$ contains the normalizer $N_G(\Gamma)$. It is a classical fact that if $G$ is a connected semisimple Lie group, with trivial center and no compact factors, and $\Gamma \leq G$ is an irreducible lattice, then either $\Gamma$ is finite index in $\text{Comm}_G(\Gamma)$, or $\text{Comm}_G(\Gamma)$ is dense in $G$ (see [Z]). Moreover Margulis [M] proved that $\Gamma$ is arithmetic if and only if $\text{Comm}_G(\Gamma)$ is dense.

A semisimple Lie group is a locally compact topological group. If $X$ is a locally finite, simply connected polyhedral complex, then the group $G = \text{Aut}(X)$ is also locally compact. It turns out that a subgroup $\Gamma \leq G$ is a uniform lattice in $G$ if and only if $\Gamma$ acts cocompactly on $X$ with finite cell stabilizers (see Section 1.1). Lattices in such groups $G$ share many properties with lattices in semisimple Lie groups, but also exhibit new and unexpected phenomena (see the surveys [Lu] and [FHT]).

In this setting, the one-dimensional case is $X$ a locally finite tree. Liu [L] proved that the commensurator of the “standard uniform lattice” $\Gamma_0$ is dense in $G = \text{Aut}(X)$; here $\Gamma_0$ is a canonical graph of finite cyclic groups over the finite quotient $G \backslash X$. In addition, Leighton [Le] and Bass–Kulkarni [BK] proved that all uniform lattices in $G$ are commensurable (up to conjugacy). Hence all uniform tree lattices have dense commensurators. In dimension two, Haglund [H1] showed that for certain 2–dimensional Davis complexes $X = X_W$, the Coxeter group $W$, which may be regarded as a uniform lattice in $G = \text{Aut}(X)$, has dense commensurator.

We consider higher-dimensional cases, focusing on regular right-angled buildings $X$ (see Section 1.4). Such buildings exist in arbitrary dimension. Examples include products of finitely many regular trees, and Bourdon’s building $I_{p,q}$, the unique 2–complex in which every 2–cell is a regular right-angled hyperbolic $p$–gon, and the link of each vertex is the complete bipartite graph $K_{q,q}$ (see [B]). The “standard uniform lattice” $\Gamma_0 \leq G = \text{Aut}(X)$, defined in Section 1.5 below, is a canonical graph product of finite cyclic groups, which acts on $X$ with fundamental domain a chamber. Our main result is:

Density Theorem. Let $G$ be the automorphism group of a locally finite regular right-angled building $X$, and let $\Gamma_0$ be the standard uniform lattice in $G$. Then $\text{Comm}_G(\Gamma_0)$ is dense in $G$.

This theorem was proved independently and using different methods by Haglund [H3].
In contrast, we show in Section 2 below that for all $G = \text{Aut}(X)$ with $X$ a locally finite polyhedral complex (not necessarily a building), and all uniform lattices $\Gamma \leq G$, the normalizer $N_G(\Gamma)$ is discrete. Hence for $G$ as in the Density Theorem, the density of $\text{Comm}_G(\Gamma_0)$ does not come just from the normalizer.

For most regular right-angled buildings $X$, it is not known whether all uniform lattices in $G = \text{Aut}(X)$ are commensurable (up to conjugacy). Januszkiewicz–Świątkowski [JS1] have established commensurability of a class of lattices in $G$ which includes $\Gamma_0$, where each such lattice is a graph product of finite groups. Hence by the Density Theorem, each such lattice has dense commensurator. For Bourdon’s building $I_{p,q}$, Haglund [H2] proved that if $p \geq 6$, then all uniform lattices in $G = \text{Aut}(I_{p,q})$ are commensurable (up to conjugacy). Thus by the Density Theorem, all uniform lattices in $G$ have dense commensurators. On the other hand, for $X$ a product of two trees, Burger–Mozes [BM2] constructed a uniform lattice $\Gamma \leq \text{Aut}(X)$ which is a simple group. It follows that $\text{Comm}_G(\Gamma) = N_G(\Gamma)$, which is discrete. Thus there are cases (when $\dim(X) \geq 2$) in which not all uniform lattices $\Gamma \leq G = \text{Aut}(X)$ can have dense commensurators. In fact, it is an open problem to determine whether the only possibilities for $\text{Comm}_G(\Gamma)$ are discreteness or density. As for commensurators of nonuniform lattices in $G = \text{Aut}(X)$, hardly anything is known, even for $X$ a tree (see [FHT]).

If the building $X$ can be equipped with a $\text{CAT}(−1)$ metric, then the Density Theorem may be combined with the commensurator superrigidity theorem of Burger–Mozes [BM1] for $\text{CAT}(−1)$ spaces, to give rigidity results for lattices in $G = \text{Aut}(X)$ which are commensurable to $\Gamma_0$. Regular right-angled buildings with piecewise hyperbolic $\text{CAT}(−1)$ metrics exist in arbitrarily high dimensions [JS2].

We now outline the proof of the Density Theorem, which is given in full in Section 4 below. Fix a basepoint $x_0 \in X$. Denote by $Y_n$ the combinatorial ball of radius $n$ about $x_0$ in $X$. We first reduce the theorem to showing that for all $g \in \text{Stab}_G(x_0)$ and for all $n \geq 0$, there is a $\gamma_n \in \text{Comm}_G(\Gamma_0)$ such that $\gamma_n$ agrees with $g$ on the ball $Y_n$. We then construct a canonical uniform lattice $\Gamma_n$ with fundamental domain the ball $Y_n$, and show that $\Gamma_n$ is a finite index subgroup of $\Gamma_0$. By considering the restriction of $g$ to $Y_n$, we are then able to build a uniform lattice $\Gamma'_n$ which contains a suitable element $\gamma_n$. By our construction, the lattice $\Gamma_n$ is a finite index subgroup of $\Gamma'_n$. That is, $\Gamma'_n$ and $\Gamma_0$ have a common finite index subgroup $\Gamma_n$, as sketched on the left of Figure 1 below. Thus $\Gamma'_n$ is commensurable to $\Gamma_0$, and so $\gamma_n$ lies in $\text{Comm}_G(\Gamma_0)$, as required.

\[\begin{array}{ccc}
\Gamma_n & \text{G(Y_n)} & \Gamma_0 \\
\Gamma'_n & \text{H(Z_n)} & \text{G(Y_0)}
\end{array}\]

Figure 1. Inclusions of lattices (left) and coverings of complexes of groups (right)

Our lattices $\Gamma_n$ and $\Gamma'_n$ are fundamental groups of complexes of groups (see [BH] and Section 1.5 below). The finite index lattice inclusions on the left of Figure 1 are induced by finite-sheeted coverings of complexes of groups, as shown on the right of Figure 1. The necessary covering theory for complexes of groups is recalled in Section 1.6 below (see also [BH] and [LT]).

To construct the sequence of lattices $\Gamma_n$, in Section 3 below we introduce a new tool, that of unfoldings of complexes of groups. The standard uniform lattice $\Gamma_0$ may be viewed as the fundamental group of a complex of groups $G(Y_0)$ over a chamber $Y_0$ of $X$. By “unfolding” along “sides” of successive unions of chambers starting from $Y_0$, and defining new local groups appropriately, we obtain a canonical family of complexes of groups $G(Y_n)$ over the combinatorial balls $Y_n \subset X$. The fundamental group $\Gamma_n$ of $G(Y_n)$ is a uniform lattice in $G = \text{Aut}(X)$, and each $\Gamma_n$ is a finite index subgroup of $\Gamma_0$. We prove these properties of unfoldings inductively by combinatorial arguments, involving careful consideration of the local structure of $X$, together with facts about Coxeter groups, and the definition of a building as a chamber system equipped with a $W$–distance function (see Section 1.4).
The other main tool in our proof of the Density Theorem is that of group actions on complexes of groups, which was introduced by the second author in [T2] (see Section 1.7 below). This theory is used to construct the sequence of lattices $\Gamma_n$, containing suitable elements $\gamma_n$, as fundamental groups of complexes of groups $H(Z_n)$, such that there are finite-sheeted coverings $G(Y_n) \to H(Z_n)$.

We describe in Section 4.2 below how our two main tools, unfoldings and group actions on complexes of groups, may be combined to construct many uniform lattices in addition to the sequences $\Gamma_n$ and $\Gamma'_n$ used in the proof of the Density Theorem. To our knowledge, the lattices so obtained are new. In particular, they do not “come from” tree lattices, unlike the lattices in [T1].

In Section 5 below, we give two further applications of the technique of unfoldings. First, in Theorem 1 below, we characterize the regular right-angled buildings $X$ such that $G = \text{Aut}(X)$ is nondiscrete (the lattice theory of $G$ is otherwise trivial). As we recall in Section 1.4, the building $X$ is determined by a right-angled Coxeter system $W = \langle S \mid (st)^{m_{st}} \rangle$, and a family of positive integers $\{q_s\}_{s \in S}$ which give the number of chambers of $X$ which meet at common faces. A polyhedral complex $L$ is said to be rigid if for any $g \in \text{Aut}(L)$, if $g$ fixes the star in $L$ of a vertex $\sigma$ of $L$, then $g$ fixes $L$. We prove:

**Theorem 1.** Let $X$ be a regular right-angled building of type $(W, S)$ and parameters $\{q_s\}$. Let $G = \text{Aut}(X)$ and let $G_0 = \text{Aut}_0(X)$ be the group of type-preserving automorphisms of $X$.

1. If there are $s, t \in S$ such that $q_s > 2$ and $m_{st} = \infty$ then $G_0$ and $G$ are both nondiscrete.
2. If for all $t \in S$ with $q_t > 2$ we have $m_{st} = 2$ for all $s \in S - \{t\}$, then $G_0$ is discrete, and $G$ is nondiscrete if and only if the nerve $L$ of $(W, S)$ is not rigid.

If all $q_s = 2$ then the building $X$ is the Davis complex for $(W, S)$, in which case this result is due to Haglund–Paulin [HP].

The second main result of Section 5 is:

**Theorem 2.** Let $G$ be the automorphism group of a regular right-angled building $X$. Then the action of $G$ on $X$ is strongly transitive.

A group $G$ is said to act strongly transitively on a building $X$ if it acts transitively on the set of pairs $(\phi, \Sigma)$, where $\phi$ is a chamber of $X$, and $\Sigma$ is an apartment of $X$ containing $\phi$ (see Section 1.4). By a theorem of Tits (see [D]), if $X$ is a thick building, it follows that the group $G$ has a $BN$–pair. For example, Bourdon’s building $I_{p,q}$ is thick for all $q \geq 3$. Theorem 2 was sketched for the case $X = I_{p,q}$ by Bourdon in [B, Proposition 2.3.3].

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1. **Background**

In Section 1.1 we briefly describe the natural topology on $G$ the automorphism group of a locally finite polyhedral complex $X$, and characterize uniform lattices in $G$. We present some necessary background on Coxeter groups and Davis complexes in Sections 1.2 and 1.3 respectively, then discuss right-angled buildings in Section 1.4. Next in Section 1.5 we recall the basic theory of complexes of groups, and use this to construct the standard uniform lattice $\Gamma_0$ in the automorphism group of a regular right-angled building $X$. Finally, Section 1.6 contains necessary definitions and results from covering theory for complexes of groups, and Section 1.7 recalls the theory of group actions on complexes of groups.

1.1. **Lattices for polyhedral complexes.** Let $G$ be a locally compact topological group. Recall that a discrete subgroup $\Gamma \leq G$ is a lattice if $\Gamma \backslash G$ carries a finite $G$–invariant measure, and that $\Gamma \leq G$ discrete is a uniform lattice if $\Gamma \backslash G$ is compact.

Let $X$ be a connected, locally finite polyhedral complex, and let $G = \text{Aut}(X)$ be the group of automorphisms, or cellular isometries, of $X$. Then $G$, equipped with the compact-open topology, is a locally compact topological group. A countable neighborhood basis of the identity in $G$ consists of
automorphisms which fix larger and larger combinatorial balls in \( X \). A subgroup \( \Gamma \) of \( G \) is discrete if and only if, for each cell \( \sigma \) of \( X \), the stabilizer \( \Gamma_{\sigma} \) is a finite group. Using a normalization of the Haar measure on \( G \) due to Serre [S], and by the same arguments as for tree lattices (see Chapter 1 of [BL]), if \( G \setminus X \) is compact, then \( \Gamma \leq G \) is a uniform lattice in \( G \) exactly when \( \Gamma \) acts cocompactly on \( X \) with finite cell stabilizers.

1.2. Coxeter groups. We recall some necessary definitions and results. Our notation and terminology in this section mostly follow Davis [D].

A Coxeter group is a group \( W \) with a finite generating set \( S \) and presentation of the form

\[
W = \langle s \in S \mid (st)^{m_{st}} = 1 \rangle
\]

where \( m_{st} = 1 \) for all \( s \in S \), and if \( s \neq t \) then \( m_{st} \) is an integer \( \geq 2 \) or \( m_{st} = \infty \), meaning that there is no relation between \( s \) and \( t \). The pair \( (W, S) \) is called a Coxeter system.

Given a Coxeter system \((W, S)\), a word in the generating set \( S \) is a finite sequence

\[
s = (s_1, \ldots, s_k)
\]

where each \( s_i \in S \). We denote by \( w(s) = s_1 \cdots s_k \) the corresponding element of \( W \). A word \( s \) is said to be reduced if the element \( w(s) \) cannot be represented by any shorter word. Tits proved that a word \( s \) is reduced if and only if it cannot be shortened by a sequence of operations of either deleting a subword of the form \((s, s)\), or replacing an alternating subword \((s, t, \ldots)\) of length \( m_{st} \) by the alternating word \((t, s, \ldots)\) of the same length \( m_{st} \) (see Theorem 3.4.2 [D]). In particular, this implies:

**Lemma 3.** Any word in \( S \) representing some \( w \in W \) must involve all of the elements of \( S \) that are used in any reduced word representing \( w \).

A Coxeter group \( W \), or a Coxeter system \((W, S)\), is said to be right-angled if all \( m_{st} \) with \( s \neq t \) are equal to 2 or \( \infty \). That is, in a right-angled Coxeter system, every pair of generators either commutes or has no relation.

**Examples 1.** Many later definitions and constructions will be illustrated by the following examples of right-angled Coxeter groups.

(1) Let \( W \) be the free product of \( n \) copies of \( \mathbb{Z}/2\mathbb{Z} \). Then \( W \) is a right-angled Coxeter group with presentation

\[
W = \langle s_1, \ldots, s_n \mid s_i^2 = 1 \rangle.
\]

In particular, if \( n = 2 \), then \( W \) is the infinite dihedral group.

(2) Let \( W \) be the free product of \( \mathbb{Z}/2\mathbb{Z} \) with the direct product \((\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z})\). Then \( W \) is a right-angled Coxeter group with presentation

\[
W = \langle s_1, s_2, s_3 \mid s_i^2 = 1, (s_2s_3)^2 = 1 \rangle.
\]

(3) Let \( W \) be the group generated by reflections in the sides of a regular right-angled hyperbolic hexagon. Then \( W \) is a right-angled Coxeter group with presentation

\[
W = \langle s_1, \ldots, s_6 \mid s_i^2 = 1, (s_i s_{i+1})^2 = 1 \rangle
\]

where the subscripts of the \( s_i \) are numbered cyclically.

1.3. Davis complexes. Let \((W, S)\) be a Coxeter system (not necessarily right-angled). In this section we recall the construction of the Davis complex \( \Sigma \) for \((W, S)\), mostly following [D].

For each subset \( T \) of \( S \), we denote by \( W_T \) the special subgroup of \( W \) generated by the elements \( s \in T \). By convention, \( W_\emptyset \) is the trivial group. A subset \( T \) of \( S \) is spherical if \( W_T \) is finite, in which case we say that \( W_T \) is a spherical special subgroup. Denote by \( S \) the set of all spherical subsets of \( S \). Then \( S \) is partially ordered by inclusion. The poset \( S_{\neq \emptyset} \) is an abstract simplicial complex, denoted by \( L \), and called the nerve of \((W, S)\). In other words, the vertex set of \( L \) is \( S \), and a nonempty set \( T \) of vertices spans a simplex \( \sigma_T \) in \( L \) if and only if \( T \) is spherical.

**Examples 2.** The nerves \( L \) of Examples 1 above are as follows.

(1) The \( n \) vertices \( \{s_1\}, \ldots, \{s_n\} \), with no higher-dimensional simplices.
A vertex \( \{s_1\} \), and an edge joining the vertices \( \{s_2\} \) and \( \{s_3\} \).

(3) A hexagon with vertices labeled cyclically \( \{s_1\}, \ldots, \{s_6\} \).

We denote by \( K \) the geometric realization of the poset \( S \). Equivalently, \( K \) is the cone on the barycentric subdivision of the nerve \( L \) of \((W, S)\). Note that \( K \) is compact and contractible, since it is the cone on a finite simplicial complex. Each vertex of \( K \) has type a spherical subset of \( S \), with the cone point having type \( \emptyset \).

For each \( s \in S \) let \( K_s \) be the union of the (closed) simplices in \( K \) which contain the vertex \( \{s\} \) but not the cone point. In other words, \( K_s \) is the closed star of the vertex \( \{s\} \) in the barycentric subdivision of \( L \). Note that \( K_s \) and \( K_t \) intersect if and only if \( m_{st} \) is finite. The family \((K_s)_{s \in S}\) is a mirror structure on \( K \), meaning that \((K_s)_{s \in S}\) is a family of closed subspaces of \( K \), called mirrors.

We call \( K_s \) the \( s \)-mirror of \( K \).

**Lemma 4** (Lemma 7.2.5, [D]). Let \( (W, S) \) be a Coxeter system and let \( K \) be the geometric realization of the poset \( S \) of spherical subsets.

1. For each spherical subset \( T \), the intersection of mirrors \( \cap_{s \in T} K_s \) is contractible.
2. For each nonempty spherical subset \( T \), the union of mirrors \( \cup_{s \in T} K_s \) is contractible.

For any spherical subset \( T \) of \( S \), we call the intersection of mirrors \( \cap_{s \in T} K_s \) a face of \( K \), and the center of this face is the unique vertex of \( K \) of type \( T \). In particular, the center of the \( s \)-mirror \( K_s \) is the vertex \( \{s\} \).

Now define an equivalence relation \( \sim \) on the set \( W \times K \) by \((w, x) \sim (w', x') \) if and only if \( x = x' \) and \( w^{-1}w' \in W_{S(x)} \). The Davis complex \( \Sigma \) for \((W, S)\) is then the quotient space:

\[
\Sigma := (W \times K) / \sim.
\]

The types of vertices of \( K \) induce types of vertices of \( \Sigma \), and the natural \( W \)-action on \( W \times K \) descends to a type-preserving action on \( \Sigma \).

We identify \( K \) with the subcomplex \((1, K)\) of \( \Sigma \). Then \( K \), as well as any one of its translates by an element of \( W \), will be called a chamber of \( \Sigma \). The subcomplexes \( K_s \) of \( K \), or any of their translates by elements of \( W \), will be called the mirrors of \( \Sigma \), and similarly for faces.

**Examples 3.** For Examples 1 above:

1. As shown in Figure 2 for \( n = 3 \), the chamber \( K \) is the cone on \( n \) vertices. The Davis complex \( \Sigma \) is the barycentric subdivision of the \( n \)-regular tree, and its mirrors are the midpoints of the edges of this tree. If \( n = 2 \) then \( \Sigma \) is homeomorphic to the real line.

2. The Davis complex \( \Sigma \) for this example is sketched in Figure 1.2 of [D].

3. The Davis complex \( \Sigma \) for this example is homeomorphic to the barycentric subdivision of the tesselation of the hyperbolic plane by regular right-angled hexagons. The mirrors are the edges of these hexagons.
1.4. Right-angled buildings. We first discuss general chamber systems and buildings in Section 1.4.1, before specializing to the right-angled case in Section 1.4.2. The local structure of right-angled buildings, which is important for our proofs, is described in Section 1.4.3. Again, we mostly follow Davis [D].

1.4.1. Chamber systems and buildings. A chamber system over a set $S$ is a set $\Phi$ of chambers together with a family of equivalence relations on $\Phi$ indexed by the elements of $S$. For each $s \in S$, two chambers are $s$–equivalent if they are equivalent via the equivalence relation corresponding to $s$, they are $s$–adjacent if they are $s$–equivalent and not equal. Two chambers are adjacent if they are $s$–adjacent for some $s \in S$. A gallery in $\Phi$ is a finite sequence of chambers $(\phi_0, \ldots, \phi_k)$ such that $\phi_{j-1}$ is adjacent to $\phi_j$ for $1 \leq j \leq k$. A chamber system is gallery-connected if any two chambers can be connected by a gallery. The type of a gallery $(\phi_0, \ldots, \phi_k)$ is the word $s = (s_1, \ldots, s_k)$, where $\phi_{j-1}$ is $s_j$–adjacent to $\phi_j$ for $1 \leq j \leq k$, and a gallery is minimal if its type is a reduced word.

Definition 5. For $(W, S)$ a Coxeter system, the abstract Coxeter complex $W$ of $W$ is the chamber system with chambers the elements of $W$, and two chambers $w$ and $w'$ being $s$–adjacent, for $s \in S$, if and only if $w' = ws$.

Definition 6. Suppose that $(W, S)$ is a Coxeter system. A building of type $(W, S)$ is a chamber system $\Phi$ over $S$ such that:

1. for all $s \in S$, each $s$–equivalence class contains at least two chambers; and
2. there exists a $W$–valued distance function $\delta : \Phi \times \Phi \to W$, that is, given a reduced word $s = (s_1, \ldots, s_k)$, chambers $\phi$ and $\phi'$ can be joined by a gallery of type $s$ in $\Phi$ if and only if $\delta(\phi, \phi') = w(s) = s_1 \cdots s_k$.

Let $\Phi$ be a building of type $(W, S)$. Then $\Phi$ is spherical if $W$ is finite. The building $\Phi$ is thick if for all $s \in S$, each $s$–equivalence class of chambers contains at least three elements; a building which is not thick is thin. The building $\Phi$ is regular if, for all $s \in S$, each $s$–equivalence class of chambers has the same number of elements.

Example 4. The abstract Coxeter complex $W$ of $W$ is a regular thin building, with $W$–distance function $\delta$ given by $\delta(w, w') = w^{-1}w'$.

Suppose $\Phi$ is a building of type $(W, S)$. An apartment of $\Phi$ is an image of the abstract Coxeter complex $W$, defined above, under a map $W \to \Phi$ which preserves $W$–distances. The building $\Phi$ has a geometric realization, which we denote by $X$, and by abuse of notation we call $X$ a building of type $(W, S)$ as well. By definition of the geometric realization, for each chamber of $\Phi$, the corresponding subcomplex of $X$ is isomorphic to the chamber $K$ defined in Section 1.3 above, and for each apartment of $\Phi$, the corresponding subcomplex of the building $X$ is isomorphic to the Davis complex $\Sigma$ for $(W, S)$. The copies of $\Sigma$ in $X$ are referred to as the apartments of $X$, and the copies of $K$ in $X$ are the chambers of $X$. Note that each vertex of $X$ thus inherits a type $T$ a spherical subset of $S$. The copies of $K_s$, $s \in S$, in $X$ are the mirrors of $X$, so that two chambers in $X$ are $s$–adjacent if and only if their intersection is a mirror of type $s$. The faces of $X$ are its subcomplexes which are intersections of mirrors. Each face has type $T$ a spherical subset of $S$, and a face of type $T$ contains a unique vertex of type $T$, called its center.

The building $X$ may be metrized as follows:

Theorem 7 (Davis, Moussong, cf. Theorems 18.3.1 and 18.3.9 of [D]). Let $(W, S)$ be a Coxeter system and let $X$ be a building of type $(W, S)$.

1. The building $X$ may be equipped with a piecewise Euclidean structure, such that $X$ is a complete CAT(0) space.
2. The building $X$ can be equipped with a piecewise hyperbolic structure which is CAT(–1) if and only if $(W, S)$ satisfies Moussong’s Hyperbolicity Condition:

   a) there is no subset $T \subset S$ such that $W_T$ is a Euclidean reflection group of dimension $\geq 2$; and
   b) there is no subset $T \subset S$ such that $W_T = W_{T'} \times W_{T''}$ for nonspherical subsets $T', T'' \subset S$. 


Unless stated otherwise, we equip buildings $X$ with the CAT(0) metric of Part (1) of Theorem 7.

1.4.2. Right-angled buildings. In this section we specialize to right-angled buildings. A building $X$ of type $(W, S)$ is right-angled if $(W, S)$ is a right-angled Coxeter system. Note that part (2) of Theorem 7 above implies that a piecewise hyperbolic CAT($-1$) structure exists for a right-angled building $X$ if and only if the nerve $L$ has no squares without diagonals (“satisfies the no-□ condition”).

The following result classifies regular right-angled buildings.

**Theorem 8** (Proposition 1.2, [HP]). Let $(W, S)$ be a right-angled Coxeter system and $\{q_s\}_{s \in S}$ a family of cardinalities. Then, up to isometry, there exists a unique building $X$ of type $(W, S)$, such that for all $s \in S$, each $s$–equivalence class of $X$ contains $q_s$ chambers.

In the 2–dimensional case, this result is due to Bourdon [B]. According to [HP], Theorem 8 was proved by M. Globus, and was known also to M. Davis, T. Januszkiewicz, and J. Świątkowski. We will refer to a right-angled building $X$ as in Theorem 8 as a building of type $(W, S)$ and parameters $\{q_s\}$. In Section 1.5 below, we recall a construction, appearing in Haglund–Paulin [HP], of regular right-angled buildings $X$ as universal covers of complexes of groups.

The following definition will be important for our proofs below.

**Definition 9.** Let $X$ be a building of type $(W, S)$. Fix $K$ some chamber of $X$. We define the combinatorial ball $Y_n$ of radius $n$ in $X$ inductively as follows. For $n = 0$, $Y_0 = K$, and for $n \geq 1$, $Y_n$ is the union of $Y_{n-1}$ with the set of chambers of $X$ which have nonempty intersection with $Y_{n-1}$.

**Examples 5.** (1) Let $(W, S)$ be the free product of $n$ copies of $\mathbb{Z}/2\mathbb{Z}$, as in part (1) of Examples 1 above. For $1 \leq i \leq n$ let $q_i = q_{s_i} \geq 2$ be a positive integer. Then the right-angled building $X$ of type $(W, S)$ and parameters $\{q_i\}$ is a locally finite tree. Each mirror $K_i = K_{s_i}$ is a vertex of $X$ of valence $q_i$. The remaining vertices of $X$ are the centers of chambers and have valence $n$. If $n = 2$ then $X$ is the barycentric subdivision of the $(q_1, q_2)$–biregular tree, and each chamber of $X$ is the barycentric subdivision of an edge of this tree. Figure 3 depicts the combinatorial ball $Y_2$ of radius 2 in $X$ for an example with $n = 3$.

![Figure 3](image-url)
In low dimensions, there are right-angled buildings $X$ which are also hyperbolic buildings, meaning that their apartments are isometric to a (fixed) tessellation of hyperbolic space $\mathbb{H}^n$. For this, let $P$ be a compact, convex, right-angled polyhedron in $\mathbb{H}^n$; such polyhedra $P$ exist only for $n \leq 4$, and this bound is sharp (Potyagailo–Vinberg [PV]). Let $(W, S)$ be the right-angled Coxeter system generated by reflections in the codimension one faces of $P$, and let $X$ be a building of type $(W, S)$. By Theorem 7 above, $X$ may be equipped with a piecewise hyperbolic structure which is CAT($-1$). Moreover, in this metric the apartments $\Sigma$ of $X$ are the barycentric subdivision of the tessellation of $\mathbb{H}^n$ by copies of $P$. Thus $X$ is a hyperbolic building. For example, Bourdon’s building $I_{p,q}$ (see [B]) is of type $(W, S)$ and parameters $\{q_s\}$, where $W$ is generated by reflections in the sides of $P$ a regular right-angled hyperbolic $p$–gon ($p \geq 5$), and each $q_s = q \geq 2$. Figure 4 below shows the combinatorial ball $Y_1$ of radius 1 in $X = I_{6,3}$.

1.4.3. **Local structure of right-angled buildings.** In our proofs below, we will rely on the following observations concerning the links of vertices in right-angled buildings.

Let $X$ be a regular right-angled building of type $(W, S)$ and parameters $\{q_s\}_{s \in S}$. Suppose $\sigma$ is a vertex of $X$, of type a maximal spherical subset $T$ of $S$. Then the link of $\sigma$ in $X$, denoted by $\text{Lk}_\sigma(X)$, is the (barycentric subdivision of the) join of $|T|$ sets of points, denoted $V_t$, of cardinalities $|V_t| = q_t$ for each $t \in T$. For example, the link of each vertex of Bourdon’s building $I_{p,q}$ is the complete bipartite graph $K_{q,q}$, which may be thought of as the join of 2 sets of $q$ points. In fact, $\text{Lk}_\sigma(X)$ is a (reducible) spherical building, of type $(W_T, T)$.

Now consider $\phi$ a chamber of $X$ such that the vertex $\sigma$ is in $\phi$. Denote by $k_\phi$ the subcomplex of the link $\text{Lk}_\sigma(X)$ corresponding to simplices in $X$ which are contained in the chamber $\phi$. For example, in Bourdon’s building $I_{p,q}$, $k_\phi$ is an edge of the graph $K_{q,q}$. By abuse of terminology, we call $k_\phi$ a maximal simplex of $\text{Lk}_\sigma(X)$. (This is justified by recalling that the chamber $\phi = K$ is the cone on the barycentric subdivision of the nerve $L$, hence $\phi$ is homeomorphic to the cone on $L$. Moreover, the maximal simplices of $L$ are correspond precisely to the maximal spherical subsets of $S$.)

Two chambers $\phi$ and $\phi'$ of $X$ containing $\sigma$ are adjacent in $X$ if and only if the corresponding maximal simplices $k_\phi$ and $k_{\phi'}$ in $\text{Lk}_\sigma(X)$ share a codimension one face in $\text{Lk}_\sigma(X)$. Hence, a gallery of chambers in $X$, each chamber of which contains $\sigma$, corresponds precisely to a gallery of maximal simplices in the spherical building $\text{Lk}_\sigma(X)$.

1.5. **Basic theory of complexes of groups.** In this section we sketch the theory of complexes of groups, due to Haefliger [BH]. The sequence of examples in this section constructs the regular right-angled building $X$ of Theorem 8 above, as well as the standard uniform lattice $\Gamma_0$ in $\text{Aut}(X)$. 

![Figure 4. The combinatorial ball $Y_1$ of radius 1 in Bourdon’s building $I_{6,3}$.](image-url)
A complex of groups is constructed over a space or set $Y$ belonging to various different categories, including simplicial complexes, polyhedral complexes, or, most generally, scwols (small categories without loops):

Definition 10. A scwol $X$ is the disjoint union of a set $V(X)$ of vertices and a set $E(X)$ of edges, with each edge $a$ oriented from its initial vertex $i(a)$ to its terminal vertex $t(a)$, such that $i(a) \neq t(a)$. A pair of edges $(a, b)$ is composable if $i(a) = t(b)$, in which case there is a third edge $ab$, called the composition of $a$ and $b$, such that $i(ab) = i(b)$, $t(ab) = t(a)$, and if $i(a) = t(b)$ and $i(b) = t(c)$ then $(ab)c = a(bc)$ (associativity).

We will always assume scwols are connected (see Section 1.1, [BH]).

Definition 11. An action of a group $G$ on a scwol $X$ is a homomorphism from $G$ to the group of automorphisms of the scwol (see Section 1.5 of [BH]) such that for all $a \in E(X)$ and all $g \in G$:

1. $g \cdot i(a) \neq t(a)$; and
2. if $g \cdot i(a) = i(a)$ then $g \cdot a = a$.

Suppose $X$ is a right-angled building of type $(W, S)$, as defined in Section 1.4 above. Recall that each vertex $\sigma \in V(X)$ has a type $T \in S$. The edges $E(X)$ are then naturally oriented by inclusion of type. That is, the edge $a$ joins a vertex $\sigma$ of type $T$ to a vertex $\sigma'$ of type $T'$, with $i(a) = \sigma$ and $t(a) = \sigma'$, if and only if $T \subseteq T'$. It is clear that the sets $V(X)$ and $E(X)$ satisfy the properties of a scwol. Moreover, if $Y$ is a subcomplex of $X$, then the sets $V(Y)$ and $E(Y)$ also satisfy Definition 10 above. By abuse of notation, we identify $X$ and $Y$ with the associated scwols. Note that a group of type-preserving automorphisms of $X$ acts according to Definition 11, and that if $G = \text{Aut}(X)$ is not type-preserving we may replace $X$ by a barycentric subdivision, with suitably oriented edges, on which $G$ does act according to Definition 11.

We now define complexes of groups over scwols.

Definition 12. A complex of groups $G(Y) = (G_{\sigma}, \psi_a, g_{a,b})$ over a scwol $Y$ is given by:

1. a group $G_{\sigma}$ for each $\sigma \in V(Y)$, called the local group at $\sigma$;
2. a monomorphism $\psi_a : G_{t(a)} \rightarrow G_{i(a)}$ along the edge $a$ for each $a \in E(Y)$; and
3. for each pair of composable edges, a twisting element $g_{a,b} \in G_{t(a)}$, such that
   $$\text{Ad}(g_{a,b}) \circ \psi_b = \psi_a \circ \psi_b$$
   where $\text{Ad}(g_{a,b})$ is conjugation by $g_{a,b}$ in $G_{t(a)}$, and for each triple of composable edges $a, b, c$
   the following cocycle condition holds:
   $$\psi_a(g_{b,c})g_{a,b,c} = g_{a,b}g_{a,b,c}.$$

A complex of groups is simple if each $g_{a,b}$ is trivial.

Let $X$ be a regular right-angled building of type $(W, S)$ and parameters $\{q_s\}_{s \in S}$, where each $q_s$ is an integer $q_s \geq 2$. We construct $X$ and the standard uniform lattice $\Gamma_0 < \text{Aut}(X)$ using a simple complex of groups $G_X(Y_0)$, which we now define.

Definition 13 (Compare [HP], p. 160). Let $K = Y_0$ be the cone on the barycentric subdivision of the nerve $L$ of $(W, S)$ (see Section 1.3 above). The simple complex of groups $G_X(Y_0)$ over $Y_0$ is defined as follows. For each $s \in S$ let $G_s$ be the cyclic group $\mathbb{Z}/q_s\mathbb{Z}$. The local group at the vertex of type $\emptyset$ of $Y_0$ is the trivial group. The local group at the vertex of type $T$ a nonempty spherical subset of $S$ is defined to be the direct product
   $$G_T := \prod_{s \in T} G_s.$$

All monomorphisms between local groups are natural inclusions, and all $g_{a,b}$ are trivial.

Figures 5 and 6 below show this complex of groups for the right-angled Coxeter systems in Examples 1 above.
Suppose a group $G$ acts on a scwol $X$, as in Definition 11 above. Then the quotient $Y = G \setminus X$ also has the structure of a scwol, and the action of $G$ on $X$ induces a complex of groups $G(Y)$ over $Y$, as follows. Let $p : X \to Y$ be the natural projection. For each $\sigma \in V(Y)$, choose a lift $\overline{\sigma} \in V(X)$ with $p(\overline{\sigma}) = \sigma$. The local group $G_\sigma$ of $G(Y)$ is then defined to be the the stabilizer of $\overline{\sigma}$ in $G$, and the monomorphisms $\psi_\sigma$ and the elements $g_{\sigma,i}$ are defined using further choices. A complex of groups is developable if it is isomorphic (see Definition 14 below) to a complex of groups $G(Y)$ induced by such an action.

Complexes of groups, unlike graphs of groups, are not in general developable. We now discuss a sufficient condition for developability. Let $Y$ be a scwol equipped with the metric structure of a polyhedral complex. An example is $Y$ a subcomplex of a right-angled building $X$. Each vertex $\sigma$ of $Y$ has a local development in $G(Y)$, which is, roughly speaking, a simplicial complex determined combinatorially by the cosets in $G_\sigma$ of the local groups at vertices adjacent to $\sigma$. The local group $G_\sigma$ acts naturally on the local development at $\sigma$, with quotient the star of $\sigma$ in $Y$. (The links of local developments for the complex of groups $G_X(Y_0)$ are described in the next example.) The metric on $Y$ induces a metric on the local development at $\sigma$. We say that $G(Y)$ has nonpositive curvature if, for every $\sigma \in V(Y)$, this induced metric on the local development at $\sigma$ is locally CAT(0). A nonpositively curved complex of groups $G(Y)$ is developable (Theorem 4.17, [BH]).

**Example 6.** We continue the notation of Definition 13 above, and show that $G_X(Y_0)$ is nonpositively curved and thus developable. By Section 4.20 of [BH], it is enough to check that the local development at each vertex $\sigma$ of $Y_0$, of type $T$ a maximal spherical subset of $S$, is locally CAT(0). By Gromov’s Link Condition (see [BH]), for this, it suffices to show that the link of the local development at $\sigma$ in $G_X(Y_0)$ is CAT(1). Now, for each proper subset $T'$ of $T$, there is a unique vertex of $Y_0$ adjacent to $\sigma$ of type $T'$. In particular, for each $t \in T$, there is a unique vertex of $Y_0$ adjacent to $\sigma$ of type $T - \{t\}$. It follows, by the construction of $G_X(Y_0)$ and Section 4.20 of [BH], that the link of the local development at $\sigma$ is the join of $|T|$ sets of points, of respective cardinalities $|G_T/G_{T - \{t\}}| = q_t$. That is, the link of the local development at $\sigma$ is the same as the link of a vertex of type $T$ in the building.
As described in Section 1.4.3 above, the vertices of type $T$ in $X$ have links which are spherical buildings. So these links are CAT(1). Hence $G_X(Y_0)$ is nonpositively curved, and thus developable.

The fundamental group $\pi_1(G(Y))$ of a complex of groups $G(Y)$ is defined so that if $G(Y)$ is a simple complex of groups and $Y$ is simply connected, then $\pi_1(G(Y))$ is isomorphic to the direct limit of the family of groups $G_\sigma$ and monomorphisms $\psi_a$.

**Example 7.** Since the chamber $Y_0 = K$ is contractible, the fundamental group $\Gamma_0 := \pi_1(G_X(Y_0))$ is the graph product of the finite cyclic groups $(G_s)_{s \in S}$. That is, $\Gamma_0$ is the quotient of the free product of the groups $(G_s)_{s \in S}$ by the normal subgroup generated by all commutators of the form $[g_s, g_t]$ with $g_s \in G_s$, $g_t \in G_t$ and $m_{st} = 2$.

If $G(Y)$ is a developable complex of groups, then it has a universal cover $\widetilde{G(Y)}$. This is a connected, simply-connected scwol, equipped with an action of $\pi_1(G(Y))$, so that the complex of groups induced by the action of the fundamental group on the universal cover is isomorphic to $G(Y)$. For each vertex $\sigma$ of $Y$, the star of any lift of $\sigma$ in $\widetilde{G(Y)}$ is isomorphic to the local development of $G(Y)$ at $\sigma$.

**Example 8.** By the discussion in Example 6 above, the complex of groups $G_X(Y_0)$ is developable. By abuse of notation, denote by $X$ the universal cover of $G_X(Y_0)$. Since the vertices of $Y_0$ are equipped with types $T \in S$, the complex of groups $G_X(Y_0)$ is of type $(W, S)$ in the sense defined in Section 1.5 of Gaboriau–Paulin [GP]. As discussed above, the links of vertices of $Y_0$ in their local development are CAT(1) spherical buildings. By an easy generalization of Theorem 2.1 of [GP], it follows that the universal cover $X$ is a building of type $(W, S)$. (Section 3.3 of [GP] treats the case of right-angled hyperbolic buildings.) By construction, the building $X$ is regular, with each mirror of type $s$ contained in exactly $q_s = |G_s|$ distinct chambers. Hence by Theorem 8 above, $X$ is the unique regular right-angled building of type $(W, S)$ and parameters $\{q_s\}$.

Let $G(Y)$ be a developable complex of groups over $Y$, with universal cover $X$ and fundamental group $\Gamma$. We say that $G(Y)$ is faithful if the action of $\Gamma$ on $X$ is faithful, in which case $\Gamma$ may be identified with a subgroup of $\text{Aut}(X)$. If $X$ is locally finite, then with the compact-open topology on $\text{Aut}(X)$, by the discussion in Section 1.1 above the subgroup $\Gamma$ is discrete if and only if all local groups of $G(Y)$ are finite, and a discrete subgroup $\Gamma$ is a uniform lattice in $\text{Aut}(X)$ if and only if the quotient $Y \cong \Gamma \backslash X$ is compact.

**Example 9.** Since the local group in $G_X(Y_0)$ at the vertex of type $\emptyset$ of $Y_0$ is trivial, the fundamental group $\Gamma_0$ acts faithfully on the universal cover $X$. Since $G_X(Y_0)$ is a complex of finite groups, $\Gamma_0$ is discrete, and since $Y_0$ is compact, $\Gamma_0$ is a uniform lattice in $\text{Aut}(X)$.

We call $\Gamma_0$ the standard uniform lattice.

### 1.6. Covering theory for complexes of groups

In this section we state necessary definitions and results from covering theory for complexes of groups. As in Section 1.5 above, all references to [BH] are to Chapter III.C.

We first recall the definitions of morphisms and coverings of complexes of groups. In each of the definitions below, $Y$ and $Z$ are scwols, $G(Y) = (G_\sigma, \psi_a)$ is a simple complex of groups over $Y$, and $H(Z) = (H_\tau, \theta_a, h_{a, \tau})$ is a complex of groups over $Z$. (We will only need morphisms and coverings from simple complexes of groups $G(Y)$.)

**Definition 14.** Let $f : Y \to Z$ be a morphism of scwols (see Section 1.5 of [BH]). A morphism $\Phi : G(Y) \to H(Z)$ over $f$ consists of:

1. a homomorphism $\phi_\sigma : G_\sigma \to H_{f(\sigma)}$ for each $\sigma \in V(Y)$, called the local map at $\sigma$; and
2. an element $\phi(a) \in H_{f(\ell(f(a)))}$ for each $a \in E(Y)$, such that the following diagram commutes

\[
\begin{array}{ccc}
G_{\ell(a)} & \xrightarrow{\psi_a} & G_{f(\ell(a))} \\
\downarrow{\Phi_{\ell(a)}} & & \downarrow{\phi_{f(\ell(a))}} \\
H_{f(\ell(f(a)))} & \xrightarrow{\text{Ad}(\phi(a)) \circ \theta_{f(\ell(a))}} & H_{f(f(\ell(a)))}
\end{array}
\]
and for all pairs of composable edges \((a, b)\) in \(E(Y)\),
\[
\phi(ab) = \phi(a) \psi_n(\phi(b)) h_{f(a), f(b)}.
\]

A morphism is \textit{simple} if each element \(\phi(a)\) is trivial. If \(f\) is an isomorphism of scwols, and each \(\phi_\sigma\) an isomorphism of the local groups, then \(\Phi\) is an \textit{isomorphism of complexes of groups}.

**Definition 15.** A morphism \(\Phi : G(Y) \rightarrow H(Z)\) over \(f : Y \rightarrow Z\) is a covering of complexes of groups if further:

1. each \(\phi_\sigma\) is injective; and
2. for each \(\sigma \in V(Y)\) and \(b \in E(Z)\) such that \(t(b) = f(\sigma)\), the map of cosets

\[
\prod_{a \in f^{-1}(b), t(a) = \sigma} G_\sigma / \psi_n(G_{i(a)}) \rightarrow H_{f(\sigma)} / \theta_n(H_{i(b)})
\]

induced by \(g \mapsto \phi_\sigma(g) \phi(a)\) is a bijection.

We will need the following general result on functoriality of coverings, which is implicit in [BH], and stated and proved explicitly in [LT].

**Theorem 16.** Let \(G(Y)\) and \(H(Z)\) be complexes of groups over scwols \(Y\) and \(Z\) and let \(\Phi : G(Y) \rightarrow H(Z)\) be a covering of complexes of groups. If \(G(Y)\) has nonpositive curvature (hence is developable) then \(H(Z)\) has nonpositive curvature, hence \(H(Z)\) is developable. Moreover, \(\Phi\) induces a monomorphism of fundamental groups
\[
\eta : \pi_1(G(Y)) \rightarrow \pi_1(H(Z))
\]
and an \(\eta\)–equivariant isomorphism of universal covers
\[
\widehat{G(Y)} \rightarrow \widehat{H(Z)}.
\]

See [LT] for the definition of an \(n\)–sheeted covering of complexes of groups, and the result that if \(G(Y) \rightarrow H(Z)\) is an \(n\)–sheeted covering then the monomorphism \(\eta : \pi_1(G(Y)) \rightarrow \pi_1(H(Z))\) in Theorem 16 above embeds \(\pi_1(G(Y))\) as an index \(n\) subgroup of \(\pi_1(H(Z))\).

### 1.7. Group actions on complexes of groups.

The theory of group actions on complexes of groups was introduced in [T2]. Let \(G(Y)\) be a complex of groups. An \textit{automorphism} \(\Phi : G(Y) \rightarrow G(Y)\) is an isomorphism of \(G(Y)\) as a group under composition, denoted \(\text{Aut}(G(Y))\). A group \(H\) \textit{acts on} \(G(Y)\) if there is a homomorphism \(\rho : H \rightarrow \text{Aut}(G(Y))\). If \(H\) acts on \(G(Y)\), then in particular \(H\) acts on the scwol \(Y\) in the sense of Definition 11 above, so we may say that the \(H\)–action on \(Y\) extends to an \textit{action} on \(G(Y)\). Denote by \(\Phi^h\) the automorphism of \(G(Y)\) induced by \(h \in H\). We say that the \(H\)–action on \(G(Y)\) is \textit{by simple morphisms} if each \(\Phi^h\) is a simple morphism.

**Theorem 17** (Thomas, Theorem 3.1 of [T2] and its proof). Let \(G(Y)\) be a simple complex of groups over a connected scwol \(Y\). Suppose that the action of a group \(H\) on \(Y\) extends to an action by simple morphisms on \(G(Y)\). Then the \(H\)–action on \(G(Y)\) induces a complex of groups \(H(Z)\) over \(Z = H \backslash Y\), well-defined up to isomorphism of complexes of groups, such that:

- if \(G(Y)\) is faithful and the \(H\)–action on \(Y\) is faithful then \(H(Z)\) is faithful;
- there is a covering of complexes of groups \(G(Y) \rightarrow H(Z)\); and
- if \(H(Z)\) is developable and \(H\) fixes a point of \(Y\), then \(H \hookrightarrow \pi_1(H(Z))\).

In particular, if the covering \(G(Y) \rightarrow H(Z)\) is finite-sheeted, as occurs for example if \(G(Y)\) is a complex of finite groups over a finite scwol \(Y\), then \(\pi_1(G(Y))\) is a finite index subgroup of \(\pi_1(H(Z))\).
2. Discreteness of normalizers

Let $G$ be the group of automorphisms of a locally finite polyhedral complex $X$ (not necessarily a building), and suppose $G \backslash X$ is compact. In this section we show that for any uniform lattice $\Gamma$ of $G$, the normalizer $N_G(\Gamma)$ is discrete. Recall from Section 1.1 above that a uniform lattice $\Gamma$ in $G = \text{Aut}(X)$ acts cocompactly on $X$, and fix a compact fundamental domain $D$ for this action.

Lemma 18. The centralizer of $\Gamma$ in $G$, denoted $Z_G(\Gamma)$, is discrete in $G$.

Proof. Suppose otherwise. Then there is a sequence $g_k \to \text{Id}_X$ with $\text{Id}_X \neq g_k \in Z_G(\Gamma)$. Since $D$ is compact, it follows that for $k$ sufficiently large $g_k|_D = \text{Id}_D$. Let $x \in X$. As $D$ is a $\Gamma$-fundamental domain, $x \in \gamma D$ for some $\gamma \in \Gamma$, that is, $\gamma^{-1}x \in D$. It follows that $g_k(\gamma^{-1}x) = \gamma^{-1}x$, so $\gamma^{-1}g_kx = \gamma^{-1}x$ as $g_k \in Z_G(\Gamma)$. Thus $g_kx = x$ for all $x \in X$ so $g_k = \text{Id}_X$, a contradiction. □

Proposition 19. The uniform lattice $\Gamma$ is a finite index subgroup of its normalizer $N_G(\Gamma)$. In particular, $N_G(\Gamma)$ is discrete in $G$.

Proof. By Lemma 18, it follows directly from Proposition 6.2(c) of [BL] that $N_G(\Gamma)$ is also discrete. Since $\Gamma < N_G(\Gamma)$, the group $N_G(\Gamma)$ is also a uniform lattice in $G$. The ratio of covolumes of $\Gamma$ and $N_G(\Gamma)$ gives the index of $\Gamma$ in $N_G(\Gamma)$. In particular, this index is finite. □

We now sketch an alternative argument for $N_G(\Gamma)$ being discrete, which was suggested to us by G. Christopher Hruska, and uses the theory of group actions on complexes of groups (Section 1.7 above). A uniform lattice $\Gamma$ of $G = \text{Aut}(X)$ is the fundamental group of a complex of groups $G(Y)$, where $Y = \Gamma \backslash X$ is compact and the local groups of $G(Y)$ are finite. Thus the group $\text{Aut}(G(Y))$ of automorphisms of $G(Y)$ is a finite group. Any element $g \in N_G(\Gamma)$ induces an automorphism of $Y$, and this automorphism extends to an action on the complex of groups $G(Y)$ (not necessarily by simple morphisms). The induced action of $g$ on $G(Y)$ is trivial if and only if $g \in \Gamma$, so we have an isomorphism $N_G(\Gamma)/\Gamma \to \text{Aut}(G(Y))$, hence $N_G(\Gamma)$ is discrete.

3. Unfoldings

We now introduce the technique of “unfolding”, which will be used in our proofs in Sections 4 and 5 below. Let $X$ be a regular right-angled building. We first, in Section 3.1, define clumps, which are a class of subcomplexes of $X$ that includes the combinatorial balls $Y_n \subset X$. For each clump $C$ we then construct a canonical complex of groups $G_X(C)$ over $C$, and we define a clump $C$ to be admissible if $G_X(C)$ is developable with universal cover $X$. In Section 3.2, we define the unfolding of a clump $C$. The main result of this section is Proposition 27, which shows that if $C$ is admissible then any unfolding of $C$ is also admissible. Finally, in Section 3.3, we prove in Proposition 29 that if $C$ is a clump obtained by a finite sequence of unfoldings of the chamber $Y_0$, then there is a covering of complexes of groups $G_X(C) \to G_X(Y_0)$. As a corollary, we obtain a sequence $\Gamma_n$ of uniform lattices in $G = \text{Aut}(X)$ such that each $\Gamma_n$ has fundamental domain $Y_n$, and is of finite index in the standard uniform lattice $\Gamma_0$.

3.1. Complexes of groups over clumps. Let $X$ be a regular right-angled building of type $(W,S)$. In this section, we define clumps, and for each clump $C$ construct a canonical complex of groups $G_X(C)$ over $C$.

We will say that two mirrors of $X$ are adjacent if the face which is their intersection has type $T$ with $|T| = 2$. Since $(W,S)$ is right-angled, it is immediate that:

Lemma 20. If two adjacent mirrors are of the same type, then they are contained in adjacent chambers. If two adjacent mirrors are of different types, then there is a chamber of $X$ which contains both of these mirrors.

Definition 21. Let $X$ be a regular right-angled building of type $(W,S)$.

- A clump in $X$ is a gallery-connected union of chambers $C$ such that at least one mirror of $C$ is contained in only one chamber of $C$.
- The boundary of a clump $C$, denoted $\partial C$, is the union of all the mirrors in $C$ each of which is contained in only one chamber of $C$. 
Two mirrors in \( \partial C \) are type-connected if they are of the same type and are equivalent under the equivalence relation generated by adjacency.

If \( C \) is a clump, then a maximal union of type-connected mirrors in \( \partial C \) will be called a type-connected component or side of \( C \), and the type of the side is the type of the mirrors in the side.

Let \( C \) be a clump in \( X \). For a vertex \( \sigma \in V(C) \) of type \( T \), the boundary type of \( \sigma \) is the subset \( \{ s \in T | \text{an } s\text{-mirror containing } \sigma \text{ is contained in } \partial C \} \).

Note that if \( \sigma \not\in \partial C \), then the boundary type of \( \sigma \) is \( \emptyset \).

We now define a simple complex of groups \( G_X(C) \) over \( C \). For \( s \in S \), let \( G_s \) be the cyclic group \( \mathbb{Z}/q_s\mathbb{Z} \). For a vertex \( \sigma \) in \( C \), we denote by \( G_\sigma(C) \) the local group at \( \sigma \) in \( G_X(C) \). Then \( G_X(C) \) is defined as follows:

- The local group \( G_\sigma(C) \) at each vertex \( \sigma \in C - \partial C \) is trivial.
- The local group \( G_\sigma(C) \) at a vertex \( \sigma \in \partial C \) of boundary type \( T \) is the direct product \( G_T := \prod_{s \in T} G_s \).
- The monomorphisms \( \psi_a \) are natural inclusions, for each edge \( a \) in \( C \).
- The twisting elements \( g_{a,b} \) are all trivial.

A clump \( C \) is admissible if \( G_X(C) \) is developable and its universal cover is (isomorphic to) \( X \). If \( C \) is an admissible clump, then we may identify \( \tilde{C} \) with a fundamental domain in \( X \) for the fundamental group of \( G_X(C) \). The preimage or lift of a vertex \( \sigma \in V(X) \) in \( C \) is the unique vertex \( \sigma' \) of \( \tilde{C} \) which is in the same orbit as \( \sigma \) under the action of the fundamental group of \( G_X(C) \) on \( X \). Lifts of edges and of chambers in \( C \) are defined similarly.

**Example 10.** The chamber \( Y_0 = K \) is an admissible clump since \( G_X(Y_0) \) is precisely the defining complex of groups for the standard uniform lattice \( \Gamma_0 \) (see Section 1.5 above).

**Example 11.** Figure 7 below depicts the complex of groups \( G_X(C) \) over a clump \( C \) in the product \( X = T_{q_s} \times T_{q_t} \) of regular trees of valences \( q_s \) and \( q_t \) respectively. This clump is nonadmissible, since the link of the vertex \( \sigma \) in the local development of \( G_X(C) \) at \( \sigma \) is not a complete bipartite graph, so this link is not the same as the link of a vertex in \( X \).
3.2. Unfolding along a side of an admissible clump. Given an admissible clump, we now define a process, called unfolding, that yields larger admissible clumps. In particular, as shown in Lemma 22 below, by starting with $Y_0$ and iterating this process, one can obtain each of the combinatorial balls $Y_n$. The main result of this section is Proposition 27 below, which shows that if $C$ is admissible then any unfolding of $C$ is admissible. Hence each $Y_n$ is admissible. Let $C$ be an admissible clump in $X$, and let $K$ be a side of $C$ of type $u$. The unfolding of $C$ along $K$ is the combing $U_K(C) := C \cup \{ \text{chambers } \phi \subset X \mid \text{the } u-\text{mirror of } \phi \text{ is contained in } K \}$. 

Lemma 22. The combinatorial ball $Y_n$ can be obtained by a sequence of unfoldings beginning with a base chamber $Y_0$.

Proof. By induction, it suffices to show that the combinatorial ball $Y_n$ can be obtained from $Y_{n-1}$ by a sequence of unfoldings. Let $K_1, K_2, \ldots, K_k$ denote the sides of $Y_{n-1}$. First unfold $Y_{n-1}$ along $K_1$ to obtain a new clump $U_{K_1}(Y_{n-1})$. For $i > 1$, if $K_i$ does not intersect $K_1$, then $K_i$ is also a side of $U_{K_1}(Y_{n-1})$. Otherwise, replace $K_i$ by the side of $U_{K_1}(Y_{n-1})$ containing $K_i$. Then unfold along the (potentially extended) side $K_2$. Iterating this process, the clump $C$ obtained by unfolding along each of the (extended) sides $K_1, K_2, \ldots, K_k$ is the combinatorial ball $Y_n$. Figure 8 below illustrates this process for obtaining $Y_1$ from $Y_0$ in Bourdon’s building $I_{6,3}$. □

We say that a vertex $\sigma$ in a clump $C \subset X$ is fully interior if every chamber in $X$ containing $\sigma$ is in $C$. Note that if some $q_1 > 2$, then a vertex can be in $C - \partial C$ without being fully interior. However, if $C$ is admissible, then since $X$ is the universal cover of $G_X(C)$, but interior local groups in $G_X(C)$ are all trivial, it follows that every interior vertex of $C$ is fully interior.

We call the local development at a vertex $\sigma$ in $G_X(C)$ complete if it is the same as the local development of a vertex of the same type in $G_X(Y_0)$, that is, if it is the star of $\sigma$ in $X$. We note that:

Lemma 23. If $\sigma$ is a vertex of $C$ such that $\sigma$ is contained in only one chamber of $C$, then the local development of $G_X(C)$ at $\sigma$ is complete.

We next prove several lemmas which will be used in this section and in Section 3.3 below.

Lemma 24. Let $C$ be an admissible clump and let $\sigma \in \partial C$ be a vertex of type $T$ and boundary type $T_{\partial C}$. If $s \in T_{\partial C}$, then every mirror of type $s$ in $C$ containing $\sigma$ is actually contained in $\partial C$, and there are exactly $\prod_{t \in T - T_{\partial C}} q_t$ such mirrors.

Proof. Since $C$ is admissible, the local development of $G_X(C)$ at $\sigma$ is complete, so the link $Lk_\partial(C)$ of $\sigma$ in $C$ is the quotient of the link $Lk_\partial(X)$ of $\sigma$ in $X$ by the action of the local group $G_{T_{\partial C}}$. Now, as discussed in Section 1.4.3 above, $Lk_\partial(X)$ is the join of $|T|$ sets of vertices $V_t$ for $t \in T$, of cardinalities respectively $|V_t| = q_t$. By construction of $G_X(C)$, the action of the local group $G_{T_{\partial C}} = \prod_{t \in T_{\partial C}} G_t$ on $Lk_\partial(X)$ is transitive on each set $V_t$ with $t \in T_{\partial C}$, and is trivial on the sets $V_t$ for $t \notin T_{\partial C}$. It follows that $Lk_\partial(C)$ is also a join of $|T|$ sets of vertices: it is the join of a singleton set for each $t \in T_{\partial C}$, along with the sets $V_t$ for $t \notin T_{\partial C}$. For each $s \in T_{\partial C}$, the faces in $Lk_\partial(C)$ corresponding to the $s$-mirrors of $C$ which contain $\sigma$ are precisely those faces in $Lk_\partial(X)$ which are a join of $|T| - 1$ vertices: the singleton sets corresponding to each $t \in T_{\partial C} - \{s\}$, together with one vertex from each of the sets $V_t$ for $t \notin T_{\partial C}$. There are $\prod_{t \in T - T_{\partial C}} q_t$ such faces. Now, by construction of $G_X(C)$, a face $k_s$ in $Lk_\partial(C)$ of type $s \in T_{\partial C}$ corresponds to a mirror in the boundary of $C$ if and only if its stabilizer in $G_{T_{\partial C}}$ is nontrivial. Since the action of $G_{T_{\partial C}}$ fixes each vertex in the sets $V_t$ for $t \notin T_{\partial C}$, it follows that all such mirrors must be on the boundary of $C$. □

Note that Lemma 24 implies that for an admissible clump $C$, the boundary type of a vertex $\sigma$ of type $T$ is actually equal to $\{s \in T \mid \text{all } s-\text{mirrors containing } \sigma \text{ are contained in } \partial C\}$. This is not necessarily true in nonadmissible clumps. For example, in Figure 7, $s$ and $t$ are in the boundary type of $\sigma$ even though neither every $s$- nor every $t$-mirror in $C$ containing $\sigma$ is contained in $\partial C$. 


Suppose $\mathcal{C}'$ is an admissible clump, that $\mathcal{K}$ is a side of $\mathcal{C}'$ of type $u$, and that $\mathcal{C} = U_\mathcal{K}(\mathcal{C}')$. Let $\sigma$ be a vertex in $\partial \mathcal{C}$ of type $T$ and let $\sigma'$ be the lift of $\sigma$ to $\mathcal{C}'$.

**Lemma 25.** If $T_{\partial \mathcal{C}'}$ denotes the boundary type of $\sigma'$ in $\mathcal{C}'$ and $T_{\partial \mathcal{C}}$ is the boundary type of $\sigma$ in $\mathcal{C}$, then $T_{\partial \mathcal{C}'} \subset T_{\partial \mathcal{C}}$.

**Proof.** Suppose $s \in T - T_{\partial \mathcal{C}}$. Then there are at least two $s$–adjacent chambers in $\mathcal{C}$ whose intersection contains $\sigma$. The lifts of these chambers to $\mathcal{C}'$ are then $s$–adjacent chambers in $\mathcal{C}'$ containing $\sigma'$. Hence $s \in T - T_{\partial \mathcal{C}'}$. □

Let $\text{Ch}_\mathcal{K}$ denote the set of chambers in $\mathcal{C} = U_\mathcal{K}(\mathcal{C}')$ that are not also in $\mathcal{C}'$, that is, $\text{Ch}_\mathcal{K}$ is the set of “new chambers” in $\mathcal{C}$. A *sheet* of chambers in $\text{Ch}_\mathcal{K}$ is an equivalence class of chambers under the equivalence relation generated by $S - \{u\}$ adjacency in $\text{Ch}_\mathcal{K}$. So two chambers in $\text{Ch}_\mathcal{K}$ are in the same sheet if and only if there is a gallery of chambers in $\text{Ch}_\mathcal{K}$ such that the type of each adjacency is in $S - \{u\}$.

**Lemma 26.** If $\mathcal{K}$ is a side of $\mathcal{C}'$ of type $u$, there are $q_u - 1$ sheets in $\text{Ch}_\mathcal{K}$.

**Proof.** Choose $K_u \subset \mathcal{K}$ a mirror of type $u$. There are $q_u - 1$ chambers in $\text{Ch}_\mathcal{K}$ glued along $K_u$. Call these chambers $\phi_1, \phi_2, \ldots, \phi_{q_u - 1}$. Since $\mathcal{K}$ is type-connected, any $\phi \in \text{Ch}_\mathcal{K}$ is in the same sheet as
some $\phi_i$. Now suppose there are $1 \leq i \neq j \leq q_u - 1$ such that $\phi_i$ and $\phi_j$ are in the same sheet. Then there is a gallery of chambers in $\text{Ch}_K$ from $\phi_i$ to $\phi_j$ with the type of each consecutive adjacency being an element of $S - \{u\}$. The chambers $\phi_i$ and $\phi_j$ are $u$-adjacent, since they are both glued to the mirror $K_u$, so the sequence $(\phi_i, \phi_j)$ is also a gallery in $X$. By the definition of a building, and, more specifically, using the $W$-valued distance function, it follows that $u$ is equal to a product of elements in $S - \{u\}$. This is a contradiction, since $u \notin W_{S - \{u\}}$. Hence there are exactly $q_u - 1$ sheets in $\text{Ch}_K$, namely the equivalence classes of each of $\phi_1, \phi_2, \ldots, \phi_{q_u - 1}$.

□

We now prove the main result of this section, that unfolding preserves admissibility.

**Proposition 27.** Let $C_0$ be an admissible clump in $X$. If $C$ is a clump obtained from $C_0$ through a finite sequence of unfoldings, then $C$ is an admissible clump.

**Proof.** By induction, it suffices to show that if $C'$ is an admissible clump and $K$ is a side of $C'$ of type $u$, then the clump $C = U_K(C')$ is admissible, that is, that $G_X(C)$ is developable with universal cover $X$. We will show that for each maximal spherical subset $T \subset S$, the local development at each vertex $\sigma \in C$ of type $T$ is complete. It will then follow that $G_X(C)$ is developable with universal cover $X$, by similar arguments to those used for $G_X(Y_0)$ in Section 1.5 above.

Let $\sigma$ be a vertex of $C$ of type $T$ a maximal spherical subset of $S$. If $\sigma \in C' - K$, then the set of chambers in $C$ containing $\sigma$ is the same as the set of chambers in $C'$ containing $\sigma$. Thus the local development of $G_X(C)$ at $\sigma$ is the same as that of $G_X(C')$ at $\sigma$, since the neighboring local groups are also all the same in the two complexes of groups. Hence by induction the local development at $\sigma$ is complete.

Thus it remains to consider the local developments of vertices in the side $K$ of $C'$ and in $C - C'$. We consider separately the three cases:

- **Case 1:** $\sigma \in C - C'$
- **Case 2:** $\sigma \in K - K \cap \partial C$
- **Case 3:** $\sigma \in K \cap \partial C$

as depicted in Figure 9 below.

**Figure 9.** A clump $C'$ in $I_{6,3}$, a side $K$ of $C'$, the unfolding $C = U_K(C')$, and the three cases for vertices on the boundary of $C$ used in the proof of Proposition 27.

**Case 1:** Suppose $\sigma \in C - C'$ is a vertex of type $T$, and let $T_{\partial C}$ be the boundary type of $\sigma$ in $C$. If $\sigma$ is contained in only one chamber of $C$, then by Lemma 23 above we are done. Otherwise, we first prove:

**Lemma 28.** Let $\sigma \in C - C'$ and suppose $\sigma$ is contained in more than one chamber of $C$. Then there is a unique vertex of type $T - T_{\partial C}$ adjacent to $\sigma$ in $C$. 
Proof. Let $\phi_1$ and $\phi_2$ be two chambers of $C$ which contain $\sigma$. Let $\phi'_1$ and $\phi'_2$ be the lifts of $\phi_1$ and $\phi_2$ respectively to $C'$, and let $\sigma'$ be the lift of $\sigma$ to $C'$.

Since $C'$ is admissible, the link $\text{Lk}_{\sigma'}(C')$ of $\sigma'$ in $C'$ is a join. Therefore there is a gallery $\beta'$ in $C'$ from $\phi'_1$ to $\phi'_2$, of type say $T$, such that each chamber of the gallery $\beta'$ contains the vertex $\sigma'$. Without loss of generality, we may assume that $\beta'$ is a minimal gallery. Thus $T$ is a reduced word. The vertex $\sigma'$ is of type $T$, and every chamber in the gallery $\beta'$ contains $\sigma'$, hence every letter in the reduced word $T$ must be an element of $T$.

The two chambers $\phi'_1$ and $\phi'_2$ both contain a mirror in the side $K$. Since $K$ is a type-connected component of mirrors of $C'$, there is also a gallery $\alpha'$ in $C'$ from $\phi'_1$ to $\phi'_2$, such that every chamber in the gallery $\alpha'$ contains a mirror in the side $K$. Let $s'$ be the type of the gallery $\alpha'$. Since $K$ is of type $u$, it follows that every letter in $s'$ commutes with $u$.

We now have two galleries $\alpha'$ and $\beta'$ from $\phi'_1$ to $\phi'_2$ in $C'$, of respective types $s'$ and $T$. By Lemma 3 above, since $T$ is reduced, every letter in $T$ appears in $s'$. Hence every letter in $T$ is contained in $T$ and commutes with $u$.

Now every letter in the type $T$ of $\beta'$ commutes with $u$, and the initial chamber $\phi_1$ of $\beta'$ contains a mirror in the side $K$. So by induction, every chamber in the gallery $\beta'$ contains a mirror in the side $K$.

We claim that every letter in $T$ is actually contained in $T - \text{Lk}_C$, where $\text{Lk}_C$ is the boundary type of $\sigma$ in $C$. So suppose there is some $t \in \text{Lk}_C$ such that $t$ appears in the reduced word $T$. Denote by $T_{\beta'}$ the boundary type of $\sigma'$ in $C'$. By Lemma 25, we have that $T_{\beta'} \subseteq T_{\text{Lk}_C}$. So assume first that $t \in T_{\text{Lk}_C}$. By Lemma 24, since $C'$ is admissible, every mirror of $C'$ of type $t$ which contains $\sigma'$ is in the boundary $\partial C'$. But the gallery $\beta'$ is contained in $C'$, and every chamber in $\beta'$ contains the vertex $\sigma'$, so the gallery $\beta'$ cannot cross any mirror of type $t$ which also contains $\sigma'$. So $t$ cannot be contained in $T_{\text{Lk}_C}$.

We now have $t \in T_{\beta'} - T_{\text{Lk}_C}$. Since $t \in T_{\beta'}$, by definition there must be some chamber $\tilde{\phi}$ of $C$ which contains $\sigma$, such that the $t$-mirror of $\tilde{\phi}$ is only contained in one chamber of $C$. Let $\tilde{\phi}$ be a chamber of $X$ which is $t$-adjacent to $\tilde{\phi}$, and note that $\tilde{\phi}$ is not in $C$. Let $\tilde{\phi}'$ be the lift of $\tilde{\phi}$ to $C'$. Since $\tilde{\phi}$ is in $\text{Ch}_C$, the chambers $\tilde{\phi}$ and $\tilde{\phi}'$ are $u$-adjacent. Since $C'$ is admissible and $t \notin T_{\text{Lk}_C}$, there is a chamber share $\tilde{\phi}'$ of $C'$ such that $\tilde{\phi}'$ is $t$-adjacent to $\tilde{\phi}'$. Now, the letter $t$ commutes with $u$, and $\tilde{\phi}'$ has its $u$-mirror contained in the side $K$ of $C'$. Hence the chamber $\tilde{\phi}'$ also has its $u$-mirror contained in the side $K$. Consider the gallery $(\tilde{\phi}, \tilde{\phi}', \tilde{\phi}')$ in $X$. This gallery has type $(t, u, t)$. Since $t$ commutes with $u$, we have $tut = t^2u = u$. Hence $\tilde{\phi}$ and $\tilde{\phi}'$ are $u$-adjacent. Therefore the $u$-mirror of $\tilde{\phi}$ is contained in $K$. But this implies that $\tilde{\phi}$ is in $C$, a contradiction. We conclude that $t \in T - T_{\text{Lk}_C}$, as claimed.

We now have a minimal gallery $\beta'$ of type $T$ from $\phi'_1$ to $\phi'_2$ in $C'$, such that every chamber in the gallery $\beta'$ contains $\sigma'$, every chamber in $\beta'$ contains a mirror in the side $K$, and every letter in $T$ is contained in $T - T_{\text{Lk}_C}$ and commutes with $u$.

Next consider the gallery $\alpha$ from $\phi_1$ to $\phi_2$ obtained by concatenating the galleries $(\phi_1, \phi'_1)$, $\beta'$ and $(\phi'_2, \phi_2)$. Let $s$ be the type of $\alpha$. Then since every letter in $T$ commutes with $u$,

$$w(s) = u w(t') u = u^2 w(t') = w(t').$$

Since $T$ is a reduced word, it follows that there is a gallery, say $\beta$, in $X$ from $\phi_1$ to $\phi_2$ of type $T$. But every letter in $T$ commutes with $u$, so every chamber in $\beta$ has a mirror contained in the side $K$. Thus the gallery $\beta$ is contained in $C$. That is, there is a minimal gallery $\beta$ from $\phi_1$ to $\phi_2$ in $C$, of type $T$, such that every letter in $T$ is in $T - T_{\text{Lk}_C}$.

Let $\sigma^1_{T - T_{\text{Lk}_C}}$ and $\sigma^2_{T - T_{\text{Lk}_C}}$ be the vertices of types $T - T_{\text{Lk}_C}$ in $\phi_1$ and $\phi_2$ respectively. Then since every letter in $T$ is in $T - T_{\text{Lk}_C}$, every chamber in the gallery $\beta$ contains $\sigma^1_{T - T_{\text{Lk}_C}}$. In particular, the chamber $\phi_2$ contains $\sigma^1_{T - T_{\text{Lk}_C}}$. Hence $\sigma^1_{T - T_{\text{Lk}_C}} = \sigma^2_{T - T_{\text{Lk}_C}}$. We conclude that there is a unique vertex of type $T - T_{\text{Lk}_C}$ adjacent to $\sigma$ in $C$. 

By Lemma 28 and Lemma 24 above, the link $\text{Lk}_\sigma(C)$ is a join of $|T - T_{\text{Lk}_C}|$ sets of vertices $V_t$ of cardinality $q_t$ for each $t \in T - T_{\text{Lk}_C}$, and a singleton $\{v_s\}$ for each $s \in T_{\text{Lk}_C}$. This is
Suppose finally that the lattices obtained via a sequence of unfoldings starting with \( K_0 \). This completes the proof of Proposition 27.

**Case 2:** Suppose \( \sigma \in K \cap (K \cap \partial C) \). Recall that the side \( K \) has type \( u \). Let \( s \in S \) be in the boundary type of \( \sigma \) in \( C' \). Then there is a mirror \( K_s \subset \partial C' \) of type \( s \) containing \( \sigma \). If \( s \neq u \), then \( K_s \subset \partial C' \), so \( \sigma \in \partial C' \), a contradiction. Hence the boundary type of \( \sigma \) in \( C' \) is \( \{ u \} \). So the local group \( G_{\sigma}(C') \) at \( \sigma \) in \( G_X(C') \) is \( G_u \). Note that if the type of \( \sigma \) in \( X \) is also \( \{ u \} \), then \( \sigma \) is the center of a \( u \)-mirror in \( K \), so all the chambers in \( X \) containing \( \sigma \) are in \( C' \), by definition of the unfolding across \( K \). Suppose then that the type of \( \sigma \) is not \( \{ u \} \). Let \( \sigma_u \) be a vertex of type \( u \) in \( C' \) that is adjacent to \( \sigma \). Since \( \sigma_u \) is in \( K \), the local group at \( \sigma_u \) in \( G_X(C') \) is also \( G_u \), so in particular has index 1 in the local group \( G_{\sigma}(C') = G_u \). By induction, the local development at \( \sigma \) in \( G_X(C') \) is complete, so it follows that every vertex of type \( u \) adjacent to \( \sigma \) is in \( C' \). That is, every mirror of type \( u \) containing \( \sigma \) is in \( C' \). Thus every chamber of \( X \) containing \( \sigma \) is either in \( C' \) or is adjacent to \( C' \) along \( K \). Hence every such chamber is contained in \( C \), so \( \sigma \) is fully interior in \( C \), and it follows that the local development of \( G_X(C) \) at \( \sigma \) is complete.

**Case 3:** Suppose finally that \( \sigma \in K \cap \partial C \) and let \( T_{\partial C'} \) be the boundary type of \( \sigma \) in \( C' \). Then the boundary type of \( \sigma \) in \( C \) is \( T_{\partial C} = T_{\partial C'} - \{ u \} \) so its local group in \( G_X(C) \) is \( G_{T_{\partial C}} = G_{T_{\partial C'}} / G_u \). Now, since interior vertices of \( C \) have trivial local groups in \( G_X(C) \), the number of chambers in the local development of \( G_X(C) \) at \( \sigma \) is

\[
|G_{T_{\partial C}}| \cdot \# \{ \text{chambers in } C \text{ containing } \sigma \}.
\]

By Lemma 24, the number of chambers in the admissible clump \( C' \) containing \( \sigma \) is \( |G_{T_{\partial C'}}| \). So by unfolding, we see that there are precisely \( q_u \cdot |G_{T_{\partial C'}}| \) chambers in \( C \) containing \( \sigma \).

It follows that the number of chambers in the local development of \( G_X(C) \) at \( \sigma \) is precisely \( |G_T| \). In fact, we can describe the local structure at \( \sigma \).

Since \( C' \) is admissible, the link \( Lk_\sigma(C') \) of \( \sigma \) in \( C' \) is \( G_{T_{\partial C}} \setminus Lk_\sigma(X) \). This is the join of the sets \( V_t \) for \( t \in T - T_{\partial C'} \) and singletons \( \{ v_t \} \) for \( t \in T_{\partial C'} \). Since the local construction of \( C \) from \( C' \) at \( \sigma \) consists of adding \( q_u - 1 \) chambers along each \( u \)-mirror in \( K \) containing \( \sigma \), it follows that the link \( Lk_\sigma(C) \) of \( \sigma \) in \( C \) is as in Lemma 24 above; it is the join of the \( |T| \) sets of vertices \( V_t \) for \( t \in T - T_{\partial C} \) and \( \{ v_t \} \) for \( t \in T_{\partial C} \). It follows that the local development of \( G_X(C) \) at \( \sigma \) is complete, as required.

This completes the proof of Proposition 27.

### 3.3. Unfoldings of \( G_X(Y_0) \) cover \( G_X(Y_0) \)

Recall from Section 1.5 above that the standard uniform lattice \( \Gamma_0 \) is the fundamental group of the complex of groups \( G_X(Y_0) \) over a single chamber \( Y_0 \). In this section, we show that uniform lattices obtained via a sequence of unfoldings starting with \( G_X(Y_0) \) are finite index subgroups of \( \Gamma_0 \). The main result is the following proposition:

**Proposition 29.** Let \( C_0 = Y_0 \), and suppose that, for all \( r > 0 \), \( C_r \) is a clump obtained by unfolding \( C_{r-1} \) along a side \( K_{r-1} \). Then there is a covering of complexes of groups \( G_X(C_r) \to G_X(C_0) \). In particular, the fundamental group of \( G_X(C_r) \) is a finite index subgroup of \( \Gamma_0 \).

By Lemma 22 above, the combinatorial balls \( Y_n \subset X \) can be obtained by a sequence of unfoldings of \( Y_0 \). Let \( \Gamma_n \) be the fundamental group of \( G_X(Y_n) \). Then \( \Gamma_n \) is a uniform lattice in \( \text{Aut}(X) \), and Proposition 29 immediately implies:

**Corollary 30.** The lattices \( \Gamma_n \) are finite index subgroups of \( \Gamma_0 \).

A key step in the proof of Proposition 29 is provided by Proposition 31 below, the proof of which is at the end of this section. It will be convenient to think of all groups \( G_T \) for \( T \subset S \) as natural subgroups of the direct product \( G_S := \prod_{s \in S} G_s \).

**Proposition 31.** Let \( C_r \) be as in Proposition 29 above. Let \( p : C_r \to C_0 \) be the natural morphism of scwols which sends a vertex of \( C_r \) to the unique vertex of \( C_0 \) of the same type. Then there is an edge
For each pair of composable edges \( \lambda : E(C_r) \to G_S = \prod_{s \in S} G_s \) satisfying all of the following:

1. \( \lambda(a) \in G_{p(t(a))} \) for each \( a \in E(C_r) \).
2. For each pair of composable edges \( (a, b) \) in \( E(C_r) \),
   \[ \lambda(ab) = \lambda(a)\lambda(b). \]
3. For each \( \sigma \in V(C_r) \) and \( b \in E(C_r) \) such that \( t(b) = p(\sigma) \), the map
   \[ \prod_{a \in p^{-1}(b)} G_{\sigma(C_r)/G_{\sigma(a)}(C_r)} \to G_{p(\sigma)}(C_0)/G_{\sigma(b)}(C_0) \]
   induced by \( g \mapsto g\lambda(a) \) is a bijection.

Proof of Proposition 29. We construct a covering \( \Lambda : G_X(C_r) \to G_X(C_0) \) over the natural morphism \( p : C_r \to C_0 \). The local maps \( \lambda_\sigma \) are defined to be the identity map (if \( \sigma \) is of type the empty set, or if the boundary type of \( \sigma \) equals its regular type), or natural inclusions (if \( \sigma \) is an interior vertex of type \( T \) not the empty set, or if the boundary type of \( \sigma \) is a proper subset of its regular type). Note that the maps \( \lambda_\sigma \) so defined are injective; by abuse of notation, we write \( g \) for \( \lambda_\sigma(g) \).

We now use the edge labeling \( \lambda \) provided by Proposition 31 above to complete the definition of \( \Lambda \). Since all local groups are abelian and the local maps \( \lambda_\sigma \) are the identity or natural inclusions, the morphism diagram (see (2) of Definition 14) commutes no matter what the value of the \( \lambda(a) \). From the properties of \( \lambda \) guaranteed by Proposition 31, it thus follows that \( \Lambda \) is a covering of complexes of groups.

Proof of Proposition 31. We proceed by induction on \( r \) and write \( \lambda^r \) for the labeling of the edges of \( C_r \). See Figure 10 for an example. Given an edge \( a \in E(C_r) \) such that \( t(a) \) is a vertex of type \( T \), we will choose an element \( \lambda^r(a) \) of \( G_T \subset G_S \). Recall that \( G_T \) is the direct product of the cyclic groups \( G_t \) for \( t \in T \). So, we can think of an element of \( G_T \) as an ordered \( |T| \)-tuple of elements of the cyclic groups \( G_t \). To define \( \lambda^r(a) \), it thus suffices to define elements \( \lambda^r_t(a) \in G_t \) for each \( t \in T \). We will refer to \( \lambda^r_t(a) \) as the \( t \)-component of \( X'(a) \).

To begin the induction, let \( a \in E(C_0) \), with \( t(a) \) of type \( T \). We define \( \lambda^0(a) \) to be the identity element in \( G_T \). Properties (1)–(3) in the statement of Proposition 31 then hold trivially for \( r = 0 \) with this labeling.
Suppose now that we inductively have a labeling $\lambda^{r-1}$ of $E(C_{r-1})$ satisfying properties (1)–(3) in the statement of Proposition 31, and suppose $C_r$ is obtained from $C_{r-1}$ by unfolding along a side $K = K_{r-1}$ of type $u$. We first use the labeling $\lambda^{r-1}$ to label the edges of $C_{r-1} \subset C_r$. That is, for all $a \in E(C_{r-1})$, define $\lambda^r(a) := \lambda^{r-1}(a)$.

Next, since $C_{r-1}$ is admissible, as in the proof of admissibility of unfoldings (Proposition 27 above), we may think of $C_r$ as a subcomplex of the universal cover of $G_X(C_{r-1})$. For each edge $a \in E(C_r) - E(C_{r-1})$ there is then a unique preimage $a' \in E(C_{r-1})$. Define $\lambda^r(a) := \lambda^{r-1}(a')$. If $t(a) \notin K$ then this is the labeling we choose for $a$, that is, we set $\lambda^r(a) := \hat{\lambda}^r(a)$. If $t(a) \in K$ but $i(a) \notin K$, then for $s \in S - \{u\}$ we define the $s$–component of $\lambda^r(a)$ to be the same as that of $\hat{\lambda}^r(a)$. The $u$–component $\lambda_u^r(a)$ is then defined as follows.

Choose $K_u \subset K$ a mirror of type $u$ and let $c \in E(C_{r-1})$ be the edge with initial vertex of type $\emptyset$ and terminal vertex the center of $K_u$. Put $g = \lambda_u^{r-1}(c) \in G_u$, that is, $g$ is the $u$–component of $\lambda^{r-1}(c)$. There are $q_u - 1$ chambers in $Ch_K$ glued along $K_u$. Call these chambers $\phi_1, \phi_2, \ldots, \phi_{q_u-1}$. For each of these new chambers, we assign distinct elements of $G_u - \{g\}$, say $g_i$ is assigned to $\phi_i$ for $1 \leq i \leq q_u - 1$, so that $G_u - \{g\} = \{g_1, 1 \leq i \leq q_u - 1\}$. Now, for all edges $a \in E(\phi_j)$ such that $t(a) \in K_u$ but $i(a) \notin K_u$, we define the $u$–component $\lambda_u^r(a) := g_j$. We then extend these $u$–components along the $q_u - 1$ sheets of new chambers described in Lemma 26 above. That is, for a chamber $\phi \in Ch_K$ in the same sheet as $\phi_j$, and for $a \in E(\phi)$ such that $t(a) \in K$ but $i(a) \notin K$, we define $\lambda_u^r(a) := g_j$.

We must verify that this is well-defined. Suppose $a \in E(\phi) \cap E(\phi')$ for some other $\phi' \in Ch_K$. We will show that $\phi$ and $\phi'$ are in the same sheet. Consider the link $Lk_{i(a)}(C_r)$ of $i(a)$ in $C_r$. As in the proof of Proposition 27 above, since $C_r$ is admissible, this is the join of sets of vertices. In particular, the chambers $\phi$ and $\phi'$ correspond to maximal simplices $k_\phi$ and $k_{\phi'}$ in this join. A gallery in $C_r$ from $\phi$ to $\phi'$ and containing $i(a)$ then corresponds to a sequence of maximal simplices in $Lk_{i(a)}(C_r)$ from $k_\phi$ to $k_{\phi'}$, which sequentially intersect along codimension one faces, that is, to a gallery in $Lk_{i(a)}(C_r)$. Such a sequence exists since $Lk_{i(a)}(C_r)$ is a join. Hence there is a gallery in $C_r$ from $\phi$ to $\phi'$ each chamber of which contains the vertex $i(a)$. Since $i(a) \notin K$, this gallery cannot cross $K$. It follows that $\phi$ and $\phi'$ are in the same sheet, as required. Thus our assignment of the $u$–component of $\lambda^r(a)$, for edges $a \in Ch_K$ with $t(a) \in K$ but $i(a) \notin K$, is well-defined.

This completes the definition of the labeling $\lambda^r$. We now verify that $\lambda^r$ satisfies properties (1)–(3) in the statement of Proposition 31.

For (1), suppose $a \in E(C_r)$. That $\lambda^r(a) \in G_{\rho(t(a))}$ follows immediately from the above construction.

For (2), for each pair of composable edges $(a, b)$ in $E(C_r)$ we must show that $\lambda^r(ab) = \lambda^r(a)\lambda^r(b)$. If both $a$ and $b$ are in $E(C_{r-1})$, then this follows by induction. Since pairs of composable edges occur in chambers, the only other possibility is that $a$ and $b$ are edges in the same chamber in $Ch_K$. It suffices to check that $\lambda^r_s(ab) = \lambda^r_s(a)\lambda^r_s(b)$ for all $s \in S$. Let $a'$ and $b'$ be the preimages of $a$ and $b$ in $E(C_{r-1})$. By induction, $\lambda^{r-1}(a'b') = \lambda^{r-1}(a')\lambda^{r-1}(b')$. The only possible difference between the labels $\lambda^r(a)$ and $\lambda^{r-1}(a')$ is in the $u$–component, and similarly for $b$ and $ab$ (recall that the side $K$ along which we unfolded is of type $u$). Hence it suffices to show that $\lambda^r_u(ab) = \lambda^r_u(a)\lambda^r_u(b)$.

By construction of $\lambda^r$, the only edges whose labels have different $u$–components from those of their preimages are edges with terminal but not initial vertex in $K$. For these edges, we have shown that the $u$–component is determined by the chamber containing the edge. Moreover, for a pair of composable edges $(a, b)$, either none of $a$, $b$, and $ab$ have terminal but not initial vertex in $K$, or $ab$ and exactly one of $a$ and $b$ do. In the latter case, by construction, the $u$–component of $ab$ is equal to the $u$–component of the other edge $(a$ or $b$ but not both) with terminal but not initial vertex in $K$. It follows that $\lambda^r(ab) = \lambda^r(a)\lambda^r(b)$, as required.
Finally, for property (3) in the statement of Proposition 31, we show that for each \( \sigma \in V(C_r) \) and \( b \in E(C_0) \) such that \( t(b) = p(\sigma) \), the map of cosets

\[
\nu_\sigma = \nu_{r,\sigma} : \left( \prod_{a \in p^{-1}(b)} G_{\sigma}(C_r)/G_{i(a)}(C_r) \right) \rightarrow G_{p(\sigma)}(C_0)/G_{i(b)}(C_0)
\]

induced by \( g \mapsto g\lambda^r(a) \) is a bijection. For this, we assume that:

\( \sigma \) has type \( T \) and \( i(b) \) has type \( U \).

So the codomain of \( \nu_\sigma \) is \( G_T/G_U \), and if \( a \in p^{-1}(b) \) then \( i(a) \) has type \( U \).

Suppose \( \sigma \in C_{r-1} - \mathcal{K} \). Then by induction and the construction of \( \lambda^r \), \( \nu_\sigma = \nu_{r,\sigma} \) is bijective.

Suppose next that \( \sigma \in \mathcal{K} \). Recall that \( \mathcal{K} \) is a side of type \( U \). If \( T_{\partial C_r} \) is the boundary type of \( \sigma \) in \( C_{r-1} \) and \( T_{\partial C_r} \), the boundary type of \( \sigma \) in \( C_r \), then \( T_{\partial C_r} \cup \{ u \} = T_{\partial C_{r-1}} \). Hence for all \( \sigma \in \mathcal{K} \),

\[
G_{\sigma}(C_r) \times G_u = G_{\sigma}(C_{r-1}).
\]

Denote by \( p_{r-1} : C_{r-1} \rightarrow C_0 \) and \( p_r : C_r \rightarrow C_0 \) the natural type-preserving morphisms of sewols.

Assume first that \( u \in U \). Then by Lemma 24 above, since \( C_{r-1} \) is admissible,

\[
\{ a \in p_{r-1}(b) \mid t(a) = \sigma \} = \{ a \in p_r^{-1}(b) \mid t(a) = \sigma \} \subset \mathcal{K}.
\]

For all edges \( a \) in this set, by construction \( \lambda^r(a) = \lambda^{r-1}(a) \) and

\[
G_{i(a)}(C_r) \times G_u = G_{i(a)}(C_{r-1}).
\]

By induction, the map \( \nu_{r-1,\sigma} \) is bijective. Therefore by Equations (1) and (2) it follows that \( \nu_{r,\sigma} \) is bijective, as required.

Now assume that \( u \notin U \). Then for all \( a \in p_r^{-1}(b) \) with \( t(a) = \sigma \), we have \( i(a) \notin \mathcal{K} \). Consider an edge \( a' \in p_r^{-1}(b) \subset C_{r-1} \) with \( t(a') = \sigma \). Since \( i(a') \in C_{r-1} - \mathcal{K} \), we now have \( G_{i(a')}(C_r) = G_{i(a')}(C_{r-1}) \). After unfolding, there are \( q_u - 1 \) images of \( a' \) in \( Ch_\mathcal{K} \), which we denote by \( a_2, \ldots, a_{q_u} \). Put \( a' = a_1 \). Then by construction, \( G_u = \{ \lambda_u(a_1), \lambda_u(a_2), \ldots, \lambda_u(a_{q_u}) \} \), and for each \( 1 \leq j \leq q_u \) we have \( G_{i(a_j)}(C_r) = G_{i(a_j)}(C_{r-1}) \). Using Equation (1) above, there is thus a natural bijection

\[
\zeta_{a'} : \left( \prod_{j=1}^{q_u} G_{\sigma}(C_r)/G_{i(a_j)}(C_r) \right) \rightarrow G_{\sigma}(C_{r-1})/G_{i(a')}(C_{r-1})
\]

induced by \( g \mapsto g\lambda_u(a_j) \). Note also that, by construction of the labeling \( \lambda^r \), we have

\[
\lambda^r(a_j) = \lambda_u(a_j)\lambda^{r-1}(a')g_j
\]

for some element \( g_j \in G_u \).

Let \( \zeta \) be the disjoint union of the maps \( \{ \zeta_{a'} \mid a' \in p_r^{-1}(b), t(a') = \sigma \} \). Then \( \zeta \) is a bijection from the domain of \( \nu_{r,\sigma} \) to the domain of \( \nu_{r-1,\sigma} \). By induction \( \nu_{r-1,\sigma} \) is bijective. By Equation (3) above, the map \( \nu_{r,\sigma} \) factors through \( \zeta \). Hence \( \nu_{r,\sigma} \) is bijective, as required.

We have now proved that \( \nu_{r,\sigma} \) is a bijection for all \( \sigma \in C_{r-1} \). For \( \sigma \in C_r - C_{r-1} \), let \( \sigma' \) denote the unique preimage of \( \sigma \) in \( C_{r-1} \).

If \( \sigma \in C_r - (C_{r-1} \cup \partial C_r) \), then the local structure at \( \sigma \) in \( C_r \) (meaning the set of edges with terminal vertex \( \sigma \), the local groups at the initial vertices of these edges, and the labels of these edges) is identical to that at \( \sigma' \) in \( C_{r-1} \). It follows by induction that \( \nu_{r,\sigma} \) is bijective.

It remains to prove that \( \nu_{r,\sigma} \) is bijective for \( \sigma \in \partial C_r - (\partial C_r \cap C_{r-1}) \). (Note that \( \sigma \) is the same kind of vertex as in Case 1 in the proof of Proposition 27 above.) Let \( T_{\partial C_r} \) be the boundary type of \( \sigma \) in \( C_r \).

**Lemma 32.** Suppose \( a \in E(C_r) \) with \( t(a) = \sigma \), and that \( i(a) \) is of type \( U' \) where \( U \subset U' \subset T \).
Then the boundary type $U'_{\partial C_r}$ of $i(a)$ in $C_r$ is given by

$$U'_{\partial C_r} = U' \cap T_{\partial C_r}.$$ 

In particular, for all such edges $a$, the local group at $i(a)$ in $G_X(C_r)$ is the same.

Proof. If $U'$ is the empty set then the boundary type $U'_{\partial C_r} \subset U'$ is also empty and we are done. So suppose there is some $s \in U'$. Then there is an $s$–mirror $K_s$ in $C$, which contains $i(a)$. Since $U' \subset T$ and $t(a) = \sigma$, the mirror $K_s$ also contains $\sigma$. By Lemma 24, since $C_r$ is admissible, $s$ is in $T_{\partial C_r}$ if and only if $K_s \subset \partial C_r$. It follows that $s$ is in the boundary type of $i(a)$ if and only if $s$ is also in the boundary type of $\sigma$. □

Lemma 33. Let $U' = (T - T_{\partial C_r}) \cup U$. Then there is a unique vertex of type $U'$ in $C_r$ adjacent to $\sigma$.

Proof. Since $U' \subset T$ and $C_r$ is a gallery-connected union of chambers, there is at least one such vertex, say $\tau$. By definition, the local group at $\sigma$ in $G_X(C_r)$ is $G_{T_{\partial C_r}}$, and the local group at $\tau$ in $G_X(C_r)$ is $G_{U_{\partial C_r}}$. Since $G_X(C_r)$ is admissible, there are thus vertices of type $U'$ adjacent to $\tau$ in $C_r$. But by admissibility of $G_X(C_0)$, the total number of vertices of type $U'$ adjacent to $\sigma$ in $X$ is $|G_T/G_U| = \prod_{s \in T - U'} q_s$. Since by Lemma 32 above

$$T - U' = T_{\partial C_r} - U'_{\partial C_r},$$

it follows that $\tau$ is unique. □

For a subset $R \subset S$, the projection of an element $g \in G_S$ to $R$, or the $R$–projection of $g$, is the projection of the ordered $|S|$–tuple $g$ to the components corresponding to $R$. To simplify notation, write $p = p_r : C_r \to C_0$.

Lemma 34. The map $\nu_{r, \sigma}$ is bijective if and only if the set of labels

$$\{\lambda'(a) \mid a \in p^{-1}(b), t(a) = \sigma\}$$

has pairwise distinct projections to $T - (T_{\partial C_r} \cup U)$.

Proof. By admissibility of $C_r$, the two sets

$$\prod_{a \in p^{-1}(b)} G_{i(a)}(C_r)/G_{i(a)}(C_r)$$

are finite sets of the same size. So $\nu_r = \nu_{r, \sigma}$ is a bijection if and only if it is injective. Note that $G_r(C_r) = G_{T_{\partial C_r}}$, and that by Lemma 32 above, for all $a \in p^{-1}(b)$ with $t(a) = \sigma$, we have $G_{i(a)}(C_r) = G_{U_{\partial C_r}} = G_{U'_{\partial C_r}}$.

Let $a_1$ and $a_2$ be distinct edges in $p^{-1}(b)$ with $t(a_1) = t(a_2) = \sigma$. Suppose $g, g' \in G_r(C_r)$. Now $\nu_r(gG_{i(a_1)}(C_r)) = \nu_r(g'G_{i(a_2)}(C_r))$ if and only if $g\lambda'(a_1)G_U = g'\lambda'(a_2)G_U$. Since $G_T$ is abelian, this equality of cosets holds if and only if $g = g'g^{-1}\lambda'(a_1)g'\lambda'(a_2)G_U$. But $G_{U'_{\partial C_r}}$ is empty and we are done. So $\nu_r$ is injective, then in particular, putting $g' = 1$, it follows that for all $g \in G_{T_{\partial C_r}}$, we have $\lambda'(a_1)g\lambda'(a_2) \notin gG_U$. Hence $\lambda'(a_1)\lambda'(a_2) \notin G_{T_{\partial C_r} \cup U}$. That is, $\lambda'(a_1)$ and $\lambda'(a_2)$ have distinct $T - (T_{\partial C_r} \cup U)$ projections.

Conversely, suppose $\nu_r$ is not injective. Then there are edges $a_1$ and $a_2$ and elements $g, g' \in G_{T_{\partial C_r}}$ such that $\lambda'(a_1)g\lambda'(a_2) \notin gG_U$. Then $\lambda'(a_1)\lambda'(a_2) \notin G_{T_{\partial C_r} \cup U}$ and so the two labels $\lambda'(a_1)$ and $\lambda'(a_2)$ have the same $T - (T_{\partial C_r} \cup U)$ projections. □
Thus to prove that $\nu_r$ is a bijection, it suffices by Lemma 34 to show that for each pair of distinct edges $a_1, a_2 \in E(C_r)$ with $p(a_1) = p(a_2) = b$ and $t(a_1) = t(a_2) = \sigma$, the labels $\lambda'(a_1)$ and $\lambda'(a_2)$ have distinct $T - (T_{\partial C_r} \cup U)$ projections. Let $a_1', a_2'$ and $\sigma'$ be the lifts of $a_1$, $a_2$, and $\sigma$, respectively, to $C_{r-1}$, and let $T_{\partial C_{r-1}}$ be the boundary type of $\sigma'$ in $C_{r-1}$. By induction and Lemma 34, the $T - (T_{\partial C_{r-1}} \cup U)$ projections of $\lambda^{-1}(a_1')$ and $\lambda^{-1}(a_2')$ are distinct. Since $\sigma \notin K$, the labels $\lambda'(a_1)$ and $\lambda'(a_2)$ are the same as the labels $\lambda^{-1}(a_1')$ and $\lambda^{-1}(a_2')$, respectively. Hence the $T - (T_{\partial C_{r-1}} \cup U)$ projections of $\lambda'(a_1)$ and $\lambda'(a_2)$ are distinct.

Now let $\tau$ be the unique vertex type of $U' = (T - T_{\partial C_r}) \cup U$ in $C_r$ adjacent to $\sigma$, as guaranteed by Lemma 33 above. Let $d$ be the edge of $C_r$ with $i(d) = \tau$ and $t(d) = \sigma$. Since $U \subset U' \subset T$, there are edges $c_1$ and $c_2$ of $C_r$ such that $i(c_1) = i(a_1)$ and $i(c_2) = i(a_2)$ are vertices of type $U$, and $t(c_1) = t(c_2) = \tau$ is of type $U'$. We then have compositions of edges $a_1 = dc_1$ and $a_2 = dc_2$, so by the already proved property (2) of the labeling $\lambda'$, we find that $\lambda'(a_1) = \lambda'(d)\lambda'(c_1)$ and $\lambda'(a_2) = \lambda'(d)\lambda'(c_2)$. Thus $\lambda'(a_1)\lambda'(a_2)^{-1} = \lambda'(c_1)\lambda'(c_2)^{-1} \in G_{U'}$. Note that by definition of $U'$ and Lemma 32, $T_{\partial C_r} \cap U' = T_{\partial C_r} \cap U = U_{\partial C_r}$. So $\lambda'(a_1)$ and $\lambda'(a_2)$ have the same $T - U' = T_{\partial C_r} - U_{\partial C_r}$ projections. Since they have different $T - (T_{\partial C_{r-1}} \cup U)$ projections, it follows from Lemma 25 above that they have different $T - (T_{\partial C_r} \cup U)$ projections, as required.

This completes the proof of Proposition 31.  

\[ \Box \]

4. Proof of the Density Theorem

We are now ready to complete the proof of the Density Theorem. The main results we use are those of Sections 1.6 and 1.7 above, on coverings of complexes of groups and group actions on complexes of groups, and those of Section 3 above, on unfoldings. After proving the Density Theorem, in Section 4.2 below we sketch how these techniques may be used to construct uniform lattices in addition to those needed for the proof.

4.1. Proof of the Density Theorem. Let $X$ be a regular right-angled building of type $(W, S)$ with parameters $\{q_s\}$ (see Section 1.4). Let $G = Aut(X)$ and let $\Gamma_0 \leq G$ be the standard uniform lattice (see Section 1.5). Let $Y_n$ be the combinatorial ball in $X$ of radius $n \geq 0$, and let $x_0$ be the center of the chamber $Y_0$. We first establish the following reduction:

**Lemma 35.** To prove the Density Theorem, it suffices to show that for any $g \in Stab_G(x_0)$, and for any integer $n \geq 0$, there is a $\gamma = \gamma_n \in Comm_G(\Gamma_0)$ such that

$$g|_{Y_n} = \gamma|_{Y_n}.$$  

**Proof.** Let $G_X(Y_0)$ be the complex of groups defined in Section 1.4 above, with fundamental group $\Gamma_0$. Let $G_0 = Aut_0(X)$ be the group of type-preserving automorphisms of $X$. Then $G_0 \backslash X$ is the chamber $Y_0$. With the piecewise Euclidean metric on $X$ provided by Theorem 7 above, the action of the full automorphism group $G$ on $X$ must preserve the cardinality of types of faces in $X$. Hence the quotient $G \backslash X$ is a further quotient of $Y_0$, by the action of the finite (possibly trivial) group of permutations $H := \{ \varphi \in \text{Sym}(S) \mid q_{\varphi(s)} = q_s \text{ and } m_{\varphi(s)\varphi(t)} = m_{st} \text{ for all } s, t \in S \}$.

Let $Z_0 = H \backslash Y_0 = G \backslash X$. By construction of $G_X(Y_0)$, the action of $H$ on $Y_0$ naturally extends to an action by simple morphisms on the complex of groups $G_X(Y_0)$. Let $H(Z_0)$ be the complex of groups induced by the $H$–action on $G_X(Y_0)$. Let $\Gamma_0'$ be the fundamental group of $H(Z_0)$. By Theorem 17 above, there is an induced finite-sheeted covering of complexes of groups $G_X(Y_0) \to H(Z_0)$. Hence by covering theory for complexes of groups, $\Gamma_0'$ is a finite index subgroup of $\Gamma_0$. In particular, $\Gamma_0'$ is commensurable to $\Gamma_0$. So $Comm_G(\Gamma_0') = Comm_G(\Gamma_0)$.

Since $G \backslash X = Z_0 = \Gamma_0' \backslash X$, it follows that

$$G \backslash X = Comm_G(\Gamma_0') \backslash X = Comm_G(\Gamma_0) \backslash X.$$  

Thus we have equality of orbits $G \cdot x_0 = Comm_G(\Gamma_0) \cdot x_0$, and so

$$G = Comm_G(\Gamma_0) \cdot \text{Stab}_G(x_0).$$
Hence to show that $\text{Comm}_G(\Gamma_0)$ is dense in $G$, it is enough to show that $\text{Comm}_G(\Gamma_0) \cap \text{Stab}_G(x_0)$ is dense in $\text{Stab}_G(x_0)$. And for this, it suffices to prove the statement of this lemma. 

To continue with the proof of the Density Theorem, fix $g \in \text{Stab}_G(x_0)$ and $u \geq 0$, and let $G_X(Y_n)$ be the canonical complex of groups over the combinatorial ball $Y_n$, as defined in Section 3.1 above. Let $H_n$ be the finite group obtained by restricting the action of $\text{Stab}_G(x_0)$ on $X$ to $Y_n$, and note that $g|_{Y_n} \in H_n$.

**Proposition 36.** The action of $H_n$ on $Y_n$ extends to an action by simple morphisms on the complex of groups $G_X(Y_n)$.

**Proof.** We first show:

**Lemma 37.** For all sides $K$ of $Y_n$ and all $h \in H_n$, the image $h.K$ is also a side of $Y_n$. That is, the action of $H_n$ takes sides to sides.

**Proof.** Let $K$ be a side of $Y_n$, of type $t \in S$. The automorphism $h$ of $Y_n$ preserves the boundary $\partial Y_n$, so for each mirror $K_t$ contained in $K$, the mirror $h.K_t$ is in $\partial Y_n$ as well. Also, $h$ preserves adjacency of mirrors in $Y_n$ (recall that two mirrors are adjacent if their intersection is of type $T$ with $|T| = 2$). Thus it suffices to show that if two $t$-mirrors $K_t$ and $K'_t$ of $K$ are adjacent, then the mirrors $h.K_t$ and $h.K'_t$ have the same type.

Let $\phi_t$ and $\phi_t'$ be the chambers of $Y_n$ containing $K_t$ and $K'_t$, respectively. As $K_t$ and $K'_t$ are adjacent and of the same type, there is a unique $s \in S$, with $m_{st} = 2$, such that $\phi_t$ is $s$-adjacent to $\phi_t'$. Thus the images $h.\phi_t$ and $h.\phi_t'$ are $s$-adjacent, for some $s \in S$. Hence $(h.\phi_t, h.\phi_t')$ is a gallery of type $s$ in $Y_n$.

Suppose the type of $h.K_t$ is $u$ and that of $h.K'_t$ is $u'$, with $u \neq u'$. Since the mirrors $h.K_t$ and $h.K'_t$ are adjacent and of distinct types, there is a chamber $\phi$ of $X$ (not necessarily in $Y_n$) which contains both $h.K_t$ and $h.K'_t$. Thus there is a gallery $(h.\phi_t, \phi, h.\phi_t')$ of type $(u, u')$ in $X$. But by the definition of the $W$-distance function on $X$, this means $s = uu'$, which is impossible. Hence $u = u'$, as required. 

We now, for each $h \in H_n$, define a simple isomorphism of complexes of groups $\Phi^h = (\phi^h) : G_X(Y_n) \to G_X(Y_n)$. For each $s \in S$, fix a generator $g_s$ of the cyclic group $G_s = \mathbb{Z}/q_s\mathbb{Z}$. Let $\sigma$ be a vertex of $Y_n$. By definition of the complex of groups $G_X(Y_n)$, if $\sigma$ is in $Y_n - \partial Y_n$ then $G_\sigma(Y_n)$ is the trivial group. Now the vertex $h.\sigma$ is in the boundary $\partial Y_n$ if and only if $\sigma \in \partial Y_n$, so for all $\sigma \in \partial Y_n$ we may define the local map $\phi^h_\sigma : G_\sigma(Y_n) \to G_{h.\sigma}(Y_n)$ to be the trivial isomorphism. 

If $\sigma$ is in $\partial Y_n$ then

$$G_\sigma(Y_n) = G_{T_{\partial Y_n}} = \prod_{t \in T_{\partial Y_n}} \langle g_t \rangle.$$

To define the local map $\phi^h_\sigma$ for $\sigma \in \partial Y_n$, let $T_{\partial Y_n}$ be the boundary type of $\sigma$ in $Y_n$ and let $U_{\partial Y_n}$ be the boundary type of $h.\sigma$ in $Y_n$. Let $t \in T_{\partial Y_n}$. Then $\sigma$ is contained in a side $K$ of $Y_n$ of type $t$. By Lemma 37 above, the image $h.K$ is a side of $Y_n$. Denote by $u_t$ the type of the side $h.K$. Since $h$ is an automorphism of $Y_n$, the map $t \mapsto u_t$ is a bijection $T_{\partial Y_n} \to U_{\partial Y_n}$. Since $h$ is the restriction of an automorphism of $X$ to $Y_n$, for all $t \in T_{\partial Y_n}$, we have $q_t = q_{u_t}$. We may thus define the local map $\phi^h_\sigma : G_\sigma(Y_n) \to G_{h.\sigma}(Y_n)$ to be the isomorphism of groups $G_{T_{\partial Y_n}} \to G_{U_{\partial Y_n}}$ induced by $g_t \mapsto g_{u_t}$ for each $t \in T_{\partial Y_n}$.

Recall that all monomorphisms $\psi$ along edges in the complex of groups $G_X(Y_n)$ are the identity or natural inclusions. Using this, it is not hard to verify that $\Phi^h$ so defined is a simple morphism of complexes of groups. Since $h$ is an automorphism of $Y_n$ and each local map $\phi^h_\sigma$ an isomorphism of groups, it follows that $\Phi^h$ is a simple isomorphism of the complex of groups $G_X(Y_n)$. Moreover, for all $h, h' \in H_n$, from the definition of composition of simple morphisms (see [BH]), it is immediate that $\Phi^h \circ \Phi^{h'} = \Phi^{hh'}$. Hence the group $H_n$ acts on the complex of groups $G_X(Y_n)$ by simple morphisms.

To finish proving the Density Theorem, let $\Gamma_n$ be the fundamental group of $G_X(Y_n)$. By Corollary 30 above, $\Gamma_n$ is a finite index subgroup of $\Gamma_0$. Let $Z_n = H_n \backslash \Gamma_n$, let $H(Z_n)$ be the complex of
groups induced by the action of $H_n$ on $G_X(Y_n)$, and let $\Gamma'_n$ be the fundamental group of $H(Z_n)$. Since the induced covering of complexes of groups $G_X(Y_n) \to H(Z_n)$ is finite-sheeted, $\Gamma'_n$ is a finite index subgroup of $\Gamma_n$. Therefore $\Gamma'_n$ and $\Gamma_0$ are commensurable. Now the group $H_n$ is, by definition, the restriction of the group $\text{Stab}_G(x_0)$ to the combinatorial ball $Y_n$. Hence $H_n$ fixes the basepoint $x_0$, and so by Theorem 17 above, $H_n$ injects into $\Gamma'_n$. Since $g|Y_n \in H_n$, it follows that there is an element $\gamma \in \Gamma'_n$ such that $\gamma|Y_n = g|Y_n$. But since $\Gamma'_n$ and $\Gamma_0$ are commensurable, $\gamma \in \text{Comm}_G(\Gamma_0)$. By the reduction established in Lemma 35 above, this completes the proof of the Density Theorem.

4.2. Constructing other lattices using unfoldings and group actions on complexes of groups. Let $X$ be a regular right-angled building and let $G = \text{Aut}(X)$. In this section we sketch how the techniques of unfoldings and group actions on complexes of groups may be combined to construct uniform lattices in $G$ in addition to the sequences $\Gamma_n$ and $\Gamma'_n$ above.

Let $Y$ be any subcomplex of $X$ obtained by unfolding the chamber $Y_0$ infinitely many times. Let $G(Y) = G_X(Y)$ be the canonical complex of groups over $Y$ defined in Section 3.1 above. By Proposition 27 above, the fundamental group $\Gamma'$ of $G(Y)$ is a uniform lattice in $G$. The possible fundamental domains $Y$ for $\Gamma$ include many subcomplexes which are not combinatorial balls in $X$.

Now suppose $H$ is any (finite) group of automorphisms of the subcomplex $Y$. As in Proposition 36 above, the action of $H$ on $Y$ extends to an action by simple morphisms on the complex of groups $G(Y)$. Let $\Gamma'$ be the fundamental group of the induced complex of groups over $H/Y$. Then $\Gamma'$ is also a uniform lattice in $G$. This construction thus yields many additional uniform lattices in $G$.

5. Further applications of unfoldings

In this section we give two further applications of the technique of unfoldings, which was developed in Section 3 above. Let $X$ be a regular right-angled building of type $(W,S)$ and parameters $\{q_s\}$, as defined in Section 1.4 above. Let $G = \text{Aut}(X)$ and let $G_0 = \text{Aut}_0(X)$ be the group of type-preserving automorphisms of $X$. In Section 5.1 we determine exactly when $G$ and $G_0$ are nondiscrete groups (Theorem 1 of the introduction). We then in Section 5.2 prove Theorem 2 of the introduction, which states that $G$ acts strongly transitively on $X$.

5.1. Discreteness and nondiscreteness of $G$ and $G_0$. Let $L$ be a polyhedral complex. Recall that $L$ is rigid if for any $g \in \text{Aut}(L)$, if $g$ fixes the star in $L$ of a vertex $\sigma \in V(L)$, then $g = \text{Id}_L$. If $L$ is not rigid it is said to be flexible. For example, a complete graph is rigid, while a complete bipartite graph $L = K_{q,q}$, with $q > 2$, is flexible.

The following statement is equivalent to Theorem 1 above.

Theorem 38. Let $X$ be a regular right-angled building of type $(W,S)$ and parameters $\{q_s\}$. Let $G = \text{Aut}(X)$ and let $G_0 = \text{Aut}_0(X)$ be the group of type-preserving automorphisms of $X$. Suppose $W$ is infinite and let $L$ be the nerve of $(W,S)$.

1. If there are $s,t \in S$ such that $q_s > 2$ and $m_{st} = \infty$ then $G_0$ and $G$ are both nondiscrete.
2. If all $q_s = 2$, then $G_0$ is discrete, and $G$ is nondiscrete if and only if $L$ is flexible.
3. If there is some $q_s > 2$, and for all $t \in S$ with $q_t > 2$ we have $m_{st} = 2$ for all $s \in S - \{t\}$, then $G_0$ is discrete, and $G$ is nondiscrete if and only if $L$ is flexible.

Note that if the Coxeter group $W$ is finite then the building $X$ is finite, so both $G$ and $G_0$ are finite groups.

Proof. Several results of [T1] imply that in Case (1), the group $G_0$ is nondiscrete. For example, the set of covolumes of lattices in $G_0$ contains arbitrarily small elements. Since a subgroup of a discrete group is discrete, the full automorphism group $G$ is thus nondiscrete as well.

Suppose next that all $q_s = 2$. Then $X$ is just the Davis complex $\Sigma$ for $(W,S)$. Assume $g_0 \in G_0$ fixes a chamber $\phi$ of $X$ pointwise. Then for each $s \in S$, since $q_s = 2$ there is a unique chamber $\phi_s$ of $X$ such that $\phi_s$ is $s$-adjacent to $\phi$. Since $g_0$ preserves types and fixes $\phi$ pointwise, the element $g_0$ fixes each adjacent chamber $\phi_s$ pointwise as well. By induction, $g_0$ fixes the building $X$ pointwise. Hence $G_0$ is discrete. Haglund–Paulin [HP] proved that the full automorphism group $G = \text{Aut}(X) = \text{Aut}(\Sigma)$ is nondiscrete exactly when the nerve $L$ of $(W,S)$ is flexible.
Suppose finally that we are in Case (3). Then in particular the set \( T := \{ t \in S \mid q_t > 2 \} \) is a nonempty spherical subset of \( S \). Let \( \mathcal{C} \) be the clump obtained by unfolding the chamber \( Y_0 \) along all of its mirrors of types \( t \in T \) (in some order). More precisely, \( \mathcal{C} \) is the clump obtained by unfolding \( Y_0 \) along some sequence of (possibly extended) sides of types \( t \in T \), as in the proof of Lemma 22 above. By Proposition 27 above, the complex of groups \( G_X(\mathcal{C}) \) is admissible. Hence \( \mathcal{C} \) is a strict fundamental domain for the action of a uniform lattice \( \Gamma := \pi_1(G_X(\mathcal{C})) \) on \( X \), and so we may think of \( X \) as tessellated by copies of \( \mathcal{C} \).

By Lemma 4 above, since \( T \) is a nonempty spherical subset of \( S \), the union of mirrors \( \bigcup_{t \in T} K_t \) of \( Y_0 \) is contractible and thus connected. Therefore, every mirror of \( \mathcal{C} \) of type \( t \in T \) is in the interior of \( \mathcal{C} \). Thus every side of \( \mathcal{C} \) is of type \( s \in S - T \).

Now suppose \( g_0 \in G_0 \) fixes \( \mathcal{C} \) pointwise. Let \( \phi \) be a chamber of \( X \) which is \( s \)-adjacent to a chamber in \( \mathcal{C} \), for some \( s \in S - T \). Then since \( q_s = 2 \) and \( g_0 \) is type-preserving, \( g_0 \) must fix the chamber \( \phi \) pointwise. For each \( t \in T \), let \( K_{\phi,t} \) be the \( t \)-mirror of \( \phi \). By hypothesis, \( m_{s,t} = 2 \), so the mirror \( K_{\phi,t} \) of \( \phi \) is adjacent to \( \Sigma \) in \( \partial \mathcal{C} \). Thus any chamber of \( X \) which is \( t \)-adjacent to \( \phi \) is \( s \)-adjacent to a chamber in \( \mathcal{C} \). Since \( q_s = 2 \), it follows that any chamber of \( X \) which is \( t \)-adjacent to \( \phi \), for \( t \in T \), must also be fixed pointwise by the element \( g_0 \). Hence \( g_0 \) fixes pointwise the copy of \( \mathcal{C} \) in \( X \) which contains the chamber \( \phi \).

We have shown that for all \( s \in S - T \), every copy of \( \mathcal{C} \) in \( X \) which is \( s \)-adjacent to the original clump \( \mathcal{C} \) is also fixed pointwise by \( g_0 \). By induction, \( g_0 = \text{Id}_X \) in \( \mathcal{C} \). Thus the group \( G_0 \) of type-preserving automorphisms of \( X \) is discrete. The proof that \( G = \text{Aut}(X) \) is nondiscrete if and only if \( L \) is flexible is by similar arguments to those of Haglund–Paulin [HP].

### 5.2. Strong transitivity.

We conclude by proving Theorem 2 of the introduction. We will actually show:

**Theorem 39.** Let \( X \) be a regular right-angled building of type \( (W,S) \) and parameters \( \{q_s\} \), and let \( G_0 = \text{Aut}_0(X) \). Let \( x_0 \) be the center of the chamber \( Y_0 \).

1. The group \( H_0 := \text{Stab}_{G_0}(x_0) \) acts transitively on the set of apartments containing \( Y_0 \).
2. The group \( G_0 \) acts transitively on the set of pairs

\[ \{ (\phi, \Sigma) \mid \Sigma \text{ is an apartment of } X \text{ containing the chamber } \phi \} \]

**Corollary 40.** The group \( G \) acts strongly transitively on \( X \).

**Proof of Theorem 39.** Since \( G_0 \) acts transitively on the set of chambers of \( X \), it is enough to show (1).

We fix an increasing sequence of subcomplexes \( \mathcal{C}_n \) of \( X \) such that \( \mathcal{C}_n \) is a clump obtained by \( n \) unfoldings of \( \mathcal{C}_0 = Y_0 \), and \( X = \bigcup_{n=0}^{\infty} \mathcal{C}_n \).

**Lemma 41.** Let \( \Sigma \) and \( \Sigma' \) be distinct apartments of \( X \) which contain \( Y_0 \). Let \( N \geq 1 \) be the smallest integer such that \( \Sigma \cap \mathcal{C}_N \neq \Sigma' \cap \mathcal{C}_N \). Then there is an element \( h_N \in H_0 \) such that \( h_N \) fixes pointwise the clump \( \mathcal{C}_{N-1} \), and \( h_N(\Sigma \cap \mathcal{C}_N) = \Sigma' \cap \mathcal{C}_N \).

**Proof.** Suppose \( \mathcal{C}_N \) is obtained from \( \mathcal{C}_{N-1} \) by unfolding along a side \( K \) of type \( u \). Recall from Lemma 26 above that \( \text{Ch}_K \), the set of “new chambers” in \( \mathcal{C}_N \), consists of \( q_u - 1 \) sheets. Since \( \Sigma \cap \mathcal{C}_N \neq \Sigma' \cap \mathcal{C}_N \), the sets of chambers \( \Sigma \cap \text{Ch}_K \) and \( \Sigma' \cap \text{Ch}_K \) belong to different sheets in \( \text{Ch}_K \).

Now, for each sheet in \( \text{Ch}_K \), the set of chambers in this sheet is in bijection with the set of mirrors in \( K \). Hence, for any two sheets in \( \text{Ch}_K \), there is a type-preserving element \( h'_N \in \text{Aut}(\mathcal{C}_N) \) such that \( h'_N \) fixes \( \mathcal{C}_{N-1} \) pointwise, and \( h'_N \) exchanges these two sheets.

Since \( \Sigma \cap \mathcal{C}_{N-1} = \Sigma' \cap \mathcal{C}_{N-1} \), the set of mirrors in \( K \) contained in \( \Sigma \) is equal to the set of mirrors in \( K \) contained in \( \Sigma' \). Thus \( h'_N \) exchanges the sets of chambers \( \Sigma \cap \text{Ch}_K \) and \( \Sigma' \cap \text{Ch}_K \). So \( h'_N \) fixes \( \mathcal{C}_{N-1} \) pointwise, and \( h'_N(\Sigma \cap \mathcal{C}_N) = \Sigma' \cap \mathcal{C}_N \).

Consider the group \( \langle h'_N \rangle \) generated by \( h'_N \). By similar arguments to the proof of Proposition 36 above, the action of \( \langle h'_N \rangle \) on \( \mathcal{C}_N \) extends to an action by simple morphisms on the complex of groups \( G_X(\mathcal{C}_N) \). Since the group \( \langle h'_N \rangle \) fixes \( \mathcal{C}_{N-1} \) pointwise, in particular it fixes the point \( x_0 \). By Theorem 17 above, the group \( \langle h'_N \rangle \) thus injects into the fundamental group of the induced complex of groups. Denote by \( h_N \) the image of \( h'_N \) in this fundamental group. By construction, \( h_N \) fixes \( \mathcal{C}_{N-1} \) pointwise, hence \( h_N \in H_0 \), and \( h_N(\Sigma \cap \mathcal{C}_N) = \Sigma' \cap \mathcal{C}_N \) as required. \(\square\)
Let \( \Sigma \) and \( \Sigma' \) be two apartments of \( X \) which contain \( Y_0 \). For each \( n \geq 0 \) we will construct an element \( h_n \in H_0 \) such that

1. \( h_n(\Sigma \cap C_n) = \Sigma' \cap C_n \), and
2. for all \( m \geq 0 \), we have \( h_{n+m}|_{C_m} = h_n|_{C_m} \).

Note that, since \( C_n \subset C_{n+1} \) for all \( n \geq 0 \), to prove (2) it suffices to show that for all \( n \geq 0 \), \( h_{n+1}|_{C_n} = h_n|_{C_n} \).

To construct the sequence \( \{h_n\} \), let \( N \geq 1 \) be the smallest integer such that \( \Sigma \cap C_N \neq \Sigma' \cap C_N \). For each \( 0 \leq n < N \) we define \( h_n \in H_0 \) to be \( h_n = \text{Id}_X \). Let \( h_N \) be the element of \( H_0 \) constructed in Lemma 41 above. Then for each \( 0 \leq n \leq N \) we have \( h_n(\Sigma \cap C_n) = \Sigma' \cap C_n \), and for all \( 0 \leq n < N \) we have \( h_{n+1}|_{C_n} = h_n|_{C_n} \).

For \( n \geq N \), assume inductively that for \( k \geq 0 \) there are elements \( h_N, h_{N+1}, \ldots, h_{N+k} \) in \( H_0 \) such that \( h_{N+k}(\Sigma \cap C_{N+k}) = (\Sigma' \cap C_{N+k}) \), and \( h_{N+k}|_{C_{N+k+1}} \). To construct the next element \( h_{N+k+1} \), note that since

\[
\Sigma \cap C_{N+k} = \Sigma' \cap C_{N+k},
\]
the apartments \( h_{N+k}\Sigma \) and \( \Sigma' \) have the same intersection with \( C_{N+k} \). If in addition the apartments \( h_{N+k}\Sigma \) and \( \Sigma' \) have the same intersection with the next clump \( C_{N+k+1} \), we put \( h_{N+k+1} = h_{N+k} \) and are done. If not, then \( N + k + 1 \) is the smallest integer such that the apartments \( h_{N+k}\Sigma \) and \( \Sigma' \) have distinct intersection with \( C_{N+k+1} \). Hence by Lemma 41 above, there is an element \( h' \in H_0 \) such that \( h' \) fixes pointwise \( C_{N+k} \), and \( h'(h_{N+k}\Sigma \cap C_{N+k+1}) = \Sigma' \cap C_{N+k+1} \). We then define \( h_{N+k+1} \) to be the product \( h' h_{N+k} \), and have that

\[
h_{N+k+1}(\Sigma \cap C_{N+k+1}) = \Sigma' \cap C_{N+k+1}.
\]

Since \( h' \) fixes pointwise \( C_{N+k} \), the restriction of \( h_{N+k+1} \) to \( C_{N+k} \) is the same as that of \( h_{N+k} \). Hence the element \( h_{N+k+1} \) has the required properties. We have thus constructed a sequence \( \{h_n\} \) satisfying (1) and (2) above.

By definition of the topology on \( G_0 \), the compact subgroup \( H_0 \) of \( G_0 \) is complete. The sequence \( \{h_n\} \) in \( H_0 \) that we have constructed is a Cauchy sequence, by (2) above. Hence there is an element \( h \in H_0 \) such that \( h\Sigma = \Sigma' \). We conclude that \( H_0 \) acts transitively on the set of apartments containing \( Y_0 \).

\[\square\]

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