AN INFRASOLVMANIFOLD WHICH DOES NOT BOUND

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Abstract. Orientable 4-dimensional infrasolvmanifolds bound orientably. We show that every non-orientable 4-dimensional infrasolvmanifold $M$ with $\beta = \beta_1(M; \mathbb{Q}) > 0$ or with geometry $\text{Nil}^3$ or $\text{Sol}^3 \times \mathbb{E}^1$ bounds. However there are $\text{Sol}^4_1$-manifolds which are not boundaries. The question remains open for $\text{Nil}^3 \times \mathbb{E}^1$-manifolds. Any possible counter-examples have severely constrained fundamental groups. We also find simple cobounding 5-manifolds for all but five of the 74 flat 4-manifolds, and investigate which flat 4-manifolds embed in $\mathbb{R}^n$, for $n = 5, 6$ or 7.

1. INTRODUCTION

Flat $n$-manifolds are boundaries [8]. This result has been extended to restricted classes of infranilmanifolds [7, 12]. We shall show that it does not extend to all infrasolvmanifolds. Since every 3-manifold bounds, and every orientable 3-manifold bounds orientably, dimension 4 is the first case of interest. Here there is a geometric simplification. Every 4-dimensional infrasolvmanifold is either a mapping torus or the union of two twisted $I$-bundles. Simple algebraic arguments show that every such mapping torus bounds, while a geometric construction applies to many of the others. We find severe constraints on possible counter-examples, which lead to a $\text{Sol}^4_1$-manifold which is not a boundary. In the latter part of the paper we seek explicit constructions of 5-manifolds with boundary a given flat 4-manifold, and we consider also the related question of which flat 4-manifolds embed in low codimensions.

Every infrasolvmanifold is finitely covered by a quotient $\Gamma \backslash S$, where $\Gamma$ is a discrete cocompact subgroup of a 1-connected solvable Lie group $S$ [1]. Such quotients are parallelizable, and so the rational Pontrjagin classes of infrasolvmanifolds are 0. In particular, orientable 4-dimensional infrasolvmanifolds have signature $\sigma = 0$. Therefore they

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bound orientably, and those with \( w_2 = 0 \) bound as \( \text{Spin} \)-manifolds, since \( \Omega_4 \) and \( \Omega_4^{\text{Spin}} \) are detected by \( \sigma \).

Non-orientable bordism is detected by Stiefel-Whitney numbers. In our context, the only Stiefel-Whitney class of interest is \( w_4 \). It follows easily that every 4-dimensional infrasolvmanifold \( M \) with \( \beta = \beta_1(M; \mathbb{Q}) > 0 \) bounds non-orientably. (This class includes all \( \text{Sol}^{4}_{m,n} \)-manifolds with \( m \neq n \) and all \( \text{Sol}^{4}_{3} \)-manifolds.) If \( \beta = 0 \) then \( \pi_1(M) \cong A \ast_C B \), where \( A, B \) and \( C \) are fundamental groups of 3-dimensional infranilmanifolds and \( [A : C] = [B : C] = 2 \). In §4–§9 we use a construction based on mapping cylinders of double covers to show that many such manifolds bound. In particular, all \( \text{Nil}^{4} \) and \( \text{Sol}^{3} \times \mathbb{E}^{1} \)-manifolds bound. We do not yet have a complete result for the remaining two geometries.

In §10 we show that if \( \beta \geq 2 \) (and in many cases with \( \beta = 1 \)) then \( M \) is also the total space of an \( S^{1} \)-bundle over a closed 3-manifold, and so bounds the associated disc bundle. If the \( S^{1} \)-bundle space \( M \) is orientable then so is the disc bundle space. In §11 we show that the mapping cylinder construction applies to most of the 24 flat 4-manifolds which are not \( S^{1} \)-bundle spaces. Closed hypersurfaces in euclidean spaces bound. In §12 we show that, with one possible exception, all flat 4-manifolds embed in \( \mathbb{R}^{7} \), while between 24 and 56 embed in \( \mathbb{R}^{6} \) and between 11 and 13 embed in \( \mathbb{R}^{5} \).

### 2. SOLVABLE LIE GEOMETRIES OF DIMENSION 4

If \( G \) is a group let \( G' \), \( \zeta G \) and \( \sqrt{G} \) denote its commutator subgroup, centre and Hirsch-Plotkin radical, respectively. Let \( G^{ab} = G/G' \) be the abelianization, and let \( I(G) = \{ g \in G \mid \exists n > 0, g^n \in G' \} \) be the isolator subgroup. This is clearly a characteristic subgroup, since \( G/I(G) \) is the maximal torsion-free abelian quotient of \( G \). If \( S \) is a subset of \( G \) then \( \langle S \rangle \) shall denote the subgroup of \( G \) generated by \( S \), and \( \langle \langle S \rangle \rangle \) shall denote the normal closure of \( \langle S \rangle \). We use the notation of Chapter 8 of [9] for automorphisms of flat 3-manifold groups.

Every 4-dimensional infrasolvmanifold is geometric. There are six relevant families of geometries: \( \mathbb{E}^{4} \), \( \text{Nil}^{4} \), \( \text{Nil}^{3} \times \mathbb{E}^{1} \), \( \text{Sol}^{4}_{0} \), \( \text{Sol}^{4}_{1} \) and \( \text{Sol}^{4}_{m,n} \). (The final family includes the product geometry \( \text{Sol}^{3} \times \mathbb{E}^{1} = \text{Sol}^{4}_{m,0} \), for all \( m > 0 \), as a somewhat exceptional case.)

Let \( G \) be a 1-connected solvable Lie group of dimension 4 with a left invariant metric, corresponding to a geometry \( \mathbb{G} \) of solvable Lie type. Let \( \text{Isom}(\mathbb{G}) \) be the group of isometries, and let \( K_{G} < \text{Isom}(\mathbb{G}) \) be the stabilizer of the identity in \( G \). Let \( \pi < \text{Isom}(\mathbb{G}) \) be a discrete subgroup which acts freely and cocompactly on \( G \), and let \( M = \pi \backslash G \). If \( \beta = \)}
$\beta_1(M; \mathbb{Q}) \geq 1$ then $M$ is the mapping torus of a self-diffeomorphism of a $\text{E}^3$-, $\text{Nil}^3$- or $\text{Sol}^3$-manifold. If $\beta = 1$ the mapping torus structure is essentially unique. If $\beta \geq 2$ then $M$ also fibres over the torus $T$, with fibre $T$ or the Klein bottle $Kb$.

All orientable $\text{Sol}_3^2$-manifolds are coset spaces $\pi \setminus \tilde{G}$ with $\pi$ a discrete subgroup of a 1-connected solvable Lie group $\tilde{G}$, which in general depends on $\pi$. (See page 138 of [9].) In all other cases, the translation subgroup $G \cap \pi$ is a lattice in $G$, and is a characteristic subgroup of $\pi$ [4]. If $G$ is nilpotent then $G \cap \pi = \sqrt{\pi}$; in general, $\sqrt{\pi} \leq G \cap \pi$, and the holonomy $\pi/G \cap \pi$ is finite.

If $g : X \to X$ is a self-homeomorphism let $M(g) = X \times [0,1]/(z,0) \sim (g(z),1)$ be the mapping torus of $g$, and let $[x,t]$ be the image of $(x,t)$ in $M(g)$. If $f : Y \to Z$ let $MCyl(f)$ be the mapping cylinder of $f$.

3. Stiefel-Whitney classes and the cases with $\beta \geq 1$

We give first some simple observations on the Stiefel-Whitney classes of 4-manifolds, which we shall use to show that 4-dimensional infrasolvmanifolds with $\beta \geq 1$ are boundaries.

Lemma 3.1. Let $M$ be a closed 4-manifold and $w_i = w_i(M)$. Then $w_4 = w_2^2 + w_1^4$ and $w_1w_3 = 0$.

Proof. The Wu formulae give $v_1 = w_1, v_2 = w_2 + w^2, w_3 = Sq^1w_2$ and $w_4 = w_2^2 + w^4$, since $v_3 = v_4 = 0$. Hence $Sq^1z = w_1z$, for $z \in H^3(M; \mathbb{F}_2)$. If $x \in H^1(M; \mathbb{F}_2)$ then $Sq^1(xw_2) = x^2w_2 + xSq^1w_2$. Therefore

$$xw_3 = (w_1x + x^2)w_2 = (w_1x + x^2)^2 + (w_1x + x^2)w_1^2 = x^4 + w_1x^3.$$ 

In particular, $w_1w_3 = w^4 + w^4 = 0$. \hfill \Box

If $M$ is a 4-dimensional infrasolvmanifold then $w_4(M) = 0$, since $w_4(M) \cap [M]$ is the reduction of $\chi(M) = 0 \mod (2)$. Therefore $w_1 = w_2^2 = w_3^2$ is the only Stiefel-Whitney class of interest.

Lemma 3.2. Let $M$ be a closed n-manifold and $x \in H^1(M; \mathbb{F}_2)$. If $n > 2$ and $x^{n-1} \neq 0$ then $x^n \neq 0$.

Proof. This follows easily from the non-degeneracy of Poincaré duality, since $x^2 \neq 0$ and $H^1(M; \mathbb{F}_2)$ is generated by $x$ and Ker$(x \cup -)$. \hfill \Box

Lemma 3.3. If $N$ is a non-orientable 3-manifold then $\beta_1(N; \mathbb{Q}) > 0$.

Proof. This is clear, since $\chi(N) = 0$ and $H_3(N; \mathbb{Q}) = 0$. \hfill \Box

Similarly, if $M$ is an orientable 4-manifold with $\chi(M) = 0$ then $\beta_1(M; \mathbb{Q}) > 0$. 

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Lemma 3.4. If a manifold $M$ fibres over an $r$-manifold, with orientable fibre, then $w_1(M)^{r+1} = 0$.

Proof. This is clear, since $w_1(M)$ is induced from a class on the base of the fibration. □

Theorem 3.5. Let $M$ be a 4-dimensional infrasolvmanifold with $\beta = \beta_1(M; \mathbb{Q}) > 0$. Then $M = \partial W$ for some 5-manifold $W$.

Proof. The manifold $M$ is the mapping torus of a (based) self diffeomorphism $f$ of a closed 3-manifold $N$. Let $\pi = \pi_1(M)$ and $\nu = \pi_1(N)$. Then $\pi$ and $\nu$ are virtually polycyclic, and $\pi \cong \nu \rtimes \mathbb{Z}$, where $\theta = \pi_1(f)$.

If $N$ is not orientable then $I(\nu) < \nu$, by Lemma 3.3, and so $I(\nu) \cong \mathbb{Z}$, $\mathbb{Z}^2$ or $\pi_1(Kb) = \mathbb{Z} \rtimes \mathbb{Z}$. In the latter case $I(I(\nu)) \cong \mathbb{Z}$. In all cases, $M$ fibres over a lower-dimensional manifold with orientable fibre, and so $w_1^4 = 0$, by Lemma 3.4. Therefore all the Stiefel-Whitney numbers of $M$ are 0, and so $M = \partial W$ for some 5-manifold $W$. □

If $M$ is a non-orientable Sol$_3^4$-manifold then $\beta = 0$. There are non-orientable manifolds with $\beta > 0$ for each of the other geometries.

For all but three flat 4-manifolds, either $w_1^2 = 0$ or $w_2 = 0$ or $w_3^2 = w_2$ [10]. Hence $w_1^4 = 0$, so all Stiefel-Whitney numbers are 0, and the manifold bounds. Two more are total spaces of $S^1$-bundles, and so bound the associated disc bundles. Thus only the example with group $G_6 \ast_\phi B_4$ requires further argument. (See the next section.)

All Sol$_{m,n}^4$-manifolds (with $m \neq n$) and all Sol$_3^4$-manifolds are mapping tori of self-diffeomorphisms of $\mathbb{R}^3/\mathbb{Z}^3$. (See Corollary 8.4.1 of [9].) Thus they all bound.

We may assume henceforth that $\beta = 0$ (so the manifolds considered are not orientable) and the geometry is Nil$^3$, Nil$^3 \times \mathbb{E}^1$, Sol$_3^1$ or Sol$_3^4 \times \mathbb{E}^1$. (However we shall also consider $\mathbb{E}^4$ in some detail.)

We shall need the following more specialized lemmas later.

Lemma 3.6. Let $w : \pi \to \mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ be a homomorphism. Then $p : \pi \to G = \pi/(k^2 \mid w(k) = 0)$ induces an isomorphism $H^1(G; \mathbb{F}_2) \cong H^1(\pi; \mathbb{F}_2)$. If $p^*(uw) = 0$ in $H^2(\pi; \mathbb{F}_2)$ then $uw = 0$ in $H^2(G; \mathbb{F}_2)$.

Proof. If $p^*(uw) = 0$ in $H^2(\pi; \mathbb{F}_2)$ there is a function $f : \pi \to \mathbb{F}_2$ such that $u(g)w(g') = f(g) + f(g') - f(gg')$, for all $g, g' \in \pi$. Let $K = \text{Ker}(w)$ and $H = \langle k^2 \mid w(k) = 0 \rangle$. Then $f|_K$ is a homomorphism, and so $f(h) = 0$, for all $h \in H$. Hence $f(g) = f(gh)$, for all $g \in \pi$ and $h \in H$. Thus $f$ factors through a function $\tilde{f} : G \to \mathbb{F}_2$, and so $uw = 0$ in $H^2(G; \mathbb{F}_2)$. □

The next lemma uses the non-degeneracy of Poincaré duality.
Lemma 3.7. Let $M$ be a non-orientable closed 4-manifold with $\chi(M) = 0$, and let $w = w_1(M)$. Suppose that $H^1(M; \mathbb{F}_2) = \langle u, w \rangle$, where $u^2 = 0$. Then

(1) if $w^2 \neq 0$ and $uw \neq 0$, then $w^3 = 0$.
(2) if $w^2 \neq 0$ and $uw = 0$ then $w^4 \neq 0 \iff w_2(M) \neq 0$ or $w^2$.

Proof. (1). Since $u.uw^2 = u^2w^2 = 0$ and $w.uw^2 = Sq^1(uw^2) = u^2w^2 = 0$, we have $uw^2 = 0$, by Poincaré duality. Now $\beta_2(M, \mathbb{F}_2) = 2\beta_1(M, \mathbb{F}_2) - 2 = 2$. Since $uw.w^2 = uw.uw = 0$ but $uw \neq 0$ and $w^2 \neq 0$ we must have $uw = w^2$, by Poincaré duality. Hence $w^3 = uw^2 = 0$.

(2). Let $v = w_2(M) + w^2 = v_2(M)$. If $w_2(M) \neq 0$ or $w^2$ then $H^2(M; \mathbb{F}_2) = \langle w^2, v \rangle$. Since $\chi(M) = 0$ we have $v^2 = w_4 = 0$. Therefore $w^4 = (w^2)^2 = w^2v \neq 0$, by Poincaré duality. The converse is clear, since $v_2^2 = w_4 = 0$.

The second condition may be generalized as follows. Let $H^i = H^i(M; \mathbb{F}_2)$ for $i = 1$ and 2. If $w_i^2 \neq 0$, $w_1 \cup - : H^1 \to H^2$ has rank 1, $w_2$ is not in the image of $H^1 \cap H^1$ and $H^2 = \langle H^1 \cap H^1, w_2 \rangle$, then $w_i^4 \neq 0$. However these conditions are harder to check if $\beta_1(\pi; \mathbb{F}_2) > 2$.

There are two (flat) 4-manifolds which fibre over $T$ with fibre $Kb$, and thus bound, but for which none of the conditions $w_i^2 = 0$, $w_2 = 0$ or $w_2 = w_1^2$ hold. Thus these conditions are not necessary for a 4-manifold to bound. Nevertheless, manifolds which are not mapping tori and whose orientable double covers are not Spin 4-manifolds may provide non-bounding examples.

4. 4-MANIFOLDS WITH $\chi = \beta = 0$

If $M$ is a closed 4-manifold with $\chi(M) = 0$ and $\beta = 0$ then $M$ is non-orientable, and there is an epimorphism $f : \pi \to D_\infty$, where $D_\infty = \mathbb{Z}/2\mathbb{Z} \ast \mathbb{Z}/2\mathbb{Z}$ is the infinite dihedral group, by Lemma 3.14 of [9]. Hence $\pi \cong A \ast C B$, where $C = \text{Ker}(f)$ and $[A : C] = [B : C] = 2$. Since $D_\infty \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, the group $\pi$ has a subgroup of index 2 which is a semidirect product $C \rtimes \mathbb{Z}$. Since $\beta = 0$ the Mayer-Vietoris sequence for the homology of $\pi$ gives an epimorphism from $H_1(C; \mathbb{Q})$ to $H_1(A; \mathbb{Q}) \oplus H_1(B; \mathbb{Q})$, and so $\beta_1(A; \mathbb{Q}) + \beta_1(B; \mathbb{Q}) \leq \beta_1(C; \mathbb{Q})$.

If, moreover, $M$ is an infrasolvmanifold then $A$, $B$ and $C$ are the fundamental groups of 3-dimensional infrasolvmanifolds $X$, $Y$ and $Z$, say, and $M = MCyl(c) \cup Z MCyl(d)$, where $c : Z \to X$ and $d : Z \to Y$ are double covers. The next two lemmas are clear.

Lemma 4.1. If $c : Z \to X$ is a double cover of an $n$-manifold $X$ then $MCyl(c)$ is an $(n + 1)$-manifold with boundary $Z$. If $Z$ is connected
the mapping cylinder is orientable if and only if \( X \) is non-orientable and \( c \) is the orientable double cover.

In particular, if \( f \) is an orientation-preserving self-diffeomorphism of a 3-manifold \( N \) then \( M(f^2) \) bounds a non-orientable 5-manifold.

**Lemma 4.2.** Let \( X \) and \( Y \) be connected \((n - 1)\)-manifolds which have double covers \( c : Z \to X \) and \( d : Z \to Y \) with the same domain, and let \( M = MCyl(c) \cup_Z MCyl(d) \). Suppose that \( X \), \( Y \) and \( Z \) each bound \( n \)-manifolds \( \hat{X} \), \( \hat{Y} \) and \( \hat{Z} \), and that \( c \) and \( d \) extend to double covers \( \hat{c} : \hat{Z} \to \hat{X} \) and \( \hat{d} : \hat{Z} \to \hat{Y} \). Let \( W = MCyl(\hat{c}) \cup_Z MCyl(\hat{d}) \). Then \( \partial W = M \). If \( c \) and \( d \) are the orientable covers of non-orientable manifolds then \( W \) and \( M \) are orientable.

We shall show that this construction applies to many 4-dimensional infrasolvmanifolds.

Theorems 8.4–8.9 of [9] limit the possibilities for \( A, B \) and \( C \). In particular, if \( C \) is virtually \( \mathbb{Z}^3 \) but \( \pi \) is not virtually abelian then \( C \) has holonomy of order \( \leq 2 \). There are four such, two orientable: \( \mathbb{Z}^3 \) and \( G_2 = \mathbb{Z}^2 \ltimes_{-1} \mathbb{Z} \), and two non-orientable: \( B_1 = \mathbb{Z} \times \pi_1(K\beta) \) and \( B_2 \). Similarly, if \( C \) is a Nil\( \mathbb{Z}^3 \)-group but \( \pi \) is not virtually nilpotent then \( [C : \sqrt{C}] \leq 2 \). We shall not need to consider the possibility that \( C \) be a Sol\( \mathbb{Z}^3 \)-group.

We note also the following simple result.

**Lemma 4.3.** If \( \pi \cong A \ast C B \) where \( [A : C] = [B : C] = 2 \) and \( A, B \) and \( C \) are the groups of 3-dimensional infranilmanifolds then the holonomy of \( A \) maps injectively to the holonomy of \( \pi \). 

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If \( C = \mathbb{Z}^3 \) then \( A \) and \( B \) have holonomy of order \( \leq 2 \). Since \( \beta_1(A; \mathbb{Q}) \) and \( \beta_1(B; \mathbb{Q}) \geq 1 \) and \( \beta_1(A; \mathbb{Q}) + \beta_1(B; \mathbb{Q}) \leq 3 \), we may assume that \( A \cong G_2 \) and \( B \) is not \( \mathbb{Z}^3 \). Let \( f, g \) and \( h \) be the involutions of \( S^1 \times D^2 \) given by \( f(u, v) = (\bar{u}, \bar{v}), g(u, v) = (u, \bar{v}) \) and \( h(u, v) = (\bar{u}, uv) \), for all \((u, v) \in S^1 \times D^2 \). The boundaries of the mapping tori \( M(f), M(g) \) and \( M(h) \) are the flat 3-manifolds with groups \( G_2, B_1 \) and \( B_2 \), respectively, and in each case the mapping torus is doubly covered by \( S^1 \times D^2 \times S^1 \), with boundary the 3-torus \( \mathbb{R}^3 / \mathbb{Z}^3 \). Therefore the mapping cylinder construction shows that \( M \) is a boundary.

If \( C = G_2 \) then \( \beta_1(C; \mathbb{Q}) \geq 1 \). We may assume that \( A = G_6 \) and \( B \) is one of \( G_2, G_4, G_6, B_3 \) or \( B_4 \). If \( B = G_2 \cong C \) then the inclusion of \( C \) into \( B \) induces an isomorphism \( C/I(C) \cong B/I(B) \), and is induced by a double cover from \( M(f) \) to itself. Non-orientable 3-manifolds bound
non-orientable 4-manifolds, and their orientable double covers bound
the orientable double covers of such manifolds. If \( f \) is the involution
of \( S^1 \times D^2 \) defined above then \( M(f) \) has an orientation-preserving free
involution given by \( [u, v, t] \mapsto [-u, \bar{v}, -t] \). The quotient manifold has
boundary \( HW \), the Hantzsche-Wendt flat 3-manifold with group \( G_6 \).
Thus the mapping cylinder construction applies, provided \( B \not\cong G_4 \).

If \( C = B_1 \) or \( B_2 \) then \( A \) and \( B \) must be \( B_3 \) or \( B_4 \), and \( I(I(A)) = I(I(B)) \cong I(C) \cong \mathbb{Z} \). Hence \( \pi/I(C) \cong A/I(C) \ast_{\mathbb{Z}^2} B/I(C) \) and so
is a 3-manifold group. The manifold \( M \) is then the total space of an
\( S^1 \)-bundle. (The mapping cylinder construction can also be used here.)

There remains the possibility that \( A = G_6, B = G_4 \) and \( C = G_2 \).
In this case the holonomy group \( Z/4Z \) of \( G_4 \) does not act diagonally, and
there is no obvious construction of a 4-manifold with boundary the flat
3-manifold with group \( G_4 \). Instead we may use algebraic arguments.
The group \( \pi \) then has a presentation

\[
\langle t, x, y, z \mid xy^2x^{-1} = y^{-2}, yx^2y^{-1} = x^{-2}, z = xy, tx^2t^{-1} = x^{2m}y^{2p},
\]

\[
ty^2t^{-1} = x^{2n}y^{-2m}, tzt^{-1} = x^{-2r}y^{2s}z, t^2 = x^{2a}y^{2b}z \rangle,
\]

where \( a, b, m, n, p, \in \mathbb{Z}, r = (m - 1)a + pb, s = -na + (m + 1)b \) and
\( m^2 + np = -1 \). (We may assume also that \( 0 \leq a, b \leq 1 \).) Here
\( C = \langle x^2, y^2, z \rangle \), and \( \pi/C \cong D_\infty \) is generated by the images of \( t \) and \( x \). The automorphism of \( \sqrt{C} = \langle x^2, y^2, z^2 \rangle \) determined by conjugation
by \( tx \) has eigenvalues \( m \pm \sqrt{m^2 + 1} \). If \( m = 0 \) then \( \pi \) is virtually
abelian, and the corresponding manifold \( M \) is flat. In this case \( \pi \) is
also isomorphic to \( G_2 \ast_{\mathbb{Z}^3} B_3 \), and so \( M \) bounds. Otherwise, \( \pi \) is not
virtually nilpotent, and \( M \) is a \( \text{Sol}^3 \times \mathbb{E}^1 \)-manifold.

The generators \( t, x \) and \( y \) in this presentation represent orientation-
reversing elements of \( \pi \). If \( m \) is even, or if \( m \) is odd and \( n, p \) are both
even, then \( \pi/\pi' \cong (Z/4Z)^2 \), and so \( \omega^2 = 0 \). Thus we may assume
that \( m, n \) are odd (and hence \( p \) is even). In this case \( \pi/\pi' \cong Z/8Z \oplus
Z/2Z \), where the summands are generated by the images of \( tx^{-1} \) and
\( x \), respectively. Thus \( w = w_1 \) is projection onto the second summand.
Let \( u : \pi \to Z/2Z \) be the homomorphism determined by \( u(t) = 1 \)
and \( u(x) = 0 \). Let \( H = \langle k^2 \mid w(k) = 0 \rangle \), as in Lemma 3.6. Then
\( G = \pi/H \cong Z/4Z \oplus Z/2Z \), and so \( u^2 = 0 \) and \( uw \neq 0 \) in \( H^2(G; \mathbb{F}_2) \).
Hence \( uw \neq 0 \) in \( H^2(\pi; \mathbb{F}_2) \), by Lemma 3.6, and so \( w^3 = 0 \), by part (1)
of Lemma 3.7. Thus all such manifolds bound.

These results apply immediately to the flat 4-manifolds with \( \beta = 0 \).
In the next section we shall use them to confirm that all \( \text{Nil}^1 \)- and
\( \text{Sol}^3 \times \mathbb{E}^1 \)-manifolds are boundaries.
6. \textit{Nil}^3\text{- AND \textit{Sol}^3 \times \mathbb{E}^1\text{-MANIFOLDS}}

Let \(M\) be a \(\textit{Nil}^3\)-manifold and let \(C\) be the centralizer of \(I(\sqrt{\pi}) \cong \mathbb{Z}^2\) in \(\sqrt{\pi}\). Then \(C \cong \mathbb{Z}^3\), and \(1 < \zeta \sqrt{\pi} < I(\sqrt{\pi}) < C < \sqrt{\pi}\) is a characteristic series with all successive quotients \(\mathbb{Z}\). (See Theorem 1.5 of [9].) In particular, \(C\) is normal in \(\pi\) and \(\pi/C\) has two ends. The preimage in \(\pi\) of any finite normal subgroup of \(\pi/C\) is a flat 3-manifold group which is normal in \(\pi\). This must be \(\mathbb{Z}^3\), by Theorem 8.4 of [9], and so \(\pi/C\) has no non-trivial finite normal subgroup. Hence \(\pi/C \cong \mathbb{Z}\) or \(D_\infty\), and \([\pi : \sqrt{\pi}]\) divides 4. In particular, if \(\beta = 0\) the mapping cylinder construction of §4 applies, and so all \(\textit{Nil}^3\)-manifolds bound. (Note that since \(\zeta \sqrt{\pi} \cong \mathbb{Z}\) the result of [7] applies here if and only if either \(\pi = \sqrt{\pi}\) or \(\pi/\sqrt{\pi} = Z/2Z\) and acts by inversion on \(\zeta \sqrt{\pi}\).)

If \(M\) is a \(\textit{Sol}^3 \times \mathbb{E}^1\)-manifold then \(\sqrt{\pi} \cong \mathbb{Z}^3\) and the quotient \(\pi/\sqrt{\pi}\) has two ends. Therefore \(\pi \cong A \ast_C B\), where \(\sqrt{\pi} \leq C\), \([C : \sqrt{\pi}]\) is finite and \([A : C] = [B : C]\) = 2, since we are assuming that \(\beta = 0\). Since \(\pi\) is not virtually nilpotent, \([C : \sqrt{\pi}] \leq 2\), by Theorem 8.4 of [9]. In all cases \(M\) is a boundary, by the results of §4.

7. \textit{AMALGAMATION OVER \textit{Nil}^3\text{-MANIFOLD GROUPS}}

The other cases that we shall need to consider are when \(A\), \(B\) and \(C\) are fundamental groups of \(\textit{Nil}^3\)-manifolds. These have canonical Seifert fibrations, with base a flat 2-orbifold with no reflector curves. (There are seven such orbifolds: \(T\), \(Kb\), \(S(2, 2, 2, 2)\), \(P(2, 2)\), \(S(2, 4, 4)\), \(S(2, 3, 6)\) and \(S(3, 3, 3)\).) The quotients \(\overline{A} = A/\zeta \sqrt{A}\), \(\overline{B} = B/\zeta \sqrt{B}\) and \(\overline{C} = C/\zeta \sqrt{C}\) are the orbifold fundamental groups of the bases. If the image of \(g \in A\) generates a maximal finite cyclic subgroup of \(\overline{A}\) then \(\zeta \sqrt{A} \leq \langle g \rangle\), since \(\langle g, \zeta \sqrt{A} \rangle\) is torsion-free and virtually \(\mathbb{Z}\).

\textbf{Lemma 7.1.} Suppose that \(\pi \cong A \ast_C B\), where \(C\) is a \(\textit{Nil}^3\)-group and \(A = \langle C, t \rangle\) and \(B = \langle C, u \rangle\), with \(t^2, u^2 \in C\). Then

1. if \([\sqrt{A} : \sqrt{C}] = 2\) or if \(C = \sqrt{C}\) and \(A/\zeta \sqrt{A} \cong \mathbb{Z}^2 \times_{-I} Z/2Z\) then the automorphism of \(\sqrt{C}/\zeta \sqrt{C}\) induced by conjugation by \(tu\) has finite order;
2. if \(\pi\) is not virtually nilpotent then \(\sqrt{A} = \sqrt{B} = \sqrt{C}\);
3. if the inclusion of \(C\) into each of \(A\) and \(B\) induces isomorphisms \(C/\zeta \sqrt{C} \cong A/\zeta \sqrt{A}\) and \(C/\zeta \sqrt{C} \cong B/\zeta \sqrt{B}\) then \(M\) bounds.

\textbf{Proof.} If \([\sqrt{A} : \sqrt{C}] = 2\) then \(t \in \sqrt{A}\), and so \(t\) centralizes \(\sqrt{C}/\zeta \sqrt{C}\). If \(C\) is nilpotent and \(A/\zeta \sqrt{A} \cong \mathbb{Z}^2 \times_{-I} Z/2Z\) then \(t\) acts via \(-I\) on \(\sqrt{C}/\zeta \sqrt{C}\). Since \(u^2 \in C\) and \([C : \sqrt{C}]\) is finite, in each case some power of \(tu\) acts trivially on \(\sqrt{C}/\zeta \sqrt{C}\). Hence \(\pi\) is virtually nilpotent.
Part (2) is an immediate consequence of part (1).

The hypotheses of part (3) imply that $\pi/\zeta\sqrt{C} \cong C/\zeta\sqrt{C} \times D_\infty$. (Hence $\pi$ is virtually a product $\sqrt{C} \times \mathbb{Z}$.) Let $N = K(C, 1)$ and let $i$ be the free involution of $N \times D^2$ which is the antipodal map on the $S^1$ fibres of $N$ and reflection across a diameter of $D^2$. Then the quotient $N \times D^2/\langle i \rangle$ is a 5-manifold with boundary $M = K(\pi, 1)$.

As in the flat case, $\beta_1(A; \mathbb{Q}) + \beta_1(B; \mathbb{Q}) \leq \beta_1(C; \mathbb{Q}) \leq 2$. If $C = \sqrt{C}$ we may assume that either $A = \sqrt{A}$ and $K(B, 1)$ has base $S(2, 2, 2, 2)$, or the bases for $K(A, 1)$ and $K(B, 1)$ are $Kb$ or $S(2, 2, 2, 2)$.

If $[C : \sqrt{C}] = 2$ then $K(C, 1)$ has base $S(2, 2, 2, 2)$ or $Kb$. In the first case $K(A, 1)$ and $K(B, 1)$ have base $S(2, 2, 2, 2)$, $P(2, 2)$ or $S(2, 4, 4)$. In the second case we may assume that $K(A, 1)$ has base $P(2, 2)$ and $K(B, 1)$ has base $Kb$ or $P(2, 2)$.

**Lemma 7.2.** Suppose that $\pi \cong A \ast_C B$, where $C$ is a $\text{Nil}^3$-group and $A = \langle C, t \rangle$ and $B = \langle C, u \rangle$, with $t^2, u^2 \in C$. Then $w^2 = 0$ if either

1. $q = [\zeta\sqrt{C} : \zeta\sqrt{C} \cap \sqrt{C}]$ is even, and either $C = \sqrt{C}$ or $t^n, u^n \in \zeta\sqrt{C}$ for some $n \geq 2$; or
2. $C = \sqrt{C}$ and $K(A, 1)$ and $K(B, 1)$ fibre over $Kb$; or
3. $K(C, 1)$ has base $S(2, 2, 2, 2)$ and $K(A, 1)$ and $K(B, 1)$ both have base $S(2, 2, 2, 2)$; or
4. $K(C, 1)$ has base $S(2, 2, 2, 2)$ and $K(A, 1)$ and $K(B, 1)$ both have base $P(2, 2)$.

**Proof.** Since $\text{Nil}^3$-manifolds are orientable the orientation reversing elements of $\pi$ are of the form $xc$, where $x \in (A \cup B) \setminus C$ and $c \in C$. In each case, such elements have images in $\pi/\pi'$ of order divisible by 4.

This does not always hold if $K(A, 1)$ has base $P(2, 2)$ and $K(B, 1)$ has base $S(2, 4, 4)$. When $\zeta\sqrt{A} = \zeta\sqrt{B} = \zeta\sqrt{C}$ and $K(C, 1)$ and $K(A, 1)$ have bases $S(2, 2, 2, 2)$ and $P(2, 2)$, respectively, the automorphism of $\sqrt{C}/\zeta\sqrt{C}$ induced by $tu$ has matrix

$$\xi = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \left( \begin{array}{cc} m & p \\ n & -m \end{array} \right) = \left( \begin{array}{cc} m & p \\ -n & m \end{array} \right),$$

where $m^2 + np = 1$ if $K(B, 1)$ has base $P(2, 2)$ and $m^2 + np = -1$ if $K(B, 1)$ has base $S(2, 4, 4)$. If $m = 0$ this has finite order, and so $M$ is a $\text{Nil}^3 \times \mathbb{E}^1$-manifold. If $m = \pm 1$ and $np = 0$ then $K(B, 1)$ must also have base $P(2, 2)$, and $M$ is a $\text{Nil}^3 \times \mathbb{E}^1$-manifold if $n = p = 0$, and is a $\text{Nil}^4$-manifold if one of $n$ or $p$ is not 0. In all these cases $w^2 = 0$, and so $M$ bounds. Otherwise (if $m^2 = 1$ and $np = -2$, or if $|m| > 1$) the eigenvalues of $\xi$ are not roots of unity, and so $M$ is a $\text{Sol}^4$-manifold.
If \([C : \sqrt{C}] > 2\) then \(M\) must be a \(\text{Nil}^3 \times \mathbb{E}^1\)-manifold. These cases are considered in the next section. (In most such cases part (3) of Lemma 7.1 applies.)

The mapping cylinder construction appears to have limited applicability here. Let \(\Theta_m\) and \(\Psi_n\) be the self-diffeomorphisms of \(S^1 \times D^2\) given by \(\Theta_m(u, d) = (u, u^m d)\) and \(\Psi_n(u, d) = (\bar{u}, u^n d)\), for all \((u, d) \in S^1 \times D^2\), respectively, and let \(\theta_m = \Theta_m|_T\) and \(\psi_n = \Psi_n|_T\) be the restrictions to \(T = \partial(S^1 \times D^2)\). The mapping tori \(M(\Theta_m)\) and \(M(\Psi_n)\) are \(D^2\)-bundles over \(T\) and \(Kb\), respectively. The double covers of \(M(\Theta_m)\) are all diffeomorphic to \(M(\Theta_{2m})\), while the double covers of \(M(\Psi_n)\) are diffeomorphic to \(M(\Theta_{2n})\) or \(M(\Psi_{2n})\). In particular, if \(C = \sqrt{A} = \sqrt{B}\) and \(K(A, 1)\) and \(K(B, 1)\) each fibre over \(Kb\) then \(M\) bounds.

8. \(\text{Nil}^3 \times \mathbb{E}^1\)-manifolds

If \(M\) is an infranilmanifold with holonomy a finite 2-group which acts effectively on \(\zeta \sqrt{\pi}\) then \(M\) bounds, by Proposition 1.3 of [7]. (The hypotheses of the later result of [12] imply that \(M\) must be an orientable \(\text{Nil}^3 \times \mathbb{E}^1\)-manifold, and so this is of limited interest for our problem.)

Let \(M\) be a \(\text{Nil}^3 \times \mathbb{E}^1\)-manifold. Then \(\sqrt{\pi} \cong \Gamma_q \times \mathbb{Z}\), for some \(q \geq 1\), and so \(\zeta \sqrt{\pi} \cong \mathbb{Z}^2\) and \(\sqrt{\pi}/\zeta \sqrt{\pi} \cong \mathbb{Z}^2\). Moreover, \(I(\sqrt{\pi}) \cong \mathbb{Z}\) and \(I(\sqrt{\pi}) < \zeta \sqrt{\pi}\). Let \(\theta : \pi \to \operatorname{Aut}(\zeta \sqrt{\pi}), \bar{\theta} : \pi \to \operatorname{Aut}(\zeta \sqrt{\pi}/I(\sqrt{\pi}))\) and \(\psi : \pi \to \operatorname{Aut}(\sqrt{\pi}/\zeta \sqrt{\pi})\) be the homomorphisms induced by conjugation in \(\pi\). Since \(I(\sqrt{\pi})\) is a characteristic subgroup of \(\pi\), the image of \(\theta\) lies in the diagonal group \((\mathbb{Z}/2\mathbb{Z})^2\) of \(\text{GL}(2, \mathbb{Z})\). The manifold \(M\) is non-orientable if and only if \(\bar{\theta}\) is nontrivial. (In that case the holonomy \(\gamma = \pi/\sqrt{\pi}\) acts by inversion on the Euclidean factor of \(\text{Nil}^3 \times \mathbb{R}\).)

Let \(K = \ker(\theta)\). Then \(\sqrt{K} = \sqrt{\pi}\), since \(\pi \leq K \leq \pi\). Moreover, \(\zeta \sqrt{\pi} \leq \zeta K \leq \sqrt{K}\), and so \(\zeta K = \zeta \sqrt{\pi}\). The quotient \(K/\zeta K\) is a flat 2-orbifold group with holonomy \(\sqrt{K}/\sqrt{\pi}\). Since \(K\) acts trivially on \(\zeta K\) this orbifold is orientable, and so \(K/\sqrt{K}\) is cyclic, of order 1, 2, 3, 4 or 6. The preimage in \(\pi\) of any finite normal subgroup of \(\pi/I(\sqrt{\pi})\) is an infinite cyclic normal subgroup, and therefore is \(I(\sqrt{\pi})\). Therefore the induced action of \(\gamma\) on \(\sqrt{\pi}/I(\sqrt{\pi})\) is effective, and so \((\psi, \bar{\theta}) : \gamma \to \text{GL}(2, \mathbb{Z}) \times \mathbb{Z}^\omega\) is injective. Hence \(\gamma\) is isomorphic to a subgroup of \(D_{2m} \times Z/2\mathbb{Z}\), for \(n = 4\) or 6. All the possibilities are realized, except for the products \(D_{2n} \times Z/2\mathbb{Z}\), with \(n = 3, 4\) or 6 [5].

Although some \(\text{Nil}^3 \times \mathbb{E}^1\)-groups with \(\beta = 0\) are amalgamated free products \(\pi \cong A *_C B\) with \(A, B\) and \(C\) virtually \(\mathbb{Z}^3\), the cases with \(A = G_6\), \(B = G_4\) and \(C = G_2\) do not arise here, and so the corresponding manifolds bound. Thus we may assume that \(\pi \cong A *_C B\), where \(A, B\) and \(C\) are fundamental groups of \(\text{Nil}^3\)-manifolds. If \(K(C, 1)\) has base
$P(2, 2)$, $S(2, 4, 4)$ or $S(2, 3, 6)$ then $\overline{A} = \overline{B} = \overline{C}$, and so $M$ bounds, by part (3) of Lemma 7.1. However, if $K(C, 1)$ has base $S(3, 3, 3)$ then $K(A, 1)$ or $K(B, 1)$ could have base $S(2, 3, 6)$. In this case there are non-normal subgroups of index 3, with similar structures $\overline{A} \ast \sqrt{\overline{C}} \overline{B}$, where $K(\overline{A}, 1)$ and $K(\overline{B}, 1)$ have base $T$ or $S(2, 2, 2, 2)$. Since coverings of odd degree induce isomorphisms on cohomology with coefficients $\mathbb{F}_2$, we may further assume that $[C : \sqrt{C}] \leq 2$, and that $\gamma = \pi/\sqrt{\pi}$ is a 2-group, of order dividing 8.

If $\gamma = Z/2Z$ then $\gamma$ must act trivially on $I(\sqrt{\pi})$ and via $-I_3$ on $\sqrt{\pi}/I(\sqrt{\pi}) \cong \mathbb{Z}^3$ (since $\beta = 0$). Thus $\gamma$ acts effectively on $\zeta \sqrt{\pi}$, and so $M$ bounds, by Proposition 1.3 of [7]. Thus we may assume that either $\gamma = (Z/2Z)^2$ and $\zeta \pi = I(\sqrt{\pi})$ (i.e., $\gamma$ does not act effectively on $\zeta \sqrt{\pi}$) or $\gamma = Z/4Z, Z/4Z \oplus Z/2Z, (Z/2Z)^3$ or $D_8$.

If $C = \sqrt{C}$ then the orientable double cover of $M$ is a Spin 4-manifold. If, moreover, either $K(A, 1)$ and $K(B, 1)$ both fibre over $Kb$ or $q = [\zeta \sqrt{C} : \zeta \sqrt{C} \cap \sqrt{C}]$ is even then $w_2^2 = 0$ and so $M$ bounds, by part (1) of Lemma 7.2. If $K(C, 1)$ has base $S(2, 2, 2, 2)$ and $\sqrt{A} = \sqrt{B} = \sqrt{C}$ (and $\pi$ is virtually nilpotent) then $w_2^2 = 0$. There are mapping tori of self-diffeomorphisms of such $K(C, 1)$ which are not Spin [10]. Thus the cases when $K(A, 1)$ and $K(C, 1)$ have base $S(2, 2, 2, 2)$ may give examples of $\text{Nil}^3 \times \mathbb{E}^1$-manifolds which are not boundaries.

9. $\text{Sol}^4$-MANIFOLDS

If $M$ is a $\text{Sol}^4$-manifold then $\sqrt{\pi} \cong \Gamma_q$ for some $q \geq 1$, and $\pi/\sqrt{\pi}$ has two ends. Therefore $\pi \cong A \ast_C B$, where $[A : C] = [B : C] = 2$, $\sqrt{\pi} = \sqrt{C}$ and $[C : \sqrt{C}]$ is finite. Thus $A$, $B$ and $C$ are fundamental groups of $\text{Nil}^3$-manifolds. Since $\pi$ is not virtually nilpotent, $[C : \sqrt{C}] \leq 2$, by Theorem 8.4 of [9], and so $[A : \sqrt{\pi}]$ and $[B : \sqrt{\pi}]$ are each $\leq 4$. Moreover $\sqrt{A} = \sqrt{B} = \sqrt{C}$, by part (2) of Lemma 7.1. The possibilities are limited further by the fact that $\pi$ cannot have $\mathbb{Z}^2$ as a normal subgroup, since $\text{Sol}^4$-manifolds are not Seifert fibred. In particular, $K(C, 1)$ cannot be fibred over $Kb$, for otherwise the characteristic subgroup $I(C) \cong \mathbb{Z}^2$ would be normal in $\pi$.

If $C = \sqrt{\pi}$ then $K(A, 1)$ and $K(B, 1)$ are $S^1$-bundles over $Kb$, by part (1) of Lemma 7.1. The mapping cylinder construction then applies to show that $M$ bounds. If $[C : \sqrt{\pi}] = 2$ then $K(C, 1)$ has base $S(2, 2, 2, 2)$, and so $K(A, 1)$ and $K(B, 1)$ have bases $P(2, 2)$ or $S(2, 4, 4)$. If the bases are the same then $w_2^2 = 0$, by parts (3) and (4) of Lemma 7.2, and so $M$ bounds. There remains the possibility that $K(A, 1)$ has base $S(2, 4, 4)$ and $K(B, 1)$ has base $P(2, 2)$. 


Theorem 9.1. Let $M$ be a $\text{Sol}^4$-manifold with $\pi = \pi_1(M) \cong A \ast_C B$, where $K(A, 1)$ is Seifert fibred over $S(2, 4, 4)$ and $K(B, 1)$ is Seifert fibred over $P(2, 2)$. If $q = [\zeta \sqrt{C} : \zeta \sqrt{C} \cap \sqrt{C}]$ is odd then $M$ bounds if and only if $w_2 = 0$.

Proof. Since $K(C, 1)$ is a double cover of each of $K(A, 1)$ and $K(B, 1)$, it is Seifert fibred over $S(2, 2, 2, 2)$, and $\sqrt{\Delta} = \sqrt{B} = \sqrt{C}$. The orbifold fundamental groups of the bases $\overline{A} = \pi^{\text{orb}}(S(2, 4, 4))$ and $\overline{B} = \pi^{\text{orb}}(P(2, 2))$ have presentations $\langle a, x \mid a^4 = (a^2x)^2, [x, axa^{-1}] = 1 \rangle$ and $\langle j, u \mid j^2 = (jua^2)^2 = 1 \rangle$, and their maximal abelian normal subgroups are $\langle x, axa^{-1} \rangle$ and $\langle u^2, (ju)^2 \rangle$, respectively.

After suitable normalizations we may assume that $A$ has a presentation

$$
\langle a, x, y \mid y = axa^{-1}, [x, y] = a^4q, a^2xa^{-2} = x^{-1} \rangle,
$$

and that $C = \langle a^2, x, y \rangle$. We may then assume that $B$ has a presentation

$$
\langle j, k, x, y \mid [x, y] = j^2q, jxy^{-1} = x^{-1}, jyx^{-1} = y^{-1}, kxk^{-1} = x^my^n, j^2 = x^py^mq^n, (jk)^2 = x^py^mq^n \rangle,
$$

where $m$ is odd and $p$ and $n$ are even (since $(\frac{m}{n}, \frac{p}{m})$ must be conjugate to $(\frac{1}{n}, \frac{0}{1})$), and $ru - ts = \pm 1$. Here $C$ is the subgroup $\langle j, x, y \rangle$, and we may identify $j$ with $a^2$. Hence $\pi$ has a presentation

$$
\langle a, k, x, y \mid axa^{-1} = y, a^2xa^{-2} = x^{-1}, kxk^{-1} = x^my^na^4, j, k^2 = x^py^mq^n, (a^2k)^2 = x^py^mq^n, [x, y] = a^4q \rangle.
$$

Abelianizing this presentation gives $[x] = [y]$, $4q[a] = 0$, $2[x] = 0$, $(m + n + 1)[x] = 4q[a]$, $(m + p + 1)[x] = 4f[a]$, $2[k] = (r + s)[x] + 4q[a]$ and $2[k] = (t + u)[x] + 4(h - 1)[a]$. Since $m + n + 1$ and $m + p + 1$ are even two of these simplify to $4q[a] = 4f[a] = 0$. Moreover $2q[k] = q[x]$. Since $r + s$ and $t + u$ cannot both be even, we can solve for $[x]$ in terms of $[a]$ and $[k]$. If they are both odd then $\pi/\pi' \cong Z/4qZ \oplus Z/4Z$, where $q = h.c.f.\{q, e, f, g - h + 1\}$, and then $w_2 = 0$. Otherwise $\pi/\pi' \cong Z/4qZ \oplus Z/2Z$, where $q$ divides $h.c.f.\{q, e, f\}$, and $w_2 \neq 0$. If (say) $r + s$ is even then $2([k] - 2g[a]) = 0$ and so $ka^{-2g}$ is an orientation reversing element with image in $\pi/\pi'$ of order 2.

The projection to the quotient $\pi/\langle (a^4, (ak)^2, x) \rangle \cong D_e$ induces an isomorphism $H^1(D_e; \mathbb{F}_2) \cong H^1(\pi; \mathbb{F}_2) = \langle u, w \rangle$. Since $uw = 0$ in $H^2(D_e; \mathbb{F}_2)$ it follows that $uw = 0$ in $H^2(\pi; \mathbb{F}_2)$ also.

The orientable double cover of $M$ is the mapping torus of the self-diffeomorphism of $K(C, 1)$ corresponding to $t = ak$, and is not a Spin manifold, since $q$ is odd. (See §7 of [10].) Therefore $w_2(M) \neq 0$ or $w^2$. It now follows from part (2) of Lemma 3.7 that $w^4 \neq 0$, and so $M$ does not bound. \qed
In particular, the $\text{Sol}^4_1$-manifold $M$ whose group has presentation
\[ \langle a, k, x, y \mid axa^{-1} = y, \ a^2xa^{-2} = x^{-1}, \ kxk^{-1} = x^3y^{-4}, \ kyk^{-1} = x^2y^{-3}, \ k^2 = xy^{-1}, \ (a^2k)^2 = xy^{-2}, \ [x, y] = a^4 \rangle. \]
is not a boundary.

10. $S^1$-bundle spaces

In many cases a 4-dimensional infrasolv manifold $M$ is the boundary of the total space of a $D^2$-bundle over a 3-manifold.

In all, 50 of the 74 flat 4-manifolds are total spaces of $S^1$-bundles. The exceptions have $\beta \leq 1$, and are three with group $G_2 \rtimes \mathbb{Z}$ (all non-orientable), three with group $G_3 \rtimes \mathbb{Z}$ (all orientable), two with group $G_4 \rtimes \mathbb{Z}$ (both orientable), one with group $G_5 \rtimes \mathbb{Z}$ (orientable), twelve with group $G_6 \rtimes \mathbb{Z}$ (seven orientable) and three with $\beta = 0$ and groups $G_2 \ast_{\varphi} B_2$, $G_6 \ast_{\varphi} B_3$ and $G_6 \ast_{\varphi} B_4$ (all non-orientable). In §11 we shall show that the mapping cylinder construction applies to most of these.

Coset spaces of $\text{Nil}^3 \times \mathbb{R}$ or $\text{Sol}^3 \times \mathbb{R}$ are products $N \times S^1$, with $N$ a $\text{Nil}^3$- or $\text{Sol}^3$-coset space, respectively, and so bound $N \times D^2$. Coset spaces of $\text{Nil}^4$ or $\text{Sol}^4_1$ are also $S^1$-bundle spaces, since the action of the centre $\mathbb{R}$ induces a free $S^1$-action on the coset space. A $\text{Nil}^4$-manifold is such a coset space if and only if $\beta = 2$, while a $\text{Nil}^3 \times \mathbb{E}^1$-manifold is such a coset space if and only if $\beta = 3$. These coset spaces are orientable, and so bound orientably.

If $M$ is a $\text{Nil}^4$-manifold or a $\text{Nil}^3 \times \mathbb{E}^1$-manifold, but is not a coset space, then $\beta \leq 1$ or $\beta \leq 2$, respectively. If $M$ is non-orientable and $\beta > 0$, or if $M$ is an orientable $\text{Nil}^3 \times \mathbb{E}^1$-manifold and $\beta = 2$, then $\pi \cong \nu \rtimes_{\varphi} \mathbb{Z}$, where $\nu = \mathbb{Z}^3, G_2, B_1$ or $B_2$. (See Theorems 8.4 and 8.9 of [9].) The manifold $M$ is the mapping torus of a self-diffeomorphism of the corresponding flat 3-manifold $N$. (If $M$ is orientable then $\nu = \mathbb{Z}^3$ or $G_2$, and if $M$ is a non-orientable $\text{Nil}^4$-manifold then $\nu = \mathbb{Z}^3$.) If $\nu = \mathbb{Z}^3$ or $G_2$ then $\theta|_{I(\nu)}$ has an eigenvalue $\pm 1$, since $\pi$ is virtually nilpotent. (If $\beta = 1$ and $\nu = \mathbb{Z}^3$ the eigenvalue must be $-1$.) The quotient of $\pi$ by the corresponding infinite cyclic normal subgroup is torsion-free, and so $M$ is also the total space of an $S^1$-bundle over a closed 3-manifold. A similar result holds if $\nu = B_1$ or $B_2$, for in these cases $I(\nu) \cong \mathbb{Z}$.

Orientable $\text{Nil}^3 \times \mathbb{E}^1$- and $\text{Nil}^4$-manifolds with $\beta = 1$, and all orientable $\text{Sol}^4_1$-manifolds (which have $\beta = 1$) are mapping tori of diffeomorphisms of $\text{Nil}^3$-manifolds. If the fibre is a $\text{Nil}^3$-coset space, with group $\nu = \sqrt{\nu}$, then $\pi/I(\nu)$ is torsion-free, and so the 4-manifold is the total space of an $S^1$-bundle over a $\text{Nil}^3$-manifold. However if $\nu \neq \sqrt{\nu}$
then $\pi$ has no infinite cyclic normal subgroup with torsion-free quotient, and the manifold is not an $S^1$-bundle space.

If $M$ is a $\text{Sol}^3 \times E^1$-manifold then $\beta \leq 2$, and if $\beta = 2$ then $\pi \cong \mathbb{Z}^3 \rtimes_{\theta} \mathbb{Z}$. In this case $\theta$ has an eigenvalue 1, and so $M$ is an $S^1$-bundle space. This is also the case if $\beta = 1$ and $\pi \cong \sigma \rtimes \mathbb{Z}$, where $\sigma$ is the group of a $\text{Sol}^3$-manifold, or $\beta = 0$.

11. MAPPING CYLINDER CONSTRUCTIONS

The mapping cylinder construction of Lemma 4.1 and 4.2 apply to many of the flat 4-manifolds which are not realizable by $S^1$-bundle spaces. We note here the following variation: if $c : Z \to X$ is a double cover and $f$ is a self-diffeomorphism $X$ such that $f_* c_* \pi_1(Z) = c_* \pi_1(Z)$ then $f$ extends to a self-diffeomorphism $F$ of $MCyl(c)$, and so $M(f) = \partial M(F)$.

All the mapping tori of self-diffeomorphisms of orientable flat 3-manifolds with cyclic holonomy and $\beta = 1$ also fibre over $K\theta$, and so their groups map onto $D_\infty$. The groups $G_6 \rtimes_{\theta} \mathbb{Z}$ corresponding to the outer automorphism classes $\theta = a, ab, i$ and $ei$ also map onto $D_\infty$. The groups corresponding to $cej, abej$ and $j$ have abelianization $\mathbb{Z}$, and so Lemma 4.2 does not apply to these. The classes $ace = (ci)^2$, $bce = (ei)^2$ and and $abcej = j^4$ are squares in $\text{Out}(G_6)$ (as are $1 = 1^2$ and $ab = (cei)^2$). These bound, since $M(f^2)$ bounds the mapping cylinder of the canonical double cover of $M(f)$. (Since $cei$ and $ci$ are orientation-reversing, two of these mapping cylinders are orientable.) The classes $a, ce, cei, ci$ and $j$ are not squares, since they are orientation-reversing. The classes $i$ and $ei$ are not squares, as they have order 4 and $\text{Out}(G_6)$ has no elements of order 8. The class $cej$ is not a square, as it has order 6 and $\text{Out}(G_6)$ has no elements of order 12.

The mapping cylinder construction applies to show that each of the four flat 4-manifolds with $\beta = 0$ is a boundary. There remain five flat 4-manifolds (corresponding to $ce, cei, cej, ci$ and $j$) for which we do not yet have simple cobounding 5-manifolds, and a further two orientable flat 4-manifolds (corresponding to $abej$ and $bce$) for which we do not have simple orientable cobounding 5-manifolds.

12. EMBEDDING FLAT 4-MANIFOLDS IN $\mathbb{R}^n$

If a closed smooth $n$-manifold embeds in $\mathbb{R}^k$ then the $k$th normal Stiefel-Whitney classes $w_k(M)$ is 0, since this is the $\text{mod}(2)$ normal Euler class. (See Theorem 10.2 of [11].) This necessary condition is also sufficient when $n = 4$ and $k = 3$: a closed smooth 4-manifold
\( M \) embeds in \( \mathbb{R}^7 \) if and only if \( \varpi_3(M) = 0 \) [6]. (Note that \( \varpi_3(M) = w_3(M) + w_1(M)^3 = Sq w_2(M) + w_1(M)^3 \), by the Whitney sum theorem and the Wu formulae.) In particular, every orientable closed smooth 4-manifold embeds in \( \mathbb{R}^7 \). An orientable closed smooth 4-manifold \( M \) embeds in \( \mathbb{R}^6 \) if and only if \( w_2(M) = 0 \) and \( \sigma(M) = 0 \) [2]. However, there is as yet no general criterion for non-orientable 4-manifolds to embed in \( \mathbb{R}^6 \).

It follows from these results (and Lemma 3.1) that if a 4-dimensional infrasolvmanifold \( M \) is a boundary and \( w_3(M) = 0 \) then \( M \) embeds in \( \mathbb{R}^7 \), since \( w_1^4 = 0 \) implies \( w_1^3 = 0 \), by Lemma 3.2, and then \( \varpi_3(M) = 0 \). If \( M \) is orientable then it embeds in \( \mathbb{R}^6 \) if and only if \( w_2(M) = 0 \).

In [10] it is shown that \( w_2 \) is integral (and hence \( w_3 = 0 \)) for all but at most two flat 4-manifolds. The exceptions have groups \( \pi = G_6 \rtimes_\omega \mathbb{Z} \) or \( G_6 \rtimes_\omega \mathbb{Z} \). When \( \pi = G_6 \rtimes_\omega \mathbb{Z} \), the Wang sequence for \( \pi \) as an extension of \( \mathbb{Z} \) and the Universal Coefficient Theorem imply that \( H^2(\pi; Z/4Z) \cong (Z/4Z)^2 \) maps onto \( H^2(\pi; F_2) \). Therefore \( w_3 = Sq w_2 = 0 \). Thus, with one possible exception, every 4 flat 4-manifold embeds smoothly in \( \mathbb{R}^7 \).

Three orientable flat 4-manifolds have \( w_2 \neq 0 \); they are mapping tori of self-diffeomorphisms of \( HW \), corresponding to \( \theta = e, bce \) or \( e \) in \( Out(G_6) \). The other 24 embed in \( \mathbb{R}^6 \). Since \( \varpi_2(M) = w_2(M) + w_1(M)^2 \), non-orientable flat 4-manifolds which embed in \( \mathbb{R}^6 \) must have \( Pin^- \) structures. This condition excludes 15 of the 47 non-orientable flat 4-manifolds, but we do not know whether all the others embed in \( \mathbb{R}^6 \).

If \( M \) embeds in \( \mathbb{R}^5 \) then it bounds a compact region and is s-parallelizable. Thus \( M \) is parallelizable if also \( \chi(M) = 0 \). Moreover, if \( X \) and \( Y \) are the closures of the components of \( S^5 \setminus M \) then \( X \) and \( Y \) are connected and \( H^1(X) \oplus H^1(Y) \cong H^1(M) \). In particular, if \( \beta = 1 \) then \( M \) has an essentially unique infinite cyclic covering \( M' \), and this bounds a covering of \( X \), say. Let \( t \) generate the covering group, and let \( T \) be the maximal finite submodule of \( H_1(M; \Lambda) \). Then Poincaré duality with coefficients in the group ring \( \Lambda = \mathbb{Z}[t, t^{-1}] \) and the Universal coefficient spectral sequence together give an isomorphism \( T \cong Ext^2_\Lambda(T, \Lambda) \). This is equivalent to a non-degenerate pairing \( \ell_p : T \times T \to \mathbb{Q}/\mathbb{Z} \), with an isometric action of the covering group. When \( M' \) is homotopy equivalent to a 3-manifold this pairing is the standard torsion linking pairing on \( M' \), with the action of the covering group \( \langle t \rangle \). (In knot theory this pairing is known as the Farber-Levine pairing.) If \( M = \partial W \) and \( p \) extends to a homomorphism from \( \pi_1(W) \) to \( \mathbb{Z} \) then \( K = \text{Ker}(T \to H_1(W; \Lambda)) \) is a submodule which is its own annihilator with respect to \( \ell_p \). Hence \( \ell_p \) is metabolic.
Every closed 3-manifold $N$ embeds in $\mathbb{R}^5$ [13]. The normal bundle of an embedding $j : N \rightarrow \mathbb{R}^5$ is classified by an Euler class $e(j) \in H^2(N; \mathbb{Z}^w) \cong H_1(N; \mathbb{Z})$. If $M$ is the boundary of a regular neighbourhood of $j$ then $M$ is the total space of an $S^1$-bundle over $N$, and $e(j)$ is also the class of the corresponding extension of $\pi_1(N)$ by $\mathbb{Z}$. If $N$ is orientable the normal bundle is trivial, and so $M = N \times S^1$.

The six orientable flat 4-manifolds which are products $N \times S^1$ (with groups $G_i \rtimes \mathbb{Z}$, for $1 \leq i \leq 6$) all embed in $\mathbb{R}^5$. Since $G_{3i}^{ab} \cong \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ and $G_{4i}^{ab} \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, the flat 4-manifolds with groups $G_i \rtimes_\theta \mathbb{Z}$ (for $i = 3$ or 4) and $\beta = 1$ do not embed in $\mathbb{R}^5$. The group $G_{6i}^{ab} \cong (\mathbb{Z}/4\mathbb{Z})^2$ does not have a subgroup which is its own annihilator with respect to the torsion linking pairing of $HW$, and so no flat 4-manifold with group $G_i \rtimes \mathbb{Z}$ and $\beta = 1$ can embed in $\mathbb{R}^5$. However, such considerations do not apply to the flat 4-manifold with group $G_5 \rtimes_\theta \mathbb{Z}$ and $\beta = 1$, since $G_5^{ab} \cong \mathbb{Z}$ is torsion-free. In this case $H_1(\pi) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ is the sum of two cyclic groups. Since the corresponding flat 4-manifold $M$ has $w_2(M) = 0$ and $\sigma(M) = 0$, it embeds in $\mathbb{R}^5$, by Theorem 6.2 of [3].

If $\pi \cong \mathbb{Z}^3 \rtimes_T \mathbb{Z}$ has cyclic holonomy and $\beta = 2$, then any basis for $\pi/I(\pi) \cong \mathbb{Z}^2$ will contain at least one element whose image generates the holonomy. Therefore if $M$ embeds in $S^5$ with closed complementary regions $X$ and $Y$ there will be an infinite cyclic cover $M'$ with fundamental group an orientable flat 3-manifold group with the same holonomy, which bounds an infinite cyclic cover of $X$, say. This is again impossible if the holonomy has order 3 or 4.

The remaining six orientable flat 4-manifolds are mapping tori of self-diffeomorphisms of the half-turn flat 3-manifold, with groups $G_2 \rtimes \mathbb{Z}$, and five of these have $\beta = 1$. These also fibre over non-orientable flat 3-manifolds. In three of these cases the group is a semidirect product $\mathbb{Z} \rtimes_w B_i$, where $w = w_1(B_2)$ and $2 \leq i \leq 4$. These correspond to $S^1$-bundles with a section, i.e., to bundles with Euler class 0. We shall show that they each embed in $\mathbb{R}^5$.

If a flat 4-manifold $M$ is the boundary of a regular neighbourhood of an embedding $j$ of a non-orientable flat 3-manifold $N$ in $\mathbb{R}^5$, then $\pi = \pi_1(M)$ is a non-trivial extension of $\pi_1(N)$ by $\mathbb{Z}$, $\beta = \beta_1(N)$ and $e(j)$ must have finite order. In particular, if $\pi_1(N) = B_1$ or $B_2$ then $\pi \cong G_2 \times \mathbb{Z}$ or $\mathbb{Z} \rtimes_w B_2$. The semidirect product is the only orientable, virtually abelian extension of $B_2$ by $\mathbb{Z}$, since $H_1(B_2; \mathbb{Z})$ is torsion-free. If $\pi_1(N) = B_3$ or $B_4$ then $\beta = 1$, $\pi \cong G_2 \rtimes_\theta \mathbb{Z}$ and the holonomy is $(\mathbb{Z}/2\mathbb{Z})^2$.

Since $Kb$ embeds in $G_2$, $Kb \times S^1$ embeds in $\mathbb{R}^5$ with normal Euler class 0, and so the flat 4-manifold with group $\mathbb{Z} \rtimes_w B_1$ embeds. (This is of course $G_2 \times S^1$.) Let $R$ be the orientation preserving involution of
\(D^2 \times D^2\) which swaps the factors. Then \(R\) restricts to an orientation-reversing involution of \(T = S^1 \times S^1\), and \(M(R_T) \cong K(B_2, 1)\) embeds in \(M(R) \cong S^1 \times D^3 \subset \mathbb{R}^5\). Since this embedding can be isotoped off itself, the flat 3-manifold \(K(B_2, 1)\) embeds in \(\mathbb{R}^5\), with normal Euler class 0.

Two of the non-orientable flat 3-manifolds fibre over the torus, while the other two fibre over the Klein bottle. Let \(p_i : E_i \to F\) be the projection of the associated \(\mathbb{R}^3\)-bundle, let \(s : F \to E_i\) be the 0-section, and let \(j_i : K(B_i, 1) \to E_i\) be the natural inclusion of the unit circle bundle. Note that \(j_i\) may be isotoped to a disjoint nearby embedding. Let \(\eta_i\) be the line bundle over \(F\) with \(w_1(\eta_i) = s^*w_1(E_i)\). Then the Whitney sum \(p_i \oplus \eta_i\) is an \(\mathbb{R}^3\)-bundle over \(F\), with orientable total space \(\hat{E}_i = E(p_i \oplus \eta_i)\).

If \(i = 2\) or 4 the fibres of the projections \(p_i j_i\) have image 0 in \(H_1(B_i; \mathbb{F}_2)\), and so \(p_i j_i\) induces isomorphisms \(H^q(F; \mathbb{F}_2) \cong H^q(B_i; \mathbb{F}_2)\), for \(q \leq 2\). Since \(w_2 = w_2^2\) for any 3-manifold, by the Wu relations, the Whitney sum formula gives \(w_2(\hat{E}_i) = 0\). Regular neighbourhoods of any embedding of \(T\) or \(Kb\) in \(\mathbb{R}^5\) are \(D^3\)-bundles with parallelizable total space. Therefore if \(i = 2\) or 4 then \(\hat{E}_i\) embeds in \(\mathbb{R}^5\). Hence the flat 3-manifold \(K(B_i, 1)\) also embeds in \(\mathbb{R}^5\), with normal Euler class 0. The boundary of a regular neighbourhood is an orientable flat 4-manifold with group \(\mathbb{Z} \ltimes_w B_4\).

When \(i = 1\) or 3 it is not so clear that \(w_2(\hat{E}_i) = 0\). Instead we use more explicit constructions. We have already done this for \(i = 1\). We may embed \(Kb\) in \(S^1 \times D^3\) as the subset \(\{(u^2, x, yu) \mid u \in S^1, x, y \in \mathbb{R}, x^2 + y^2 = 1\}\). Let \(h\) be the orientation-preserving diffeomorphism of \(S^1 \times D^3\) given by \(h(u, x, y, z) = (\bar{u}, x, y, -z)\). Then \(h\) reverses the \(S^1\) factor, \(h(Kb) = Kb\) and \(h\) fixes pointwise the fibre of \(Kb\) over \(u = 1\). The mapping torus \(M(h)\) is an orientable \(D^3\)-bundle over \(Kb\), and \(M(h)(Kb) = B_3\). Since \(h|_0\) has 1-dimensional fixed point set, the boundary of \(M(h)\) is the orientable \(S^2\)-bundle over \(Kb\) with \(w_2 = 0\), and so \(w_2(M(h)) = 0\). Therefore \(M(h)\) embeds in \(\mathbb{R}^5\) as a regular neighbourhood of an embedding of \(Kb\). Hence \(K(B_3, 1)\) also embeds in \(\mathbb{R}^5\), with normal Euler class 0. The boundary of a regular neighbourhood is an orientable flat 4-manifold with group \(\mathbb{Z} \ltimes_w B_3\).

One of the three remaining groups \(G_2 \ltimes \mathbb{Z}\) has abelianization \(\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}\). The corresponding flat 4-manifold embeds in \(\mathbb{R}^5\), by Theorem 6.2 of [3]. The group is a non-split extension of \(B_4\) by \(\mathbb{Z}\), and so the normal Euler class is a non-zero torsion class.

The two undecided cases have groups with presentations
\[
\langle t, x, y, z \mid ttx^{-1} = x^{-1}yz, tgy = yt, tz^{-1}t = z^{-1}\rangle,
\]
\[
xyx^{-1} = y^{-1}, \quad xzx^{-1} = z^{-1}, \quad yz = zy
\]
and
\[
\langle t, x, y, z \mid txt^{-1} = x^{-1}, \quad tyt^{-1} = z, \quad tzt^{-1} = y, \quad xyx^{-1} = y^{-1}, \quad xzx^{-1} = z^{-1}, \quad yz = zy \rangle,
\]
respectively. These manifolds are \textit{Spin}, and so embed in \(\mathbb{R}^6\). In each case the Farber-Levine pairing is metabolic, and so provides no obstruction to an embedding in \(\mathbb{R}^5\). On the other hand, the abelianizations each need at least three generators, and so the result of [3] does not apply.

\section*{References}


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