

# PRODUCT SET PHENOMENA FOR COUNTABLE GROUPS

MICHAEL BJÖRKLUND AND ALEXANDER FISH

**ABSTRACT.** We develop in this paper general techniques to analyze local combinatorial structures in product sets of two subsets of a countable group which are "large" with respect to certain classes of (not necessarily invariant) means on the group. As applications of our methods, we extend and quantify a series of recent results by Jin, Bergelson-Furstenberg-Weiss, Beiglböck-Bergelson-Fish, Griesmer and diNasso-Lupini to general countable groups.

## 1. INTRODUCTION

**1.1. General comments.** Let  $G$  be a group and let  $\mathfrak{L}, \mathfrak{R}$  and  $\mathfrak{S}$  be given sets of subsets of  $G$ . We shall think of  $\mathfrak{L}$  and  $\mathfrak{R}$  as defining two (possibly different) classes of *large* subsets of the group and the elements of  $\mathfrak{S}$  will be regarded as the *structured* subsets of  $G$ .

In this paper, the term *product set phenomenon* (with respect to the sets  $\mathfrak{L}, \mathfrak{R}$  and  $\mathfrak{S}$ ) will refer to the event that whenever  $A \in \mathfrak{L}$  and  $B \in \mathfrak{R}$ , then their *product set*  $AB$ , defined by

$$AB := \{a \cdot b : a \in A, b \in B\} \in \mathfrak{S}.$$

If this happens, we shall say that the pair  $(\mathfrak{L}, \mathfrak{R})$  is  $\mathfrak{S}$ -*regular*.

Perhaps the first occurrence of a (non-trivial) product set phenomenon recorded in the literature is the following classical observation, which is often attributed to Steinhaus (see e.g. [23]). Let  $G$  be a locally compact group with left Haar measure  $m$  and define

$$\mathfrak{L} := \{A \in \mathcal{B}(G) : m(A) > 0\},$$

where  $\mathcal{B}(G)$  denotes the set of Borel sets of  $G$ . Let  $\mathfrak{S}$  denote the set of all subsets of  $G$  with non-empty interior. Then the pair  $(\mathfrak{L}, \mathfrak{L})$  is  $\mathfrak{S}$ -regular, that is to say, the product set of any two Borel sets with positive Haar measures contains a non-empty open set.

**1.2. Structured sets in countable groups.** This paper is concerned with product set phenomena in *countable* groups. To explain our main results, we first need to define what classes of *large* sets and *structured* sets we shall consider. We begin by describing our choices of the structured sets.

Let  $G$  be a countable group. A set  $T \subset G$  is *right thick* if whenever  $F \subset G$  is a finite subset, then there exists  $g$  in  $G$  such that

$$F \cdot g \subset T.$$

We say that a set  $C \subset G$  is *left syndetic* if it has non-trivial intersection with any right thick set, or equivalently, if there exists a finite set  $F \subset G$  such that  $FC = G$ . Let Syn

---

2010 *Mathematics Subject Classification.* Primary: 37B05; Secondary: 05C81, 11B13, 11K70 .

*Key words and phrases.* Ergodic Ramsey theory, random walks on groups, topological dynamics, additive combinatorics.

denote the set of all left syndetic subsets of  $G$  and define the class of *right piecewise left syndetic sets* PW-Syn by

$$\text{PW-Syn} := \left\{ C \cap T : C \text{ is left syndetic and } T \text{ is right thick.} \right\}.$$

Equivalently, a set  $C \subset G$  is right piecewise left syndetic if there exists a finite set  $F \subset G$  such that  $FC$  is right thick.

A particularly nice sub-class of left syndetic sets is formed by the so called *Bohr sets*. Recall that a set  $C \subset G$  is *Bohr* if there exist a compact hausdorff group  $K$ , an epimorphism  $\tau : G \rightarrow K$  (a homomorphism with dense image) and a non-empty open set  $U \subset K$  such that

$$C \supset \tau^{-1}(U).$$

Let Bohr denote the set of all Bohr sets in  $G$  and define the class PW-Bohr of *right piecewise Bohr sets* by

$$\text{PW-Bohr} := \left\{ C \cap T : C \text{ is Bohr and } T \text{ is right thick.} \right\}.$$

The classes Syn, PW-Syn, Bohr and PW-Bohr will all be used as structured subsets of  $G$  in this paper.

**1.3. Large sets in amenable groups.** The *large* subsets of  $G$  will primarily be defined in terms of means on  $G$ . Recall that a *mean* on a countable  $G$  is a positive linear functional  $\lambda$  on  $\ell^\infty(G)$  with norm one. Alternatively, we can think of a mean as a *finitely additive* probability measure  $\lambda'$  on  $G$  via the correspondence

$$\lambda'(C) = \lambda(\chi_C), \quad C \subset G,$$

where  $\chi_C$  denotes the indicator function of the set  $C$ . Let  $\mathcal{M}(G)$  denote the set of all means on  $G$ . If  $\mathcal{C} \subset \mathcal{M}(G)$ , then we define the upper and lower  $\mathcal{C}$ -density of a set  $C \subset G$  by

$$d_{\mathcal{C}}^*(C) = \sup_{\lambda \in \mathcal{C}} \lambda'(C) \quad \text{and} \quad d_{\mathcal{C}}^{\mathcal{C}}(C) = \inf_{\lambda \in \mathcal{C}} \lambda'(C)$$

respectively. We say that a set is  $\mathcal{C}$ -large if  $d_{\mathcal{C}}^*(C)$  is positive and  $\mathcal{C}$ -conull if  $d^*(C)$  equals one.

Recall that a countable group  $G$  is *amenable* if there exists a left invariant mean on  $G$ , i.e. if the set

$$\mathcal{L}_G := \left\{ \lambda \in \mathcal{M}(G) : \rho(g)^*\lambda = \lambda, \quad \forall g \in G \right\},$$

where  $\rho$  denotes the left regular representation on  $\ell^\infty(G)$  is non-empty. The class of amenable groups contains all abelian groups, all locally finite groups and all groups of subexponential growth. The free groups on at least two generators are well-known examples of non-amenable groups. The upper and lower  $\mathcal{L}_G$ -densities on an amenable group are often referred to as the *upper and lower Banach densities* in the literature.

In the very influential classical paper [9], Følner (inspired by earlier works of Bogolyubov in [6]) observed that if  $G$  is a countable amenable group and  $A \subset G$  is  $\mathcal{L}_G$ -large, then its *difference set*  $AA^{-1}$  is left syndetic and right piecewise Bohr. In fact, Følner showed that one can always find a Bohr set  $B$ , which contains the identity element, and a right thick set  $T$  with the property that  $T \cap S$  is right thick for *every* right thick set  $S$  in  $G$  such that

$$AA^{-1} \supset B \cap T.$$

More recently, Jin in [14], Bergelson, Furstenberg and Weiss in [4] (for the integers) and Bergelson, Beiglböck and the second author of this paper in [3], showed that whenever  $A$  and  $B$  are  $\mathcal{L}_G$ -large subsets of a countable amenable group  $G$ , then the product set  $AB$  is both right piecewise left syndetic and right piecewise Bohr. In other words, the pair  $(\mathcal{L}_G, \mathcal{L}_G)$  is PW-Bohr-regular (and hence PW-Syn-regular). Note however that already easy examples show that  $(\mathcal{L}_G, \mathcal{L}_G)$  is not Syn-regular.

**1.4. Pit-falls for non-amenable groups.** Let  $G$  be a countable group and suppose  $\mu$  is a probability measure on  $G$  such that  $\mu(g) = \mu(g^{-1})$  for all  $g$  and the support of  $\mu$  generates  $G$ . We shall call such a measure *adapted* and refer to the pair  $(G, \mu)$  as a *measured group*. If  $\rho$  denotes the left regular representation of  $G$  on  $\ell^\infty(G)$ , then we let  $\rho(\mu)$  denote the operator

$$\rho(\mu)\varphi(x) = \int_G \varphi(g^{-1}x) d\mu(g), \quad \varphi \in \ell^\infty(G).$$

Classical arguments (e.g. Kakutani's Fixed Point Theorem) show that the set

$$\mathcal{L}_\mu = \left\{ \lambda \in \mathcal{M}(G) : \rho(\mu)^* \lambda = \lambda \right\}$$

is non-empty for every measured group  $(G, \mu)$ . We think of  $\mathcal{L}_\mu$  as a substitute for  $\mathcal{L}_G$  when the group is not amenable. Indeed, if  $G$  is amenable, then it is well-known (see e.g. [16] or [21]) that there exists a probability measure  $\mu$  as above, such that  $\mathcal{L}_\mu = \mathcal{L}_G$ . The study of  $\mathcal{L}_\mu$ -large sets was initiated by Furstenberg and Glasner in [10].

One of the aims of this paper is to extend the results mentioned above (for amenable groups) to the setting of general countable groups and  $\mathcal{L}_\mu$ -densities. There are however some serious pit-falls in the non-amenable case which the reader should be aware of before we formulate our results.

Note that Følner's Theorem implies that whenever  $G$  is a countable *amenable* group and  $A$  and  $B$  and  $C$  are  $\mathcal{L}_G$ -large sets, then the intersection

$$AA^{-1} \cap BB^{-1} \cap CC^{-1} \quad \text{is } \mathcal{L}_G\text{-large.}$$

Let  $G$  denote the free group on three (free) generators  $\{a, b, c\}$  and let  $A$ ,  $B$  and  $C$  denote the sets of all elements in  $G$ , viewed as reduced words written from left to right, beginning with the letters  $a$ ,  $b$  and  $c$  respectively. One readily checks that  $A$ ,  $B$  and  $C$  are left syndetic, so in particular they are  $\mathcal{L}_\mu$ -large for every probability measure  $\mu$ . However,

$$AA^{-1} \cap BB^{-1} \cap CC^{-1} = \{e\},$$

which shows that Følner's Theorem does not (naively) extend to non-amenable groups. In fact, the situation is even worse. In Subsection 3.6 we shall construct a subset  $A$  of the free group on two generators, which is  $\mathcal{L}_\mu$ -large for every adapted probability measure  $\mu$  on  $G$ , such that the difference set  $AA^{-1}$  is *not* left syndetic.

**1.5. Furstenberg-Poisson means.** Let  $(G, \mu)$  be a countable measured group. A  $\mu$ -integrable function  $f$  on  $G$  is called *left  $\mu$ -harmonic* if

$$f(g) = \int_G f(hg) d\mu(h), \quad \forall g \in G.$$

Let  $\mathcal{H}_l^\infty(G, \mu)$  denote the space of all *bounded* left  $\mu$ -harmonic functions. If  $G$  is non-amenable, then it is well-known that  $\mathcal{H}_l^\infty(G, \mu)$ , equipped with the supremum norm, is a non-separable Banach space. Define the set of (left) *Furstenberg-Poisson means* on  $G$  by

$$\mathcal{F}_\mu := \left\{ \lambda \in \mathcal{L}_\mu : \lambda|_{\mathcal{H}_l^\infty(G, \mu)} = \delta_e \right\}.$$

It is easy to construct elements in  $\mathcal{F}_\mu$ . Indeed, for  $n \geq 1$ , we define the mean

$$\lambda_n(\varphi) := \frac{1}{n} \sum_{k=1}^n \int_G \varphi d\mu^{*k}, \quad \forall \varphi \in \ell^\infty(G),$$

where  $\mu^{*k}$  denotes the  $k$ -th convolution power of  $\mu$ . One readily checks that every accumulation point  $\lambda$  of the sequence  $(\lambda_n)$  in the weak\*-topology on  $\mathcal{M}(G)$  belongs to  $\mathcal{L}_\mu$  and if  $f$  is a left  $\mu$ -harmonic function on  $G$ , then  $\lambda_n(f) = f(e)$  for all  $n$ , and thus  $\lambda \in \mathcal{F}_\mu$ . In particular, if  $B \subset G$  and

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu^{*k}(B) > 0,$$

then  $B$  is  $\mathcal{F}_\mu$ -large (and hence  $\mathcal{L}_\mu$ -large).

**1.6. Fourier-Stiltjes means.** Let  $G$  be a countable group and let  $\pi$  be a unitary representation of  $G$  on a Hilbert space  $\mathcal{H}$ . Given  $x, y \in \mathcal{H}$ , the function

$$\varphi(g) = \langle y, \pi(g)x \rangle$$

is called a *matrix coefficient* of  $\pi$ . The space of all matrix coefficients on  $G$  (when  $\pi$  ranges over all unitary representations of  $G$ ) is easily seen to be a \*-sub-algebra of  $\ell^\infty(G)$ , i.e. a linear subspace which is closed under pointwise multiplication and taking complex conjugates. We shall refer to this algebra as the *Fourier-Stiltjes algebra* of  $G$  and denote it by  $B(G)$ .

A classical result due to Ryll-Nardzewski (see e.g. [12]) asserts there exists a *unique* left invariant mean  $\lambda_o$  on  $B(G)$ . We define the space  $\mathcal{FS}$  of *Fourier-Stiltjes means* on  $G$  by

$$\mathcal{FS} := \left\{ \lambda \in \mathcal{M}(G) : \lambda|_{B(G)} = \lambda_o \right\}.$$

Clearly, if  $G$  is amenable, then every left invariant mean belongs to  $\mathcal{FS}$ . In fact, it is not hard to show that  $\mathcal{L}_\mu \subset \mathcal{FS}$  for every measured group  $(G, \mu)$ . Indeed, note that if  $\varphi$  is a matrix coefficient on the form above and  $\lambda$  belongs to  $\mathcal{L}_\mu$ , then

$$\lambda(\varphi) = \langle y, \bar{x} \rangle,$$

where

$$\bar{x} := \int_G \pi(h)x d\lambda(h)$$

Note that  $\bar{x}$  satisfies

$$\begin{aligned} \int_G \pi(g)\bar{x} d\mu(g) &= \int_G \left( \int_G \pi(gh)x d\lambda(h) \right) d\mu(g) \\ &= \int_G \pi(h)x d(\rho(\mu)^*\lambda)(h) = \bar{x}. \end{aligned}$$

By the strict convexity of the unit ball in  $\mathcal{H}$ , we conclude that  $\bar{x}$  is  $\pi(G)$ -invariant, and thus the restriction of  $\lambda$  to  $B(G)$  must be left  $G$ -invariant. By the uniqueness of the left invariant mean on  $B(G)$ , the restriction must coincide with  $\lambda_o$ . In particular,

$$\mathcal{F}_\mu \subset \mathcal{L}_\mu \subset \mathcal{FS}.$$

For an alternative proof of the second inclusion, we refer the reader to [10].

We stress that the set of  $\mathcal{FS}$ -large subsets of  $G$  is considerably larger than the set of  $\mathcal{L}_\mu$ -large subsets for any adapted measure  $\mu$ . See Subsection 1.8 for further comments.

**1.7. Statements of main results.** As we have seen from the discussions above, great care needs to be taken when trying to formulate sensible extensions of Følner's Theorem to non-amenable groups. In this paper, we shall present two results in this direction. The first one partly extends the syndeticity claim in Følner's Theorem, and the second partly extends the "local" Bohr containment.

Recall that a set  $C \subset G$  is *right piecewise left syndetic* if there exists a finite set  $F \subset G$  such that  $FC$  is right thick.

**Theorem 1.1.** *Let  $(G, \mu)$  be a countable measured group. Suppose  $A \subset G$  is  $\mathcal{F}_\mu$ -large and  $B \subset G$  is  $\mathcal{L}_\mu$ -large. Then  $AB$  is right piecewise left syndetic.*

A more symmetrical statement, which does not involve  $\mathcal{F}_\mu$ -densities, can also be made. The deduction of this statement from the theorem above will be briefly explained in Subsection 3.3. However, we stress that there are more direct routes to proving this result.

**Corollary 1.1.** *Let  $(G, \mu)$  be a countable measured group. If  $A \subset G$  is  $\mathcal{L}_\mu$ -large, then the difference set  $AA^{-1}$  is right piecewise left syndetic, but not necessarily left syndetic.*

Our second theorem requires a bit more set up. Let  $G$  be a countable group and let  $B \subset G$ . The *Bebutov algebra* of  $B$ , here denoted by  $\mathcal{B}_B$ , is defined as the smallest left  $G$ -invariant sub- $C^*$ -algebra of  $\ell^\infty(G)$  containing the constant functions and the indicator function  $\chi_B$  of the set  $B$ .

We shall say that  $B$  is *strongly non-paradoxical* if there exists a left  $G$ -invariant mean  $\eta$  on  $\mathcal{B}_B$  such that  $\eta'(B)$  is positive. Let  $\mathcal{L}_B$  denote the set of all means on  $G$  whose restrictions to  $\mathcal{B}_B$  are left  $G$ -invariant. The *non-paradoxical* density of  $B$  is now defined as

$$d_{np}^*(B) = \sup_{\lambda \in \mathcal{L}_B} \lambda'(B).$$

This definition may seem rather far-fetched, so we shall take a moment here to motivate why we have introduced it. First note that if  $G$  is amenable, then  $\mathcal{B}_B$  always admits a left  $G$ -invariant mean and

$$d_{\mathcal{L}_G}^*(B) = d_{np}^*(B),$$

so the reader who wishes to stay within the class of amenable groups (for which the result which we shall state below is also new) can simply think of the non-paradoxical density as a roundabout re-packaging of the standard upper Banach density.

In order to better understand which kind of sets we wish to exclude by restricting our attention to strongly non-paradoxical sets (and to partly motivate the name), we can consider the group  $\mathbb{F}_2$ , the free group on two (free) generators, which we shall denote by

$a$  and  $b$ . Let  $A$  be the set of all group elements whose reduced form (from left to right) begins with either  $a$  or  $a^{-1}$ . Note that

$$A \cup a \cdot A \cup a^{-1} \cdot A = \mathbb{F}_2,$$

so in particular,  $A$  is left syndetic. Moreover,

$$b^m \cdot A \cap b^n \cdot A \neq \emptyset, \quad \forall m \neq n,$$

so all left translates of the form  $b^n \cdot A$  are *disjoint*. If there exists a mean  $\lambda$  on  $G$  such that  $\lambda'(g \cdot A) = \lambda'(A)$  for all  $g \in G$  (that is to say, the set  $\mathcal{L}_A$  is non-empty), then

$$\lambda' \left( \bigcup_{k=1}^n b^k \cdot A \right) = n \cdot \lambda'(A) \leq 1,$$

for all  $n$ , which clearly forces  $\lambda'(A) = 0$ . However, we also have

$$1 = \lambda'(A \cup a \cdot A \cup a^{-1} \cdot A) \leq 3 \cdot \lambda'(A),$$

and thus  $\lambda'(A) \geq \frac{1}{3}$ . This contradiction clearly implies that  $A$  is *not* strongly non-paradoxical. In essence, the notion of strong non-paradoxicality is designed to exclude exactly this kind of sets.

The *Bohr compactification*  $(bG, \iota_o)$  of a countable group  $G$  consists of a (possibly trivial) compact hausdorff group  $bG$  and a homomorphism  $\iota_o : G \rightarrow bG$  with dense image such that whenever  $K$  is a compact hausdorff group, together with a homomorphism  $\iota : G \rightarrow K$  with dense image, then there exists a surjective homomorphism  $\iota_K : bG \rightarrow K$  such that  $\iota_K \circ \iota_o = \iota$ . Note that a set  $B \subset G$  is a Bohr set according to the definition above if and only if there exists a non-empty open set  $U \subset bG$  such that  $B = \iota_o^{-1}(U)$ .

We need a final definition before we can state our second theorem. If  $K$  is a compact hausdorff group and  $U$  is a non-empty open set, then there is a finite set  $F$  such that  $FU = K$ . The minimal cardinality will be referred to as the *syndeticity index* of  $U$  in  $K$  and denoted by  $s_K(U)$ . Formally,

$$s_K(U) := \min \left\{ |F| : F \subset K \quad \text{and} \quad FU = K \right\}.$$

We shall think of  $s_K(U)$  as a measurement of the "regularity" of the open set. Intuitively, if  $U$  is a random union of small disjoint open sets (a "porous" set), then we expect the syndeticity index of this set to be quite large relative to the Haar measure of the set, while more "connected" subsets (which look more like intervals or open subgroups) are likely to have smaller syndeticity indices.

We can now state our last theorem which is new already in the case of the integers (where strong non-paradoxicality translates to  $\mathcal{L}_{\mathbb{Z}}$ -large and  $d_{np}^*$  equals the upper Banach density).

**Theorem 1.2.** *Let  $G$  be a countable group. Suppose  $A \subset G$  is  $\mathcal{FS}$ -large and  $B \subset G$  is strongly left non-paradoxical. Then there exist an open set  $U \subset bG$  with*

$$s_{bG}(U) \leq \left\lfloor \frac{1}{d_{\mathcal{FS}}^*(A) \cdot d_{np}^*(B)} \right\rfloor$$

and a right thick set  $T \subset G$  such that

$$AB \supset \iota_o^{-1}(U) \cap T.$$

In particular, if  $G$  is amenable and  $A \subset G$  is  $\mathcal{FS}$ -large, then  $AB$  is a right piecewise Bohr set for every  $\mathcal{L}_G$ -large set  $B \subset G$ .

**1.8. Connection to earlier results.** We have already mentioned in the introduction the classical papers of Bogliouboff and of Følner which have motivated a plethora of later works on product sets in groups. In this short section, we wish to draw the reader's attention to some more recent works which our main results generalize.

Theorem 1.1 is inspired by the following result of R. Jin from 2005 which was established by a clever non-standard analytic reduction to the setting of Følner's Theorem.

**Theorem 1.3.** [14] *Suppose  $A, B \subset \mathbb{Z}$  are  $\mathcal{L}_{\mathbb{Z}}$ -large subsets. Then  $AB$  is piecewise syndetic.*

More recently, M. di Nasso and Lupini [8] gave the following quantification of Jin's result (for any countable amenable group), which motivated Theorem 1.2 of this paper.

**Theorem 1.4** ([8]). *Let  $G$  be a countable amenable group and suppose  $A, B \subset G$  are  $\mathcal{L}_G$ -large subsets. Then there exists a finite set  $F \subset G$  with*

$$|F| \leq \left\lfloor \frac{1}{d_{\mathcal{L}_G}^*(A) \cdot d_{\mathcal{L}_G}^*(B)} \right\rfloor$$

such that  $FAB$  is right thick.

The question whether the product of two sets of integers with positive upper Banach densities contains translates of every finite piece of a Bohr set was raised and answered in the paper [4] by V. Bergelson, H. Furstenberg and B. Weiss, and was later extended in [3] to cover the case of products of sets in any countable amenable group by V. Bergelson, M. Beiglböck and the second author of this paper.

We shall say that a set  $C \subset G$  is *right piecewise Bohr* if there exists a Bohr set  $B \subset G$  and a right thick set  $T \subset G$  such that

$$C \supset B \cap T.$$

**Theorem 1.5** ([3]). *Let  $G$  be a countable amenable group and suppose  $A, B \subset G$  are  $\mathcal{L}_G$ -large sets. Then  $AB$  is a right piecewise Bohr set. In particular,  $AB$  is right piecewise left syndetic.*

In [13], J. Griesmer established a beautiful and far-reaching generalization of the main theorem in [3]. To explain his result, we shall need some notation. Let  $G$  be a countable amenable group. We shall say that a sequence  $(\nu_n)$  of probability measures is *equidistributed* if for any unitary representation  $(\mathcal{H}, \rho)$  of  $G$ , we have

$$\lim_n \nu_n(\rho)x = P_\rho x, \quad \forall x \in \mathcal{H},$$

in the norm topology on  $\mathcal{H}$ , where  $P_\rho$  denotes the projection onto the  $\rho$ -fixed vectors, and

$$\nu_n(\rho) := \int_G \rho(g) d\nu_n(g).$$

In particular, if  $G$  is a countable *abelian* group, then  $(\nu_n)$  is equidistributed if and only if the limit  $\hat{\nu}_n(\chi) \rightarrow 0$  holds of all non-zero characters  $\chi$  in the dual group of  $G$  as  $n \rightarrow \infty$ . Here,  $\hat{\nu}_n$  denotes the Fourier transform of the measure  $\nu_n$ .

We shall say that a subset  $A \subset G$  is *large with respect to*  $(\nu_n)$  if

$$\limsup_{n \rightarrow \infty} \nu_n(A) > 0.$$

Griesmer gives a wealth of examples in [13] of subsets of integers which are not  $\mathcal{L}_{\mathbb{Z}}$ -large, but large with respect to some equidistributed sequence.

One observes that upon considering an equidistributed sequence  $(\nu_n)$  as a sequence in the dual of  $\ell^\infty(G)$ , then its defining property implies that any accumulation point in the set  $\mathcal{M}(G)$  must belong to  $\mathcal{FS}$ . In particular, every subset of a countable group which is large with respect to an equidistributed sequence must be  $\mathcal{FS}$ -large, so the following theorem is strictly contained in our Theorem 1.2.

**Theorem 1.6** ([13]). *Let  $G$  be a countable amenable group and suppose  $A \subset G$  is large with respect to an equidistributed sequence of probability measures on  $G$  and  $B \subset G$  is  $\mathcal{L}_G$ -large. Then  $AB$  is a right piecewise Bohr set.*

**1.9. An outline of Theorem 1.1.** We shall now attempt to break down the proof of Theorem 1.1 into two main steps. The proofs of each step will be given in Section 4 and Section 5 respectively. We refer to these sections for a more detailed discussion about the notions in this chapter.

**1.9.1. Ergodicity of  $\mathcal{C}$ -densities.** The first main step in the proof of Theorem 1.1 consists of the following generalization of Lemma 3.2 in [3]. This lemma asserts the same as the proposition below but for the set  $\mathcal{L}_G$  of left invariant means on a countable amenable group instead of the set of Furstenberg-Poisson means  $\mathcal{F}_\mu$  on a countable (symmetric) measured group  $(G, \mu)$ .

Although not explicitly stated in [3], the only property of  $\mathcal{L}_G$  which is used in the proof of Lemma 3.2 in [3] is what we shall refer to as *left ergodicity*, namely that every extremal point of the weak\*-compact convex subset  $\mathcal{L}_G \subset \mathcal{M}(G)$  corresponds (under the Gelfand-Naimark map) to a left  $G$ -invariant *ergodic* probability measure on  $\beta G$ .

This property is relatively straightforward to verify for the sets  $\mathcal{L}_G$  (when  $G$  is amenable) and  $\mathcal{L}_\mu$  (for a measured group  $(G, \mu)$ ). However, it also holds for the set  $\mathcal{F}_\mu$  but this fact requires a more detailed discussion about the Furstenberg-Poisson boundary of  $(G, \mu)$  which will be outlined in Subection 3.5.

**Proposition 1.2.** *Suppose  $A \subset G$  is  $\mathcal{F}_\mu$ -large. Then,*

$$\sup d_{\mathcal{F}_\mu}^*(FA) = 1,$$

*where the supremum is taken over all finite subsets  $F \subset G$ .*

**1.9.2. A criterion for right thickness of product sets.** The second main step is inspired by Lemma 3.1 in [3], and indeed reduces to this lemma in the case when  $\mathcal{L}_G = \mathcal{L}_\mu$ . However, the proof of this step for a general measured group necessarily follows a different route than the proof given in [3]. For instance, we do not know if the proposition holds with  $\mathcal{F}_\mu$  replaced with  $\mathcal{L}_\mu$  in the first addend.

**Proposition 1.3.** *Suppose  $A, B \subset G$  satisfy*

$$d_{\mathcal{F}_\mu}^*(A) + d_{\mathcal{L}_\mu}^*(B) > 1.$$

*Then  $AB$  is right thick.*

We shall now combine the two propositions above to give a proof of Theorem 1.1.

1.9.3. *Proof of Theorem 1.1.* Suppose  $A \subset G$  is  $\mathcal{F}_\mu$ -large and  $B \subset G$  is  $\mathcal{L}_\mu$ -large. We wish to prove that there exists a finite set  $F \subset G$  such that the triple product set  $FAB$  is right thick. By Proposition 1.3, it suffices to prove that

$$d_{\mathcal{F}_\mu}^*(FA) + d_{\mathcal{L}_\mu}^*(B) > 1$$

for some finite set  $F \subset G$ , or equivalently,

$$\sup_F d_{\mathcal{F}_\mu}^*(FA) > 1 - d_{\mathcal{L}_\mu}^*(B),$$

where the supremum is taken over all finite subsets  $F \subset G$ . The result now follows from Proposition 1.2.

1.10. **An outline of Theorem 1.2.** The proof of Theorem 1.2 can be divided into three main steps, where the first step is the most important one, and the two other steps are more of a standard nature. However, since we do not know of a good reference for these steps in the generality needed here, we shall provide proofs of the necessary statements.

1.10.1. *Producing some Bohr structure.* The first step will be outlined in Section 6. For definitions and a more detailed discussion about the notation in this summary we refer the reader to Subsection 2.3.

Let  $G$  be a countable group and let  $\beta G$  denote its Stone-Ćech compactification. Given a subset  $C \subset G$ , we denote by  $\overline{C}$  its "closure" in  $\beta G$ . If  $q \in \beta G$ , then we shall write

$$\overline{C}_q := \left\{ g \in G : g \cdot q \in \overline{C} \right\} \subset G.$$

In particular, if  $\bar{e}$  denotes the "image" of the identity element  $e$  in  $\beta G$ , then  $C = \overline{C}_{\bar{e}}$ .

**Proposition 1.4.** *Suppose  $A \subset G$  is  $\mathcal{FS}$ -large and  $B \subset G$  is strongly non-paradoxical. Then there exist Borel sets  $\tilde{A}, \tilde{B} \subset bG$  with*

$$m(\tilde{A}) \geq d_{\mathcal{FS}}^*(A) \quad \text{and} \quad m(\tilde{B}) \geq d_{np}^*(B)$$

*and  $q \in \beta G$  and  $k_o \in bG$  such that*

$$\overline{AB}_q \supset \iota^{-1}(U \cdot k_o),$$

*where*

$$U := \left\{ k \in bG : m(\tilde{A} \cap k \cdot \tilde{B}^{-1}) > 0 \right\} \subset bG.$$

1.10.2. *Piecewise inclusion.* Let  $B, C \subset G$  and let  $\overline{B}, \overline{C} \subset \beta G$  denote the corresponding closures in  $\beta G$ . Suppose we have an inclusion of the form

$$\overline{C}_q \supset \overline{B}_q,$$

for some  $q \in \beta G$ . The next proposition shows that if  $\overline{B}$  is the pull-back to  $\beta G$  (under a continuous  $G$ -equivariant map) of a *Jordan measurable* subset of the Bohr compactification, then some "virtual" translate of  $B$  is "almost" completely contained in  $C$ .

**Proposition 1.5.** *Let  $C \subset G$  and suppose there exist  $q \in \beta G$  and an open Jordan measurable set  $U \subset bG$  such that*

$$\overline{C}_q \supset \iota^{-1}(U).$$

*Then there exist a right thick set  $T \subset G$  and  $k \in bG$  such that*

$$C \supset \iota^{-1}(U \cdot k) \cap T.$$

1.10.3. *Getting Jordan measurability.* The last step is the most standard one and is needed to adapt the outcome of Proposition 1.4 to the setting of Proposition 1.5. The main result here is a generalization to compact hausdorff groups of the well-known fact that every *closed* subset of a compact *metrizable* group is contained in a Jordan measurable subset with Haar measure arbitrarily close to the Haar measure of the closed set. The arguments needed to relate this generalization to the proposition below will be outlined in Section 8.

**Proposition 1.6.** *Let  $K$  be a compact hausdorff group with Haar probability measure  $m$ . If  $A, B \subset K$  are Borel sets with positive Haar measures, and*

$$U := \left\{ k \in K : m(A \cap k \cdot B) > 0 \right\} \subset K,$$

*then there exists an open Jordan measurable set  $U' \subset U$  with*

$$s_K(U') \leq \left\lfloor \frac{1}{m(A) \cdot m(B)} \right\rfloor.$$

We shall now combine the three propositions above to give a proof of Theorem 1.2.

1.10.4. *Proof of Theorem 1.2.* Let  $G$  be a countable group and suppose  $A \subset G$  is  $\mathcal{FS}$ -large and  $B \subset G$  is strongly non-paradoxical.

By Proposition 1.4 there exist Borel sets  $\tilde{A}, \tilde{B} \subset bG$  with

$$m(\tilde{A}) \geq d_{\mathcal{FS}}^*(A) \quad \text{and} \quad m(\tilde{B}) \geq d_{np}^*(B)$$

and  $q \in \beta G$  and  $k_o \in bG$  such that

$$\overline{AB}_q \supset \iota^{-1}(U \cdot k_o),$$

where

$$U := \left\{ k \in bG : m(\tilde{A} \cap k \cdot \tilde{B}^{-1}) > 0 \right\} \subset bG.$$

Proposition 1.6 guarantees that we can find an open and *Jordan measurable* subset  $U' \subset U$  with

$$s_{bG}(U') \leq \left\lfloor \frac{1}{m(\tilde{A}) \cdot m(\tilde{B}^{-1})} \right\rfloor \leq \left\lfloor \frac{1}{d_{\mathcal{FS}}^*(A) \cdot d_{np}^*(B)} \right\rfloor,$$

where the last inequality is justified by the fact that  $m(\tilde{B}^{-1}) = m(\tilde{B})$ . Now,

$$\overline{AB}_q \supset \iota^{-1}(U \cdot k_o) \supset \iota^{-1}(U' \cdot k_o),$$

so Proposition 1.5 applies with  $C = AB$ , and we conclude that there exist a right thick set  $T \subset G$  and  $k \in bG$  such that

$$AB \supset \iota^{-1}(U' \cdot k_o \cdot k) \cap T,$$

which finishes the proof.

**1.11. Organization of the paper.** The paper is organized as follows. Section 2 is devoted to basic facts about the  $C^*$ -algebraic theory of sub-algebras of the space of bounded functions on a countable group  $G$ , and to some basic measure theory on  $\beta G$ . In particular, we give characterizations of right thick sets and left syndetic sets in terms on upper and lower  $\mathcal{C}$ -densities for special classes of sets  $\mathcal{C} \subset \mathcal{M}(G)$ .

In Section 3 we develop some basic ergodic theory for measured groups which will be needed in the proofs of Proposition 1.2 and Proposition 1.3. In the final subsection of this section we outline a proof of claim about failure of syndeticity of difference sets in general countable measured groups.

In Section 4 and Section 5 we give the proofs of the assertions needed in the proof of Theorem 1.1. In Section 6 we discuss unitary representations of countable groups; in particular we make the direct sum of the Eberlein algebra more explicit. This is needed for the proof of Proposition 1.4.

In Section 7 we discuss the relevance of almost automorphic points to Proposition 1.5, and give a proof of this proposition.

In Section 8 we prove some elementary facts about Jordan measurable subsets of compact hausdorff groups needed for Proposition 1.6.

**1.12. Acknowledgements.** The authors have benefited enormously from discussions with Benjamin Weiss during the preparation of this manuscript, and it is a pleasure to thank him for sharing his many insights with us. We are also very grateful for the many inspiring and enlightening discussions we have had with Mathias Beiglböck, Vitaly Bergelson, Manfred Einsiedler, Hillel Furstenberg, Eli Glasner, John Griesmer, Elon Lindenstrauss, Fedja Nazarov, Amos Nevo, Imre Ruzsa and Jean-Paul Thouvenot.

The present paper is an outgrowth of a series of discussions between the authors which took place at Ohio State University, University of Wisconsin, Hebrew University in Jerusalem, Weizmann institute, IHP Paris, KTH Stockholm and ETH Zürich. We thank the mathematics departments at these places for their great hospitality.

The first author acknowledges support from the European Community seventh Framework program (FP7/2007-2012) grant agreement 203418 when he was a postdoctoral fellow at Hebrew university, and ETH Fellowship FEL-171-03 since January 2011.

## 2. PRELIMINARIES AND BASIC NOTIONS

**2.1. Uniform algebras.** We shall now review some basic elements in the theory of norm closed sub-algebras of  $\ell^\infty(G)$ . For more details, including the proofs of the all statements in this subsection, we refer the reader to the second chapter of the book [17]; In particular, subsections 2.9 and 2.10 therein.

2.1.1. *The Stone-Čech compactification.* Let  $G$  be a countable group, and let  $\ell^\infty(G)$  denote the algebra of bounded functions on  $G$ . We denote the complex conjugation operator on  $\ell^\infty(G)$  by  $*$ . Equipped with this involution and the supremum norm  $\|\cdot\|_\infty$ , this algebra carries the structure of a unital commutative  $C^*$ -algebra. As such, its Gelfand spectrum  $\beta G$ , i.e. the set of multiplicative functionals on  $\ell^\infty(G)$ , endowed with the weak\*-subspace topology inherited from the unit ball of the dual of  $\ell^\infty(G)$ , is a weak\*-compact hausdorff space. Moreover, by Gelfand-Naimark's Theorem, we have

$$\ell^\infty(G) \cong C(\beta G),$$

where  $C(\beta G)$  denotes the space of continuous function on  $\beta G$ .

Furthermore, the left and right action of  $G$  on itself induce left and right anti-actions on  $\ell^\infty(G)$ , and thus left and right actions by homeomorphisms on  $\beta G$ . We shall refer to  $\beta G$  as the *Stone-Čech compactification* of  $G$ .

The map  $\delta_e$  which evaluates a bounded function on  $G$  at the identity element  $e$  is a multiplicative functional on  $\ell^\infty(G)$  and thus corresponds to a point  $\bar{e}$  in  $\beta G$ . One readily shows that this point is  $G$ -transitive under both the left and right action of  $G$  on  $\beta G$ . Moreover, a left  $G$ -equivariant isomorphism between  $C(\beta G)$  and  $\ell^\infty(G)$  is implemented by the map

$$\tau(\varphi)(g) = \varphi(g \cdot \bar{e}), \quad \varphi \in C(\beta G). \quad (2.1)$$

2.1.2. *Relations with sub-algebras.* More generally, if  $\mathcal{B} \subset \ell^\infty(G)$  is a norm-closed unital sub-algebra of  $\ell^\infty(G)$ , then it is again a  $C^*$ -algebra, and its Gelfand spectrum  $\Delta(\mathcal{B})$  is a compact hausdorff space with the property that

$$\mathcal{B} \cong C(\Delta(\mathcal{B})).$$

By Urysohn's Lemma,  $\Delta(\mathcal{B})$  is second countable if and only if  $\mathcal{B}$  is separable.

If  $\mathcal{B}$  is invariant under the left anti-action of  $G$  on  $\ell^\infty(G)$ , then this anti-action induces an action of  $G$  by homeomorphisms on  $\Delta(\mathcal{B})$ . For every  $g \in G$ , the Dirac measure  $\delta_g$  at  $g$  is a multiplicative functional on  $\mathcal{B}$ . Let  $x_o$  denote the point in  $\Delta(\mathcal{B})$  corresponding to the evaluation map  $\delta_e$  and let  $X$  denote the orbit closure of  $x_o$  under the induced  $G$ -action. If  $X \neq \Delta(\mathcal{B})$ , then the complement of  $X$  is open and we can find a non-zero continuous function  $\bar{\varphi}$  on  $\Delta(\mathcal{B})$  (corresponding to a non-zero element  $\varphi$  in  $\mathcal{B}$ ) which vanishes on  $X$ . In particular,

$$g \cdot x_o(\varphi) = \varphi(g) = 0, \quad \forall g \in G,$$

which is a contradiction, since  $\varphi$  was assumed non-zero, and thus  $X = \Delta(\mathcal{B})$ . We conclude that  $x_o$  is a  $G$ -transitive point in  $\Delta(\mathcal{B})$ . By Gelfand-Naimark's Theorem, the inclusion map  $\mathcal{B} \hookrightarrow \ell^\infty(G)$  is induced by a surjective continuous map  $\pi_{\mathcal{B}} : \beta G \rightarrow \Delta(\mathcal{B})$ , i.e.  $\iota = \pi_{\mathcal{B}}^*$ , where

$$\pi_{\mathcal{B}}^* \varphi := \varphi \circ \pi_{\mathcal{B}}.$$

If  $\mathcal{B}$  is left invariant, then this map is left  $G$ -equivariant (with respect to the left action on  $\beta G$ ) and  $\pi_{\mathcal{B}}(\bar{e}) = x_o$ .

2.1.3. *Almost periodic functions.* We shall now consider a few examples of the situation outlined in the previous paragraph.

Let  $\text{AP}(G)$  denote the algebra of *almost periodic functions* on  $G$ , i.e. the  $*$ -sub-algebra of  $\ell^\infty(G)$  consisting of those bounded functions  $\varphi$  whose left anti- $G$ -orbit  $G \cdot \varphi$  in  $\ell^\infty(G)$  is pre-compact in the norm topology. Equipped with the conjugation involution and the supremum norm, this  $*$ -algebra is a sub- $C^*$ -algebra of  $\ell^\infty(G)$ , and we shall denote its Gelfand spectrum by  $bG$ .

We refer to subsection 2.10 in the book [17] for the proof that the group multiplication on  $G$  extends to a jointly continuous multiplication on  $bG$ , i.e.  $bG$  can be given the structure of a compact hausdorff group. Again, there is a natural (but this time not necessarily injective) homomorphism  $\iota : G \rightarrow bG$  given by

$$g \mapsto \delta_g,$$

where  $\delta_g$  denotes the evaluation map on  $\ell^\infty(G)$  at the element  $g$ . Furthermore, this map is universal in the sense that whenever  $K$  is a compact hausdorff group and  $\iota_o : G \rightarrow K$  is a homomorphism with a dense image, then there exists a *continuous* surjective homomorphism  $\iota_K : bG \rightarrow K$  such that

$$\iota_K \circ \iota = \iota_o.$$

We shall refer to  $(bG, \iota)$  as the *Bohr compactification* of  $G$ , and the *unique* Haar probability measure on  $bG$  will be denoted by  $m$ . Note that by construction, we have

$$\text{AP}(G) = \iota^*C(\beta G).$$

Recall that a state (or mean) on  $\ell^\infty(G)$  is a positive, norm-one functional on  $\ell^\infty(G)$ . The set  $\mathcal{M}(G)$  of all states on  $\ell^\infty(G)$  is a weak $*$ -compact and convex subset of  $\ell^\infty(G)^*$ . We shall say that  $\lambda \in \mathcal{M}(G)$  is a *Bohr mean* if the restriction of  $\lambda$  to  $\text{AP}(G)$  equals the pull-back of the Haar probability measure on  $bG$  and denote the *non-empty* weak $*$ -compact and convex subset of  $\mathcal{M}(G)$  consisting of all Bohr means by *Bohr*.

2.1.4. *Fourier-Stiljes algebras.* Let  $B(G)$  denote the *Fourier-Stiljes algebra* of  $G$ , i.e. the  $*$ -sub-algebra of  $\ell^\infty(G)$  generated by all matrix coefficients of *unitary* representations of  $G$ . Hence an element  $\varphi$  in  $B(G)$  has the form

$$\varphi(g) = \langle x, \pi(g)y \rangle, \quad g \in G,$$

where  $(\mathcal{H}, \pi)$  is some unitary representation of  $G$  on a Hilbert space  $\mathcal{H}$ , and  $x, y \in \mathcal{H}$ . We stress that  $B(G)$  is not a norm-closed subspace of  $\ell^\infty(G)$ . Its norm closure, often called the *Eberlein algebra*, will here be denoted by  $E(G)$ , and its Gelfand spectrum will be denoted by  $eG$ .

As is well-known (see e.g. Chapter 1.10 in [12]), there is a unique left and right invariant state  $\lambda_o$  on  $E(G)$ , and  $E(G)$  admits a direct sum decomposition of the form

$$E(G) = \text{AP}(G) \oplus E_o(G), \tag{2.2}$$

where

$$E_o(G) := \left\{ \varphi \in E(G) : \lambda_o(\varphi^* \varphi) = 0 \right\}.$$

The set  $\mathcal{FS} \subset \text{Bohr}(G)$  consisting of those means whose restrictions to  $E(G)$  coincide with  $\lambda_o$  will be referred to as *Fourier-Stiljes means*.

**2.2. Bauer's Maximum Principle.** Let  $G$  be a countable group and let  $\mathcal{M}(G)$  denote the set of states on  $\ell^\infty(G)$ , or alternatively, the set of means on  $G$ . If  $\lambda \in \mathcal{M}(G)$ , then we define

$$\lambda'(C) := \lambda(\chi_C), \quad C \subset G.$$

One readily checks that  $\lambda'$  is a finitely additive (but not necessarily  $\sigma$ -additive) probability measure on  $G$  and for every  $C \subset G$ , the *affine* map

$$\lambda \mapsto \lambda'(C), \quad \lambda \in \mathcal{M}(G)$$

is weak\*-continuous.

Given a set  $\mathcal{C} \subset \mathcal{M}(G)$ , we define the *upper* and *lower*  $\mathcal{C}$ -densities of a subset  $C \subset G$  by

$$d_{\mathcal{C}}^*(C) := \sup_{\lambda \in \mathcal{C}} \lambda'(C) \quad \text{and} \quad d_{\mathcal{C}}^{\mathcal{C}}(C) := \inf_{\lambda \in \mathcal{C}} \lambda'(C)$$

respectively. If  $\mathcal{C}$  is weak\*-compact, then the supremum and infimum above are attained. Furthermore, since the maps involved are affine, we can also say some things about the location of these extrema, as the following result shows.

Before we state the main proposition of this section (which is known as Brauer's Maximum Principle), we recall that the classical Krein-Milman's Theorem asserts that any non-empty weak\*-compact and convex subset  $\mathcal{C}$  of the dual of a Banach space has extremal points, i.e. elements  $\lambda$  in  $\mathcal{C}$  which do not admit a representation of the form

$$\lambda = \alpha_1 \cdot \lambda_1 + \alpha_2 \cdot \lambda_2,$$

where  $\lambda_1, \lambda_2 \in \mathcal{C}$  and  $\alpha_1, \alpha_2$  are *positive* numbers with  $\alpha_1 + \alpha_2 = 1$ .

**Proposition 2.1** (Theorem 7.69 in [1]). *Let  $\mathcal{C}$  be a weak\*-compact convex subset of the dual of a Banach space and suppose  $\varphi : \mathcal{C} \rightarrow \mathbb{R}$  is an upper semicontinuous convex function. Then  $\varphi$  attains its maximum at an extremal point in  $\mathcal{C}$ .*

Applied to the Banach space  $X = \ell^\infty(G)$ , we have the following immediate corollary, which shall be used frequently.

**Corollary 2.2.** *Let  $X$  be a Banach space and suppose  $\mathcal{C} \subset X^*$  is a weak\*-compact convex set. For every  $x \in X$ , there exist an extremal  $\lambda_o \in \mathcal{C}$  such that*

$$\lambda_o(x) = \sup_{\lambda \in \mathcal{C}} \lambda(x).$$

*In particular, for any weak\*-compact convex set  $\mathcal{C} \subset \mathcal{M}(G)$  and for any  $C \subset G$ , there exist extremal elements  $\lambda_o$  and  $\lambda_1$  in  $\mathcal{C}$  such that*

$$d_{\mathcal{C}}^*(C) = \lambda_o'(C) \quad \text{and} \quad d_{\mathcal{C}}^{\mathcal{C}}(C) = \lambda_1'(C).$$

For separable Banach spaces, this corollary can be deduced from Choquet's Theorem. However, in the case  $X$  is *not* separable (which is of crucial interest to us), there is no reason to expect that the set of extremal points of  $\mathcal{C} \subset X^*$  is even Borel measurable, and thus disintegration techniques are not directly applicable.

**2.3. Measure theory on  $\beta G$ .** There is also an important interchange between subsets of the group and *clopen* subsets of  $\beta G$  which we now describe.

2.3.1. *Set correspondences.* Recall that the map  $\tau : C(\beta G) \rightarrow \ell^\infty(G)$  given by

$$\tau(f)(g) = f(g \cdot \bar{e}), \quad g \in G,$$

is an isometric  $*$ -isomorphism. Hence, if  $C \subset G$ , then there exists a *unique*  $\varphi$  in  $C(\beta G)$  such that  $\chi_C = \tau(\varphi)$ , and thus

$$\tau(\varphi^* \varphi) = \tau(\varphi)^* \tau(\varphi) = \chi_C^2 = \tau(\varphi),$$

which implies that  $\varphi$  is idempotent in  $C(\beta G)$ . Since  $\tau$  is an isomorphism, this entails that  $\varphi$  is necessarily equal to the indicator function of a subset  $\bar{C}$  of  $\beta G$ . Since the indicator function of this set is continuous, we conclude that  $\bar{C}$  is clopen. Conversely, if  $\bar{C}$  is a clopen set, then one readily checks that  $\tau(\chi_{\bar{C}})$  is the indicator function of a subset  $C$  of  $G$ . Hence we have a one-to-one correspondence

$$C \longleftrightarrow \bar{C}$$

between subsets of  $G$  and *clopen* subsets of  $\beta G$ .

2.3.2. *Measure correspondences.* Since the map  $\tau$  above is clearly positivity preserving and unital, we see that if  $\lambda$  is a mean on  $G$ , then  $\tau^* \lambda$  is a positive unital continuous functional on  $C(\beta G)$ , and can thus be thought of as a  $\sigma$ -additive Borel probability measure on  $\beta G$ , via Riesz Representation Theorem. Conversely, any Borel probability measure  $\bar{\lambda}$  on  $\beta G$  induces a linear functional on  $C(\beta G)$  by integration, and thus to a state on  $\ell^\infty(G)$  (or equivalently, a mean on  $G$ ). Hence there is a one-to-one correspondence between means on  $G$  and probability measures on  $\beta G$  given by

$$\lambda \longleftrightarrow \bar{\lambda}.$$

2.3.3. *Beyond continuous functions.* Note that the map  $\tau$  above is clearly defined for *all* functions on  $\beta G$ , continuous or not. However, easy examples show that there may be no relation between the  $\bar{\lambda}$ -integral of a discontinuous function  $\varphi$  on  $\beta G$  and the  $\lambda$ -value of  $\tau(\varphi)$  for a given mean  $\lambda$  on  $G$ .

However, as we shall see in the following lemma, it is possible to relate the two values when  $\varphi$  is a *lower semicontinuous* function on  $\beta G$ .

**Lemma 2.3.** *Suppose  $\varphi$  is a bounded positive function on  $\beta G$  and equal to the supremum of an increasing sequence of continuous functions on  $\beta G$ . Then*

$$\lambda(\tau(\varphi)) \geq \int_{\beta G} \varphi d\bar{\lambda}, \quad \forall \lambda \in \mathcal{M}(G).$$

*Proof.* Let  $(\varphi_n)$  be an increasing sequence of continuous functions on  $\beta G$  such that

$$\varphi = \sup_n \varphi_n.$$

Fix  $\lambda \in \mathcal{M}(G)$ . Then, for every  $n$ , we have

$$\lambda(\tau(\varphi)) \geq \lambda(\tau(\varphi_n)) = \int_{\beta G} \varphi_n d\bar{\lambda}.$$

Since  $\bar{\lambda}$  is a  $\sigma$ -additive Borel probability measure on  $\beta G$ ,

$$\sup_n \int_{\beta G} \varphi_n d\bar{\lambda} = \int_{\beta G} \varphi d\bar{\lambda}$$

by monotone convergence, which finishes the proof.  $\square$

2.3.4. *Integrating open sets in  $\beta G$ .* Let us recall the notion of a *lower  $\mathcal{C}$ -density* for a countable group  $G$ . If  $\mathcal{C} \subset \mathcal{M}(G)$ , we defined the *lower  $\mathcal{C}$ -density* of a subset  $C \subset G$  by

$$d_*^{\mathcal{C}}(C) := \inf_{\lambda \in \mathcal{C}} \lambda'(C).$$

Now, if  $B \subset \beta G$  is any Borel set and  $q$  is any point in  $\beta G$ , we can construct "the return time" subset  $B_q \subset G$  by

$$B_q := \left\{ g \in G : g \cdot q \in B \right\}.$$

In particular, if  $C \subset G$  and  $\overline{C}$  denotes the corresponding clopen set in  $\beta G$ , then  $C = \overline{C}_{\bar{e}}$ , and

$$\lambda'(\overline{C}_{\bar{e}}) = \overline{\lambda}(C), \quad \forall \lambda \in \mathcal{M}(G).$$

When  $B$  is no longer clopen in  $\beta G$ , there is no reason to expect any relation of this sort. However, as we shall see in the next lemma, there is a very important lower bound, in the case when  $B$  consists of *countable* union of clopen sets.

**Lemma 2.4.** *Let  $\mathcal{C} \subset \mathcal{M}(G)$  be a weak\*-compact and convex set and suppose  $U \subset \beta G$  is a countable union of clopen sets in  $\beta G$ . Then there exists an extremal  $\lambda \in \mathcal{C}$  such that*

$$d_*^{\mathcal{C}}(U_{\bar{e}}) \geq \overline{\lambda}(U).$$

*Proof.* By Corollary 2.2, applied to the set  $C := U_{\bar{e}}^c$ , there exists an extremal  $\lambda \in \mathcal{C}$  such that

$$\lambda'(U_{\bar{e}}) = d_*^{\mathcal{C}}(U_{\bar{e}}).$$

By assumption, the indicator function  $\varphi := \chi_U$  equals the supremum of an increasing sequence of indicator functions on clopen sets (which are continuous), so by Lemma 2.3, we have

$$\lambda'(U_{\bar{e}}) = \lambda(\tau(\chi_U)) \geq \overline{\lambda}(U),$$

which finishes the proof.  $\square$

2.4. **Combinatorics on  $\beta G$ .** The next two subsection will be concerned with the problem of reading of combinatorial properties of a subset  $C \subset G$  from topological properties of  $\overline{C} \subset \beta G$  and dynamical properties of the "return time sets"

$$\overline{C}_q = \left\{ g \in G : g \cdot q \in \overline{C} \right\} \subset G,$$

for various choices of  $q \in \beta G$ .

**Lemma 2.5.** *For every  $C \subset G$  and finite  $F \subset G$ , the sets*

$$U_F(C) := \left\{ q \in \beta G : F \subset \overline{C}_q \right\} \quad \text{and} \quad U^F(C) := \left\{ q \in \beta G : \overline{C}_q \subset G \setminus F \right\}$$

*are clopen subsets of  $\beta G$ .*

*Proof.* Since  $\overline{C} \subset \beta G$  is clopen and

$$U^F(C) = U_F(C^c),$$

it suffices to prove that  $U_F(C)$  is clopen in  $\beta G$  for every  $C \subset G$  and finite  $F \subset G$ . Note that

$$U_F(C) = \bigcap_{f \in F} U_{\{f\}}(C),$$

and

$$U_{\{f\}}(C) = \left\{ q \in \beta G : f \in \overline{C}_q \right\} = f^{-1} \cdot \overline{C}, \quad \forall f \in F.$$

Since the latter sets are all clopen and  $U_F(C)$  is a finite intersection of such sets, we conclude that  $U_F(C)$  is clopen as well.  $\square$

**Lemma 2.6.** *For every  $C \subset G$ ,  $q \in \beta G$  and finite  $F \subset G$ , there exists  $g \in G$  such that*

$$C \cdot g^{-1} \cap F = \overline{C}_q \cap F.$$

*In fact, for every  $q \in \beta G$ , the set of  $p \in \beta G$  such that*

$$\overline{C}_p \cap F = \overline{C}_q \cap F$$

*is clopen.*

*Proof.* Fix  $q \in \beta G$  and a finite set  $F \subset G$  and define

$$I := F \cap \overline{C}_q \quad \text{and} \quad J := F \cap (\overline{C}_q)^c$$

By Lemma 2.5 the set

$$E_F(q) := \left\{ p \in \beta G : \overline{C}_p \cap F = \overline{C}_q \cap F \right\} = U_I(C) \cap U_J(C)$$

is clopen and clearly contains  $q$ . Since  $\bar{e} \in \beta G$  is  $G$ -transitive, we can find  $g \in G$  such that  $g \cdot \bar{e} \in E_F(q)$ , or equivalently,

$$\overline{C}_{g \cdot \bar{e}} \cap F = C \cdot g^{-1} \cap F = \overline{C}_q \cap F,$$

which finishes the proof.  $\square$

**2.4.1. Jordan measurability and local inclusion.** Recall that if  $\nu$  is a Borel probability measure on a compact hausdorff space  $X$ , then a Borel set  $C \subset X$  is  $\nu$ -Jordan measurable if the  $\nu$ -measure of the *topological boundary*

$$\partial B := \overline{B} \setminus B^\circ \subset X$$

is zero, where  $B^\circ$  denotes the interior of  $B$ .

In many of the applications in this paper, we shall require a stronger form of Jordan measurability. Let  $\mathcal{P}(X)$  denote the space of Borel probability measures on  $X$ , identified with the set of states of the  $C^*$ -algebra  $C(X)$  via the Riesz Representation Theorem. Given a subset  $\mathcal{C} \subset \mathcal{P}(X)$  we shall say that a set  $C \subset X$  is  $\mathcal{C}$ -tempered if  $C$  is  $\nu$ -Jordan measurable for *all*  $\nu \in \mathcal{C}$ .

The property of  $\mathcal{C}$ -temperedness behaves well under liftings as the following lemma shows.

**Lemma 2.7.** *Let  $\mathcal{C} \subset \mathcal{P}(X)$  and suppose  $\pi : X \rightarrow Y$  is a continuous map. If  $B \subset Y$  is  $\pi_*\mathcal{C}$ -tempered, then  $\pi^{-1}(B) \subset X$  is  $\mathcal{C}$ -tempered.*

*Proof.* This is an immediate consequence of the inclusions

$$\pi^{-1}(B^\circ) \subset \pi^{-1}(B)^\circ \subset \pi^{-1}(B) \subset \overline{\pi^{-1}(B)} \subset \pi^{-1}(\overline{B}),$$

and the monotonicity of positive measures.  $\square$

The next result will be a crucial ingredient in the proof of Proposition 1.5. A key point in the proof is the *extreme disconnectedness* of  $\beta G$ . Recall that a compact hausdorff space is *extremely disconnected* if the closure of any open subset is *clopen*. The classical fact that  $\beta G$  is extremely disconnected for *any* discrete group  $G$  can be found in [15].

Also recall that a set  $\mathcal{C} \subset \mathcal{M}(G)$  is *left saturated* if every closed left- $G$ -invariant subset  $Y \subset \beta G$  supports a probability measure of the form  $\bar{\lambda}$  for some  $\lambda \in \mathcal{C}$  with the property that whenever  $A \subset \beta G$  is a Borel set with  $\bar{\lambda}(A) = 0$ , then  $\bar{\lambda}(g \cdot A) = 0$  for all  $g \in G$ .

**Lemma 2.8.** *Let  $\mathcal{C} \subset \mathcal{M}(G)$  be a left saturated set and denote by  $\bar{\mathcal{C}}$  its image in  $\mathcal{P}(\beta G)$ . Let  $C \subset G$  and suppose  $U \subset \beta G$  is an open  $\bar{\mathcal{C}}$ -tempered set. If*

$$\bar{\mathcal{C}}_q \supset U_q$$

*for some  $q \in \beta G$ , then there exists a  $\mathcal{C}$ -conull set  $T \subset G$  such that*

$$C \supset U_\varepsilon \cap T.$$

*Proof.* Define the open set  $D := U \setminus \bar{\mathcal{C}}$ . By assumption,  $D_q$  is empty, and thus

$$Y := \beta G \setminus GD \subset \beta G$$

is a non-empty, closed and  $G$ -invariant set. Since  $\mathcal{C}$  is left saturated, there exists  $\lambda \in \mathcal{C}$  such that  $\bar{\lambda}(Y) = 1$ , and thus  $\bar{\lambda}(D) = 0$ . Since  $\bar{\mathcal{C}}$  is clopen and  $U$  is Jordan measurable with respect to  $\bar{\lambda}$ , we have

$$\bar{\lambda}(\bar{U}) = \bar{\lambda}(U) = \bar{\lambda}(U \cap \bar{\mathcal{C}}) = \bar{\lambda}(\bar{U} \cap \bar{\mathcal{C}}),$$

and hence  $\bar{\lambda}(\bar{U} \setminus \bar{\mathcal{C}}) = 0$ . Since  $\beta G$  is extremely disconnected and  $U$  is open,  $\bar{U} \setminus \bar{\mathcal{C}}$  is clopen, and thus

$$\lambda((\bar{U} \setminus \bar{\mathcal{C}})_\varepsilon) = \bar{\lambda}(\bar{U} \setminus \bar{\mathcal{C}}).$$

Define the sets

$$\bar{T} := \beta G \setminus (\bar{U} \setminus \bar{\mathcal{C}}) \quad \text{and} \quad T := \bar{T}_\varepsilon.$$

Then  $\lambda(T) = \bar{\lambda}(\bar{T}) = 1$ , so  $T$  is  $\mathcal{C}$ -conull, and

$$\bar{\mathcal{C}}_\varepsilon \supset \bar{U}_\varepsilon \cap T \supset U_\varepsilon \cap T,$$

which finishes the proof.  $\square$

**2.5. Properties of right thick sets.** Recall that a subset  $T \subset G$  is *right thick* if for every finite subset  $F \subset G$  there exists  $g \in G$  such that

$$Fg^{-1} \subset T.$$

The following lemma is a simple characterization of right thickness in a countable group  $G$  in terms of clopen subsets of  $\beta G$ .

**Lemma 2.9.** *A set  $T \subset G$  is right thick if and only if  $\bar{T}_q = G$  for some  $q \in \beta G$ .*

*Proof.* Suppose  $\overline{T}_q = G$  for some  $q \in \beta G$  and fix a finite set  $F$  in  $G$ . By Lemma 2.6, there exists  $g \in G$  such that

$$T \cdot g^{-1} \supset F,$$

and hence  $T$  is right thick.

Now suppose that  $T$  is right thick and choose an increasing sequence  $(F_n)$  of finite sets with  $\cup_n F_n = G$ . Since  $T$  is right thick, we can find a sequence  $(g_n)$  in  $G$  such that

$$T = \overline{T}_{\bar{e}} \supset F_n \cdot g_n^{-1},$$

or equivalently  $\overline{T}_{g_n^{-1} \cdot \bar{e}} \supset F_n$ , for all  $n$ . Extract a convergent subset  $(g_{n_\alpha}^{-1} \cdot \bar{e})$  with limit point  $q$ . We claim that  $\overline{T}_q = G$ . Indeed, fix any finite subset  $F \subset G$ . By Lemma 2.5, the set  $U_F(T)$  is clopen. Note that  $g_{n_\alpha} \cdot \bar{e} \in U_F(T)$  for all  $n_\alpha$  with  $F \subset F_{n_\alpha}$ . In particular,  $q \in U_F(T)$  since  $U_F(T)$  is closed. Now,  $F$  was chosen arbitrary, so we conclude that  $\overline{T}_q = G$ .  $\square$

**2.5.1. Density characterizations of right thick sets.** If  $G$  is a countable amenable group, then right thickness has a simple description in terms of invariant means. Namely, suppose  $T \subset G$  is right thick and choose a left Følner sequence  $(F_n)$  in  $G$ . Since  $T$  is right thick, we can find  $(g_n)$  such that

$$F_n \cdot g_n^{-1} \subset T, \quad \forall n.$$

Note that the sequence  $F'_n := F_n g_n^{-1}$  is still left Følner, so any left invariant mean constructed as a weak\*-accumulation point of the averages over  $(F'_n)$  will give measure one to the set  $T$ . In our terminology,  $T$  is  $\mathcal{L}_G$ -conull, where  $\mathcal{L}_G$  denotes the set of left-invariant means on  $G$ . It is not hard to establish the converse as well; that is to say, a subset of a countable amenable group is right thick if and only if it is  $\mathcal{L}_G$ -conull.

Recall that a set  $\mathcal{M}(G)$  is left saturated if every closed  $G$ -invariant subset of  $\beta G$  supports a probability measure of the form  $\bar{\lambda}$  for some  $\lambda$  in  $\mathcal{C}$  which has the property that if  $A \subset \beta G$  is a Borel set with  $\bar{\lambda}(A) = 0$ , then  $\bar{\lambda}(g \cdot A) = 0$  for all  $g \in G$ . Note that amenability of a countable group  $G$  is equivalent to  $\mathcal{L}_G$  being left saturated.

**Proposition 2.10.** *Suppose  $\mathcal{C} \subset \mathcal{M}(G)$  is a left saturated set. Then  $T \subset G$  is right thick if and only if it is  $\mathcal{C}$ -conull, i.e.  $d_{\mathcal{C}}^*(T) = 1$ .*

*Proof.* Suppose  $T$  is right thick. By Lemma 2.9 there exists  $q \in \beta G$  such that  $\overline{T}_q = G$ . Let  $Z$  denote the orbit closure of  $q$  in  $\beta G$  under the left action by  $G$  on  $\beta G$ . Note that  $\overline{T}_q = G$  implies that  $Z \subset \overline{T}$ . Since  $\mathcal{C}$  is left saturated, we can find  $\lambda \in \mathcal{C}$  with  $\bar{\lambda}(Z) = 1$ , and thus  $\lambda(T) = 1$ , which shows that  $T$  is  $\mathcal{C}$ -conull.

Now suppose that  $T$  is  $\mathcal{C}$ -conull and fix  $\lambda \in \mathcal{C}$  with  $\bar{\lambda}(\overline{T}) = 1$ . Since  $\mathcal{C}$  is left saturated,  $\bar{\lambda}$  is non-singular under the left action of  $G$  on  $\beta G$ . In particular, the support  $Z$  of  $\bar{\lambda}$  is a closed  $G$ -invariant set, and  $Z \subset \overline{T}$ . Hence, for all  $q \in Z$ , we have  $\overline{T}_q = G$ , which by Lemma 2.9 implies that  $T$  is right thick.  $\square$

**2.6. Properties of left syndetic sets.** Recall that a set  $C \subset G$  is *left syndetic* if there exists a finite set  $F \subset G$  such that  $FC = G$ . The following lemma isolates an important property of these sets.

**Lemma 2.11.** *A set  $C \subset G$  is left syndetic if and only if  $\overline{C}_q$  is non-empty for all  $q \in \beta G$ .*

*Proof.* Suppose  $C$  is left syndetic and fix a finite set  $F \subset G$  with  $FC = G$ . Then

$$F\overline{C}_q = (F\overline{C})_q = \overline{FC}_q = G$$

for all  $q \in \beta G$ . In particular,  $\overline{C}_q$  is non-empty for all  $q \in \beta G$ .

Suppose  $C$  is *not* left syndetic and note that the set

$$Z := \beta G \setminus G\overline{C}$$

is a non-empty, closed and  $G$ -invariant set. Indeed, if it was empty, then by compactness of  $\beta G$  we could find a finite set  $F$  in  $G$  such that  $F\overline{C} = \beta G$ , or equivalently,  $FC = G$ . However, this means that  $C$  is left syndetic, which contradicts our assumption. Now, for every  $q \in Z$ , we see that  $\overline{C}_q$  is empty, which finishes the proof.  $\square$

This lemma, combined with Lemma 2.9, now gives:

**Proposition 2.12.** *A set  $C \subset G$  is left syndetic if and only if its complement  $C^c$  is not right thick.*

2.6.1. *Density characterizations of left syndetic sets.* We can also establish the following analogue of Proposition 2.10.

**Proposition 2.13.** *Suppose  $\mathcal{C} \subset \mathcal{M}(G)$  is left saturated. Then  $C \subset G$  is left syndetic if and only if the lower  $\mathcal{C}$ -density of  $C$  is positive, i.e.  $d_*^{\mathcal{C}}(C) > 0$ .*

*Proof.* Let  $C$  be a left syndetic set and choose a finite set  $F$  in  $G$  such that  $FC = G$ . Suppose  $d_*^{\mathcal{C}}(C) = 0$ , or equivalently, there exists  $\lambda \in \mathcal{C}$  with  $\lambda(C) = 0$ . Since

$$1 = \lambda(FC) \leq \sum_{f \in F} \lambda(f \cdot C),$$

we conclude that  $\lambda(f \cdot C) > 0$  for at least one  $f \in F$ . Since  $\mathcal{C}$  is left saturated and thus in particular left non-singular, we conclude that  $\lambda(C) > 0$  as well, which contradicts our assumption, and hence  $d_*^{\mathcal{C}}(C) > 0$ .

Now suppose that  $\lambda(C) > 0$  for all  $\lambda \in \mathcal{C}$ . We claim that in this case,  $\overline{C}_q$  is non-empty for all  $q \in \beta G$ . By Lemma 2.11 this implies that  $C$  is left syndetic. Note that  $\overline{C}_q$  is empty if and only if  $q$  belongs to the set

$$Z := \beta G \setminus G\overline{C}.$$

Since  $\mathcal{C}$  is left saturated, the set  $Z$ , being both closed and  $G$ -invariant, is non-empty if and only if there exists  $\bar{\lambda} \in \mathcal{C}$  with  $\bar{\lambda}(Z) = 1$ , or equivalently (since  $\bar{\lambda}$  is non-singular),  $\bar{\lambda}(\overline{C}) = 0$ . This finishes the proof.  $\square$

### 3. ERGODIC THEORY OF MEASURED GROUPS

The aim of this section is to establish some basic properties of measured groups which will be used in the proofs of Proposition 1.2 and Proposition 1.3.

**3.1. Affine  $\mu$ -spaces.** Let  $G$  be a countable group and assume that  $X$  is a Banach space equipped with a homomorphism

$$\rho : G \rightarrow \text{Isom}(X),$$

where  $\text{Isom}(X)$  denotes the group of all *linear* isometries of  $X$ . Given a probability measure  $\mu$  on  $G$ , we can define a bounded operator  $\rho(\mu)$  on  $X$  by the formula

$$\langle \lambda, \rho(\mu)x \rangle = \int_G \langle \lambda, \rho(g)x \rangle d\mu(g),$$

for all  $x \in X$  and  $\lambda \in X^*$ . If  $K \subset X^*$  is a weak\*-compact and convex subset such that

$$\rho(\mu)^*K \subset K,$$

then we shall say that  $K$  is an *affine  $\mu$ -space*. Note that we do *not* assume that the representation  $\rho^*$  itself preserves the set  $K$ .

We include a proof of the following standard lemma (sometimes referred to as the Kakutani-Markov Lemma) for completeness.

**Lemma 3.1.** *Every affine  $\mu$ -space admits a fixed point.*

*Proof.* Suppose  $K \subset X^*$  is an affine  $\mu$ -space, where  $X$  is a Banach space, equipped with a representation  $\rho : G \rightarrow \text{Isom}(X)$ . Fix  $\lambda \in K$ . Since  $\rho(\mu)^*K \subset K$ , the sequence

$$\lambda_n := \frac{1}{n} \sum_{k=1}^n \rho(\mu)^{*k} \lambda$$

belongs to  $K$ , and thus admits a weak\*-cluster point  $\lambda_o \in K$  by weak\*-compactness. It is readily checked that  $\rho(\mu)^*\lambda_o = \lambda_o$ , which finishes the proof.  $\square$

The lemma above especially applies to the regular representation on  $X = C(Z)$ , where  $Z$  is a compact hausdorff space, equipped with an action of  $G$  by homeomorphisms. One readily checks that the space of regular Borel probability measures  $\mathcal{P}(Z)$  is preserved by  $\rho(\mu)^*$ , and identified with the states on  $C(Z)$ , this is a weak\*-compact and convex subset. The set of  $\rho(\mu)^*$ -fixed points in  $\mathcal{P}(Z)$  will be denoted by  $\mathcal{P}_\mu(Z)$ , and the elements in  $\mathcal{P}_\mu(Z)$  will be referred to as  *$\mu$ -stationary* (or  *$\mu$ -harmonic*) probability measures.

**Corollary 3.2.** *Let  $X$  and  $Y$  be compact  $G$ -spaces and suppose  $\pi : X \rightarrow Y$  is a  $G$ -equivariant continuous surjection. For every  $\mu \in \mathcal{P}(G)$  and  $\eta \in \mathcal{P}_\mu(Y)$ , there exists  $\nu \in \mathcal{P}_\mu(X)$  such that  $\pi_*\nu = \eta$ .*

*Proof.* Fix  $\mu$  and  $\eta$  and define

$$\mathcal{C} := \left\{ \nu \in \mathcal{P}(X) : \pi_*\nu = \eta \right\}.$$

This set is non-empty by standard arguments, and clearly weak\*-compact and convex. Moreover, since  $\eta$  is  $\mu$ -stationary and  $\pi$  is  $G$ -equivariant, this set is also an affine  $\mu$ -space (however, it is not an affine  $G$ -space!). By Lemma 3.1 there exists a  $\mu$ -fixed  $\nu$  point in  $\mathcal{C}$ , i.e. a  $\mu$ -stationary measure on  $X$  which projects onto  $\eta$  under  $\pi$ .  $\square$

3.1.1. *Properties of  $\mu$ -stationary measures.* Recall that if  $Z$  is a  $G$ -space, i.e. a compact hausdorff space equipped with an action of  $G$  by homeomorphisms, then a Borel probability measure  $\nu$  on  $Z$  is said to be *non-singular* (or quasi-invariant) if  $\nu(g \cdot N) = 0$  for all  $g \in G$ , whenever  $N \subset Z$  is a  $\nu$ -null set of  $Z$ . We now have the following proposition.

**Proposition 3.3.** *For every measured group  $(G, \mu)$  and compact  $G$ -space  $Z$ , the set  $\mathcal{P}_\mu(Z)$  of  $\rho(\mu)^*$ -fixed probability measures is non-empty. Furthermore, every  $\nu \in \mathcal{P}_\mu(Z)$  is non-singular.*

*Proof.* The first assertion is contained in Corollary 3.2 with  $Y$  equal to a one-point space, so we shall only prove that every element in  $\mathcal{P}_\mu(Z)$  is non-singular.

Fix  $\nu \in \mathcal{P}_\mu(Z)$  and suppose  $B \subset Z$  is a Borel set with  $\nu(B) = 0$ . Since

$$\rho(\mu)^{*n}\nu(B) = \int_G g_*\nu(B) d\mu^{*n}(g) = \nu(B)$$

we see that  $\nu(g^{-1}B) = 0$  for all  $g \in \text{supp}(\mu)^n$  and  $n \geq 1$  ( $G$  is countable). Since the support  $\text{supp}(\mu)$  is assumed to generate  $G$  as a semigroup (by admissibility), we conclude that  $\nu(g^{-1} \cdot B) = 0$  for all  $g \in G$ , and thus  $\nu$  is non-singular.  $\square$

3.1.2. *Basic properties of the set  $\mathcal{L}_\mu$ .* Recall that a set  $\mathcal{C} \subset \mathcal{M}(G)$  is *left saturated* if every (non-empty) closed left  $G$ -invariant subset of  $\beta G$  supports at least one *non-singular* probability measure of the form  $\bar{\lambda}$  for some  $\lambda \in \mathcal{C}$ . Also recall that  $\mathcal{L}_\mu$  denotes the set of left  $\mu$ -stationary means on  $G$ .

**Corollary 3.4.** *For every measured group  $(G, \mu)$ , the set  $\mathcal{L}_\mu$  is left saturated.*

*Proof.* Apply the proposition above to non-empty closed  $G$ -invariant subsets of the  $G$ -space  $Z = \beta G$ .  $\square$

In view of Proposition 2.10 and Proposition 2.13, we have also proved the following corollary.

**Corollary 3.5.** *For every measured group  $(G, \mu)$ , a set  $T \subset G$  is right thick if and only if it is  $\mathcal{L}_\mu$ -conull and a set  $C \subset G$  is left syndetic if and only if the lower  $\mathcal{L}_\mu$ -density is positive.*

3.2. **Basic properties of  $\mu$ -harmonic functions on  $G$ -spaces.** Let  $Y$  be a compact  $G$ -space, and fix an *admissible* (symmetric) probability measure  $\mu$  on  $G$ , i.e. a probability measure such that  $\mu(g) = \mu(g^{-1})$  for all  $g$  in  $G$  and whose support generates  $G$  as a semigroup. We shall say that a Borel function  $\varphi : Y \rightarrow \mathbb{R}$  is  *$\mu$ -harmonic* if

$$\mu * \varphi(y) := \int_Y \varphi(g \cdot y) d\mu(g) = \varphi(y), \quad \forall y \in Y.$$

The following lemma shows that  $\mu$ -harmonic semicontinuous functions exhibit substantial  $G$ -invariance. This will be a crucial ingredient in the proof of Proposition 1.3.

**Lemma 3.6.** *Suppose  $\varphi$  is a lower semicontinuous and  $\mu$ -harmonic function on  $Y$ . Then the set of minima for  $\varphi$  is non-empty, closed and  $G$ -invariant.*

*Proof.* Let  $Z$  denote the set of minima for  $\varphi$ . Since  $\varphi$  is lower semicontinuous,  $Z$  is non-empty and closed, so it suffices to show that  $Z$  is  $G$ -invariant. Fix  $y \in Z$  and suppose  $g \cdot y \notin Z$ . Since the support of  $\mu$  is assumed to generate  $G$  as a semigroup, there exists  $n \geq 1$  such that  $g$  belongs to

$$\text{supp}(\mu^{*n}) = \text{supp}(\mu)^n$$

and thus

$$\int_G \varphi(h \cdot y) d\mu^{*n}(h) > \varphi(y),$$

which contradicts our assumption that  $\varphi$  is  $\mu$ -harmonic. Hence  $g \cdot y \in Z$  and  $Z$  is  $G$ -invariant.  $\square$

For  $\mu$ -harmonic and *continuous* functions, the lemma above can be sharpened as follows.

**Lemma 3.7.** *Let  $\nu \in \mathcal{P}_\mu(Y)$  and suppose  $\varphi$  is a continuous and  $\mu$ -harmonic function on the  $G$ -space  $Y$ . Then the restriction of  $\varphi$  to the support of  $\nu$  is  $G$ -invariant.*

*Proof.* It suffices to prove that for all  $\alpha \in \mathbb{R}$ , the intersection of the support of  $\nu$  and the set

$$Z_\alpha := \{y \in Y : \varphi(y) = \alpha\} \subset Y$$

is  $G$ -invariant. For fixed  $\alpha$ , we define the function

$$\varphi_\alpha(y) := \max(\alpha, \varphi(y)), \quad y \in Y.$$

Note that  $\mu * \varphi_\alpha \geq \varphi_\alpha$  and that the set  $Z_\alpha$  coincides with the set of minima for  $\varphi_\alpha$  for all  $\alpha$ . If we can show that the restriction of  $\varphi_\alpha$  to the support of  $\nu$  is  $\mu$ -harmonic, then Lemma 3.6 implies that the intersection of  $Z_\alpha$  with the support of  $\nu$  is  $G$ -invariant.

However, since  $\mu * \varphi_\alpha - \varphi_\alpha \geq 0$  for all  $\alpha$ , and

$$\int_Y (\mu * \varphi_\alpha - \varphi_\alpha) d\nu = 0, \quad \forall \alpha \in \mathbb{R},$$

by  $\mu$ -stationarity of  $\nu$ , we conclude that  $\mu * \varphi_\alpha = \varphi_\alpha$  on the support of  $\nu$ .  $\square$

**Corollary 3.8.** *Let  $\nu \in \mathcal{P}_\mu(Y)$  and suppose  $\varphi \in L^\infty(Y, \nu)$  satisfy  $\rho(\mu)\varphi = \varphi$ . Then  $\varphi$  is essentially  $G$ -invariant.*

*Proof.* Since  $L^\infty(Y, \nu)$  is a commutative unital  $C^*$ -algebra, the Gelfand-Naimark Theorem asserts that it is isometrically isomorphic to the space of continuous functions on its (compact) Gelfand spectrum  $Z$ . Furthermore, the isometric  $G$ -action on  $L^\infty(Y, \nu)$  translates to an action of  $G$  by homeomorphisms on  $Z$ .

Note that  $\nu \in \mathcal{P}_\mu(Y)$  can be thought of as a  $\rho(\mu)$ -invariant state on  $L^\infty(Y, \nu)$ , and thus as a regular  $\mu$ -stationary Borel probability measure  $\bar{\nu}$  on  $Z$ . One readily checks that  $\bar{\nu}$  has full support.

If  $\rho(\mu)\varphi = \varphi$  for some  $\varphi \in L^\infty(Y, \nu)$ , then the image  $\bar{\varphi}$  of  $\varphi$  under the Gelfand map is a continuous  $\mu$ -harmonic function on  $Z$ , so by Lemma 3.7, it must be  $G$ -invariant on  $Z$  (since  $\bar{\nu}$  has full support). This clearly implies that  $\varphi$  is  $G$ -invariant, which finishes the proof.  $\square$

**3.3. An ergodic theorem for  $(G, \mu)$ -spaces.** Recall that if  $Y$  is a  $G$ -space, then a non-singular Borel probability measure  $\nu$  on  $Y$  is said to be *ergodic* if a  $G$ -invariant Borel set either has  $\nu$ -measure zero or one. In particular, if  $\varphi$  is a  $G$ -invariant Borel function on  $Y$ , then

$$\varphi = \int_Y \varphi d\nu, \quad \text{a.e. w.r.t. } \nu.$$

Fix a probability measure  $\mu$  on  $G$  and a  $\mu$ -stationary Borel probability measure  $\nu$  on  $Y$ . Consider the operator  $\rho(\mu)$  on  $C(Y)$  defined by

$$\rho(\mu)\varphi(y) = \int_G \varphi(g \cdot y) d\mu(g), \quad y \in Y.$$

Note that since  $\nu$  is  $\mu$ -stationary,  $\rho(\mu)$  extends to a bounded operator  $\rho_\nu(\mu)$  on  $L^1(Y, \nu)$  of norm at most one and thus also defines a bounded operator on  $L^2(Y, \nu)$  of norm at most one. However, we stress that the  $G$ -action itself is far from isometric on  $L^1(Y, \nu)$ .

For this operator, we have the following analogue of von Neumann's Ergodic Theorem.

**Lemma 3.9.** *Suppose  $\nu \in \mathcal{P}_\mu(Y)$  is ergodic. Then the  $L^2$ -limit*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \rho_\nu(\mu)^k \varphi = \int_Y \varphi d\nu,$$

*exists for all  $\varphi \in L^2(Y, \nu)$ .*

*Proof.* Note that the operator norm of  $\rho_\nu(\mu)$  on  $L^2(Y, \nu)$  is bounded by one. Hence, by Lorch's Ergodic Theorem (Theorem 1.2, Chapter 2 in [18]), the  $L^2$ -limit exists for all  $\varphi$  and is  $\rho_\nu(\mu)$ -invariant. Since  $\nu$  is ergodic, Corollary 3.8 implies that the limit is essentially constant and thus equal to the integral.  $\square$

By a simple approximation argument, using the fact that  $L^2(Y, \nu)$  is a dense subspace of  $L^1(Y, \nu)$ , we can extend the convergence asserted in the previous lemma to  $L^1$ -convergence for all  $L^1$ -functions. Furthermore, one can also prove statements about almost everywhere convergence upon referring to classical results.

**3.3.1. Estimating the upper  $\mathcal{F}_\mu$ -density.** Recall that

$$\mathcal{F}_\mu := \left\{ \lambda \in \mathcal{L}_\mu : \lambda|_{\mathcal{H}^\infty(G, \mu)} = \delta_e \right\}.$$

We have already noticed that every weak\*-accumulation point in  $\ell^\infty(G)^*$  of the sequence

$$\lambda_n := \frac{1}{n} \sum_{k=1}^n \mu^{*k},$$

belongs to  $\mathcal{F}_\mu$ . In particular, if  $Y$  is a compact  $G$ -space, then

$$d_{\mathcal{F}_\mu}^*(B_y) \geq \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \mu^{*k}(B_y)$$

for all  $y \in Y$ , where

$$B_y = \left\{ g \in G : g \cdot y \in B \right\}, \quad y \in Y.$$

The following corollary shows that there is a simple lower bound for the upper  $\mathcal{F}_\mu$ -density (and thus for the upper  $\mathcal{L}_\mu$ -density as well) for these kinds of subsets.

**Corollary 3.10.** *Suppose  $\nu \in \mathcal{P}_\mu(Y)$  is ergodic. Then, for every Borel set  $B \subset Y$ , there exists a  $\nu$ -conull Borel set  $Y' \subset Y$  such that*

$$d_{\mathcal{F}_\mu}^*(B_y) \geq \nu(B), \quad \forall y \in Y'.$$

*Proof.* It suffices to prove that there exists a conull subset  $Y' \subset Y$  such that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \mu^{*k}(B_y) > 0$$

for all  $y \in Y'$ . Since

$$\sum_{k=1}^n \mu^{*k}(B_y) = \sum_{k=1}^n \rho(\mu)^{*k} \chi_B(y),$$

where  $\rho(\mu)$  denotes the bounded operator on  $L^2(Y, \nu)$  introduced above, we have by Lemma 3.9,

$$\lim_n \frac{1}{n} \sum_{k=1}^n \rho(\mu)^{*k} \chi_B = \nu(B) > 0$$

in the  $L^2$ -norm. Hence we can extract a subsequence such that the limit exists almost everywhere along this subsequence, and this limit will be positive.  $\square$

Note that since  $\mu$  is assumed to be *symmetric*, the proof of the previous corollary also implies that sets of the form  $B_y^{-1}$  are  $\mathcal{F}_\mu$ -large, for all  $y \in Y'$ . This has an important consequence as the following corollary shows.

**Corollary 3.11.** *Suppose  $A \subset G$  is  $\mathcal{L}_\mu$ -large. Then there exists a  $\mathcal{F}_\mu$ -large set  $B \subset G$  such that  $B^{-1}$  is  $\mathcal{F}_\mu$ -large as well and*

$$AA^{-1} \supset BB^{-1}.$$

*Proof.* Let  $\lambda$  be an element in  $\mathcal{L}_\mu$  such that  $\bar{\lambda}$  is an *ergodic*  $\mu$ -stationary probability measure on  $\beta G$  and the set  $\bar{A} \subset \beta G$  has positive  $\bar{\lambda}$ -measure (such elements exist by Proposition 3.13 below). The previous corollary guarantees the existence of  $q \in \beta G$  such that both sets  $\bar{A}_q$  and  $\bar{A}_q^{-1}$  are  $\mathcal{F}_\mu$ -large.

Fix an increasing exhaustion  $(F_n)$  of  $G$  by finite sets. For every  $n$ , there exists by Lemma 2.6 an element  $g_n \in G$  such that

$$\bar{A}_q \cap F_n = (A \cap F_n \cdot g_n) \cdot g_n^{-1}$$

In particular,

$$AA^{-1} \supset (A \cap F_n \cdot g_n)(A \cap F_n \cdot g_n)^{-1} = (\bar{A}_q \cap F_n)(\bar{A}_q \cap F_n)^{-1}$$

for all  $n$ , and thus

$$AA^{-1} \supset \bar{A}_q \bar{A}_q^{-1}.$$

Hence, if we set  $B := \bar{A}_q$ , then this set satisfies the desired properties.  $\square$

**3.4. Poisson boundaries of a measured groups.** Recall that a *bounded* function  $f$  on  $G$  is *left  $\mu$ -harmonic* (note that we shall always assume that  $\mu$  is *symmetric*) if the equation

$$\mu * f(g) := \int_G f(hg) d\mu(h) = f(g)$$

holds for all  $g \in G$ . Clearly, a constant function is always left  $\mu$ -harmonic, and there are instances when constants are the only (bounded)  $\mu$ -harmonic functions (e.g. when  $G$  is a countable abelian group and  $\mu$  is any admissible probability measure on  $G$ ).

We shall denote by  $\mathcal{H}_l^\infty(G, \mu)$  the space of all bounded left  $\mu$ -harmonic functions on  $G$ . Easy examples show that the pointwise product of two left  $\mu$ -harmonic functions may fail to be left  $\mu$ -harmonic, so  $\mathcal{H}_l^\infty(G, \mu)$  is *not* a sub-algebra of  $\ell^\infty(G)$ , and thus do not quite fit onto the framework we have set up so far.

However, there is another product (which is a special case of a more general construction in operator system theory called the *Choi-Effros* product), which does turn  $\mathcal{H}_l^\infty(G, \mu)$  into an algebra. In our setting, this product, here denoted by  $\square$ , can be defined by

$$(\varphi \square \psi)(g) := \lim_{n \rightarrow \infty} \int_G \varphi(hg) \psi(hg) d\mu^{*n}(h), \quad g \in G,$$

for  $\varphi, \psi \in \mathcal{H}_l^\infty(G, \mu)$ . One can show that the limit exists by a simple use of the Martingale Convergence Theorem. We refer the reader to [2] for details about this product; especially the fact that  $(\mathcal{H}_l^\infty(G, \mu), \square)$  is a commutative  $C^*$ -algebra.

Another serious deviation from the set up so far is that  $\mathcal{H}_l^\infty(G, \mu)$  is invariant under the right regular representation  $r$  on  $\ell^\infty(G)$ , but *not* necessarily under the left regular representation. This right action on  $\mathcal{H}_l^\infty(G, \mu)$  gives rise to a (right) action of  $G$  by homeomorphisms on the Gelfand spectrum

$$B := \Delta(\mathcal{H}_l^\infty(G, \mu)),$$

which we shall consistently write as a left action, i.e.  $g \cdot b := bg^{-1}$ , for  $b \in B$  and  $g \in G$ .

Note that the evaluation map  $\delta_e$  on  $\mathcal{H}_l^\infty(G, \mu)$  is a  $r(\mu)$ -invariant state, and thus corresponds to a  $\mu$ -stationary probability measure  $\nu_o$  on  $B$ . Indeed,

$$\langle r(\mu)^* \delta_e, f \rangle = \int_G f(h) d\mu(h) = \langle \delta_e, f \rangle, \quad \forall f \in \mathcal{H}_l^\infty(G, \mu).$$

If  $f \in C(B)$ , we define  $\hat{f} \in \ell^\infty(G)$  by

$$\hat{f}(g) := \int_B f(g^{-1} \cdot b) d\nu_o(b), \quad g \in G.$$

Since  $\mu$  is symmetric,  $\hat{f}$  belongs to  $\mathcal{H}_l^\infty(G, \mu)$  as the following calculation shows.

$$\begin{aligned} \int_G \hat{f}(hg) d\mu(h) &= \int_G \int_B f(g^{-1} \cdot h^{-1} \cdot b) d\nu_o(b) d\mu(h) \\ &= \int_B f(g^{-1} \cdot b) d(\mu * \nu_o)(b) = \hat{f}(g), \quad \forall g \in G. \end{aligned}$$

Note that we can also write  $\hat{f}$  as

$$\hat{f}(g) = \langle \nu, r(g)f \rangle_{C(B)} = \langle r(g)^* \delta_e, f' \rangle_{\mathcal{H}_l^\infty(G, \mu)} = f'(g), \quad \forall g \in G,$$

where  $f'$  denotes the image of  $f \in C(B)$  in  $\mathcal{H}_l^\infty(G, \mu)$  under the Gelfand map (for the Choi-Effros product), so we conclude that  $\hat{f}$  completely determines  $f$ . This observation suggests the following proposition, whose proof can be found in [2].

**Proposition 3.12.** *The  $\mu$ -stationary probability measure  $\nu_o$  on  $B$  is ergodic and the map*

$$f \mapsto \hat{f}$$

*is an isometric  $*$ -isomorphism between  $L^\infty(B, \nu_o)$  and  $\mathcal{H}^\infty(G, \mu)$ .*

**Remark 3.1.** The Gelfand spectrum of  $\mathcal{H}_l^\infty(G, \mu)$  with the Choi-Effros product, together with the measure  $\nu_o$ , is sometimes referred to as the *Poisson boundary* of  $(G, \mu)$ . However, since this measure space is only standard when  $\mathcal{H}_l^\infty(G, \mu)$  reduces to the constants, one often prefers to choose other (countably generated) Borel representations which will yield the same  $L^\infty$ -space, and then such a space is also referred to as the Poisson boundary of  $(G, \mu)$ . In this paper, we shall only work with the above realization of the Poisson boundary.

**3.5. Left ergodicity of  $\mathcal{F}_\mu \subset \mathcal{M}(G)$ .** As we have seen, the space  $\mathcal{P}_\mu(Y)$  of  $\mu$ -stationary Borel measures on a  $G$ -space  $Y$ , is always a non-empty, weak\*-compact and convex set. In particular, by Krein-Milman's Theorem, it has extremal points. We shall denote the set of extremal points by  $\text{Ext}(\mathcal{P}_\mu(Y))$ .

Let  $\text{Erg}_\mu(Y)$  denote the set of  $\mu$ -stationary *ergodic* probability measures on  $Y$ . In the weak\*-topology, this is a nice  $G_\delta$ -set if  $Y$  is second countable. When  $Y$  is not second countable, then there is no reason to expect this set to be even Borel measurable. This is particular the case for the Poisson boundary  $B$  defined in the previous subsection. In any case, we do have the following equality of sets.

**Proposition 3.13.** *For every measured group  $(G, \mu)$  and compact  $G$ -space  $Y$ , we have*

$$\text{Ext}(\mathcal{P}_\mu(Y)) = \text{Erg}_\mu(Y).$$

*In particular,  $\text{Erg}_\mu(Y)$  is non-empty.*

*Proof.* First note that if  $\nu \in \mathcal{P}_\mu(Y)$  and  $X \subset Y$  is a  $G$ -invariant  $\nu$ -measurable set with measure

$$0 < \nu(X) < 1,$$

then the re-normalized restriction of  $\nu$  to  $X$  clearly belongs to  $\mathcal{P}_\mu(Y)$ . Hence, if  $\nu$  is *not* ergodic, it *cannot* be extremal in  $\mathcal{P}_\mu(Y)$ .

The converse is slightly more difficult. Suppose  $\nu$  is not extremal, but still ergodic, so that we can write

$$\nu = \alpha_1 \cdot \nu_1 + \alpha_2 \cdot \nu_2,$$

for some positive  $\alpha_1, \alpha_2$  with  $\alpha_1 + \alpha_2 = 1$ , where  $\nu_1, \nu_2 \in \mathcal{P}_\mu(Y)$ .

Note that both  $\nu_1$  and  $\nu_2$  must be absolutely continuous with respect to  $\nu$ , so by Radon-Nikodym's Theorem, it suffices to prove that if  $\rho \in L^1(Y, \nu)$  has the property that the  $d\nu' = \rho d\nu$  is a  $\mu$ -stationary, probability measure, then  $\rho$  is equal to one  $\nu$ -almost everywhere.

Let  $r > 0$  and let  $\rho_r$  denote the cut-off of  $\rho$  for values greater than  $r$  so that  $\rho_r$  is a bounded  $\nu$ -measurable function. Note that

$$\int_Y \varphi d\nu' = \int_Y A_n \varphi d\nu' \geq \int_Y (A_n \varphi) \rho_r d\nu$$

for all  $\varphi \in L^\infty(Y, \nu)_+$ , where

$$A_n \varphi := \frac{1}{n} \sum_{k=1}^n \rho(\mu)^{*k} \varphi.$$

Here,  $\rho(\mu)$  is the bounded operator on  $L^2(Y, \nu)$  introduced in Lemma 3.9. Since  $\nu$  is ergodic, this lemma applies, and thus the sequence  $A_n \varphi$  converges to the  $\nu$ -integral of  $\varphi$  in the  $L^2$ -norm. Hence, for all  $\varphi \in L^\infty(Y, \nu)_+$ , we have

$$\int_Y \varphi d\nu' \geq \lim_n \langle \rho_r, A_n \varphi \rangle_{L^2(Y, \nu)} = \left( \int_Y \rho_r d\nu \right) \cdot \int_Y \varphi d\nu.$$

Upon letting  $r$  tend to infinity, we see that

$$\int_Y \varphi d\nu' \geq \int_Y \varphi d\nu, \quad \forall \varphi \in L^\infty(Y, \nu)_+$$

which readily implies that  $\rho \geq 1$   $\nu$ -almost everywhere. However, since the  $\nu$ -integral equals one, we conclude that  $\rho$  must in fact be  $\nu$ -almost everywhere equal to one, which finishes the proof.  $\square$

Recall that a subset  $\mathcal{C} \subset \mathcal{L}_\mu$  is *left ergodic* if  $\tau^* \lambda$  is an *ergodic*  $\mu$ -stationary Borel probability measure on  $\beta G$  whenever  $\lambda$  is an *extremal* point in  $\mathcal{C}$ , where  $\tau : C(\beta G) \rightarrow \ell^\infty(G)$  is the  $G$ -equivariant map defined by

$$\tau(\varphi)(g) := \varphi(g \cdot \bar{e}), \quad g \in G.$$

The set of *Furstenberg-Poisson means* is defined by

$$\mathcal{F}_\mu := \left\{ \lambda \in \mathcal{L}_\mu : \lambda|_{\mathcal{H}_l^\infty(G, \mu)} = \delta_e \right\}.$$

The following corollary isolates one of the most important properties of  $\mathcal{F}_\mu$  which will be used in the proof of Theorem 1.1.

**Corollary 3.14.** *For every measured group  $(G, \mu)$ , the set  $\mathcal{F}_\mu \subset \mathcal{M}(G)$  is left ergodic.*

*Proof.* Let  $\lambda \in \mathcal{F}_\mu$  and suppose that the  $\mu$ -stationary probability measure  $\bar{\lambda}$  on  $\beta G$  is *not* ergodic. By Lemma 3.13,  $\bar{\lambda}$  is not extremal in the set  $\mathcal{P}_\mu(\beta G)$ , so we can write

$$\bar{\lambda} = \alpha \cdot \bar{\lambda}_1 + \alpha_2 \cdot \bar{\lambda}_2,$$

for some  $\bar{\lambda}_1, \bar{\lambda}_2 \in \mathcal{P}_\mu(Y)$  and *positive*  $\alpha_1, \alpha_2$  with  $\alpha_1 + \alpha_2 = 1$ . We claim that the corresponding means  $\lambda_1$  and  $\lambda_2$  must belong to  $\mathcal{F}_\mu$ , and thus  $\lambda$  is *not extremal*  $\mathcal{F}_\mu$ , which finishes the proof.

To prove this claim, we first note that restriction of  $\lambda$  to  $\mathcal{H}_l^\infty(G, \mu)$  is  $\mu$ -stationary *on the right*, i.e.

$$\lambda(r(\mu)f) = \lambda(f), \quad \forall f \in \mathcal{H}_l^\infty(G, \mu),$$

where  $r$  denotes the right regular representation on  $\mathcal{H}_l^\infty(G, \mu)$ . Note that if  $f$  is left  $\mu$ -harmonic, then so is  $r(\mu)^k f$  for all  $k$  and thus

$$f(e) = \alpha_1 \cdot \lambda_1(r(\mu)^k f) + \alpha_2 \cdot \lambda_2(r(\mu)^k f)$$

for all left  $\mu$ -harmonic functions  $f$  and for all  $k$ . In particular, upon averaging over  $k$ , we may assume that the *restrictions* to  $\mathcal{H}_l^\infty(G, \mu)$  of both  $\lambda_1$  and  $\lambda_2$  are fixed by  $r(\mu)$ .

Recall that the map  $f \mapsto \hat{f}$  from  $L^\infty(B, \nu_o)$  to  $\mathcal{H}_l^\infty(G, \mu)$  defined in Proposition 3.12 is an isometry, so the functionals

$$\nu_1(f) := \lambda_1(\hat{f}) \quad \text{and} \quad \nu_2(f) := \lambda_2(\hat{f})$$

can be viewed as regular Borel probability measures on the Poisson boundary  $B$  of  $(G, \mu)$ . Furthermore,

$$\lambda(\hat{f}) = \hat{f}(e) = \nu_o(f), \quad \forall f \in C(B).$$

We wish to prove that

$$\nu_o = \nu_1 = \nu_2.$$

Note that since

$$\nu_o = \alpha_1 \cdot \nu_1 + \alpha_2 \cdot \nu_2$$

and  $\nu_o$  is ergodic by Proposition 3.12, it suffices to show that  $\nu_1$  and  $\nu_2$  are  $\mu$ -stationary. Indeed, by Proposition 3.13,  $\nu_o$  is extremal in  $\mathcal{P}_\mu(B)$ , so such an equation must force all measures to be equal.

Since the restrictions of  $\lambda_1$  and  $\lambda_2$  to  $\mathcal{H}_l^\infty(G, \mu)$  satisfy

$$r(\mu)^* \lambda_i = \lambda_i, \quad i = 1, 2,$$

we have

$$\begin{aligned} \mu * \nu_i(f) &= \int_G \int_B f(g \cdot b) d\nu_i(b) d\mu(g) \\ &= \int_G \int_G \left( \int_B f(g \cdot h^{-1} \cdot b) d\nu_o(b) \right) d\lambda_i(h) d\mu(g) \\ &= \int_G \int_G \left( \int_B f((h \cdot g^{-1})^{-1} \cdot b) d\nu_o(b) \right) d\lambda_i(h) d\mu(g) \\ &= \int_G \left( \int_B f(h^{-1} \cdot b) d\nu_o(b) \right) d(r(\mu)^* \lambda_i)(h) \\ &= r(\mu)^* \lambda_i(\hat{f}) = \lambda_i(\hat{f}) = \nu_i(f), \end{aligned}$$

for  $i = 1, 2$  and all  $f \in C(B)$ . Thus  $\nu_1$  and  $\nu_2$  belong to  $\mathcal{P}_\mu(B)$ , which finishes the proof.  $\square$

**3.6. Failure of left syndeticity for difference sets.** We shall now construct examples of countable measured groups  $(G, \mu)$  and  $\mathcal{F}_\mu$ -large sets  $A \subset G$  with the property that their difference sets  $AA^{-1}$  are right piecewise left syndetic but *not* left syndetic.

3.6.1. *Basic setup.* Let  $G$  denote the free group on the free symbols  $\{a, b\}$ . Let  $\mathbb{T} = \mathbb{R}/\mathbb{Z}$  and fix an irrational number in  $\mathbb{T}$ . Also choose a homeomorphism  $T$  on  $\mathbb{T}$  which fixes two distinct points  $x_+$  and  $x_-$  on  $\mathbb{T}$  and has the property that for every open neighborhood  $U$  of  $x_+$  and every closed subset  $B \subset \mathbb{T}$  which does not contain  $x_-$ , there exists  $n$  such that

$$T^n(B) \subset U.$$

We define an action of  $G$  on  $\mathbb{T}$  by

$$a \cdot x = x + \alpha \quad \text{and} \quad b \cdot x = T(x), \quad x \in \mathbb{T}.$$

**Lemma 3.15.** *For every non-empty open set  $U \subset \mathbb{T}$  and proper closed subset  $B \subset \mathbb{T}$ , there exists  $g \in G$  such that  $g \cdot B \subset U$ .*

*Proof.* Since  $\alpha$  is irrational, the restricted action by the subgroup  $a^{\mathbb{Z}}$  is minimal, and thus we can find  $m, n \in \mathbb{Z}$  such that  $a^m \cdot U$  is an open neighborhood of  $x_+$  and  $a^n \cdot B$  does not contain  $x_-$ . By the defining property of  $T$ , we can now find  $k \in \mathbb{Z}$  such that

$$b^k \cdot (a^m \cdot B) \subset a^n \cdot U,$$

or equivalently,  $(a^{-n} b^k a^m) \cdot B \subset U$ .  $\square$

**Corollary 3.16.** *For every finite set  $F \subset G$  and closed subset  $A \subset \mathbb{T}$  with empty interior, there exists  $g \in G$  such that  $FA \cap g \cdot A$  is an empty subset of  $S^1$ .*

*Proof.* By Lemma 3.15 it suffices to show that  $FA$  is a proper (closed is immediate) subset for every finite set  $F \subset G$ . Indeed, if this is the case, then we can choose

$$B := A \quad \text{and} \quad U := (FA)^c$$

and find (using the previous lemma) an element  $g \in G$  such that  $g \cdot B \subset U$ , or equivalently,

$$FA \cap g \cdot A = \emptyset.$$

However, if  $FA = \mathbb{T}$  for some finite set, then by Baire's Category Theorem, at least one of  $fA$ , with  $f \in F$ , has non-empty interior. Since  $G$  acts by homeomorphisms, this would imply that  $A$  has non-empty interior, which we have assumed it does not have.  $\square$

Fix an admissible (symmetric)  $\mu \in \mathcal{P}(G)$  and choose an ergodic  $\mu$ -stationary Borel probability measure  $\nu$  on  $\mathbb{T}$ . The support of  $\nu$  is a non-empty closed  $G$ -invariant subset, so in particular it is invariant under the dense subgroup  $\mathbb{Z} \cdot \alpha \subset \mathbb{T}$ , and must thus be equal to  $\mathbb{T}$ .

Recall that if  $A$  is any subset of  $\mathbb{T}$  and  $x$  is a point in  $\mathbb{T}$ , then

$$A_x := \left\{ g \in G : g \cdot x \in A \right\} \subset G.$$

The following lemma is now an immediate consequence of Corollary 3.10.

**Lemma 3.17.** *For every Borel set  $A \subset \mathbb{T}$  with positive  $\nu$ -measure, there exists a  $\nu$ -conull Borel set  $X \subset \mathbb{T}$  such that  $A_x \subset G$  is  $\mathcal{F}_\mu$ -large for all  $x \in X$ .*

Before we can construct our examples, we need the following general result.

**Lemma 3.18.** *Let  $X$  be a compact separable hausdorff space and let  $\nu$  be a non-atomic regular Borel probability measure on  $X$  with full support. Then there exists a closed set  $B \subset X$  with empty interior and positive  $\nu$ -measure.*

*Proof.* Since  $\nu$  is a non-atomic and regular Borel probability measure, every point  $x$  in  $X$  admits a decreasing sequence  $(U_n)$  of open neighborhoods with

$$\nu(U_n) \leq \frac{1}{n}, \quad \forall n.$$

Let  $Y$  be a countable dense subset of  $X$  and fix an enumeration  $(y_n)$  of the elements in the set  $Y$ . Let  $(q_n)$  be a sequence of positive real numbers with

$$\sum_n q_n < 1.$$

Since  $\nu$  has full support, we can, for every  $n$ , choose a neighborhood  $U_n$  of  $y_n$  with  $\nu$ -measure at most  $q_n$ . If we let

$$U = \bigcup_n U_n,$$

then, since  $Y$  is dense,  $U$  is open and dense, and thus  $B = U^c$  is closed with no interior and

$$\nu(B) = 1 - \mu(B) \geq 1 - \sum_n q_n > 0,$$

which finishes the proof. □

**Remark 3.2.** The assumption that  $X$  is separable is crucial. Indeed, let  $(Z, \eta)$  be any non-atomic probability measure space and let  $X$  denote the Gelfand spectrum of  $L^\infty(Z, \eta)$ . Then  $X$  is a compact (non-separable) hausdorff space and the corresponding probability measure  $\bar{\eta}$  on  $X$  (viewing  $\eta$  as a positive linear functional on  $L^\infty(Z, \eta)$ ) is a (non-atomic) *normal* measure (with full support), i.e. for every Borel set  $B \subset X$ , we have

$$\bar{\eta}(B^o) = \bar{\eta}(B).$$

In particular, a Borel subset of  $X$  has positive  $\nu$ -measure if and only if  $B$  has non-empty interior. See Chapter 1 in [11] for references.

Since  $\nu$  has full support and  $\mathbb{T}$  is separable, the previous lemma asserts that there is a proper closed subset  $A \subset \mathbb{T}$  with positive  $\nu$ -measure, but empty interior. By Lemma 3.17 there exists a  $\nu$ -conull Borel set  $X \subset \mathbb{T}$  such that the sets  $A_x \subset G$  are  $\mathcal{F}_\mu$ -large for all  $x \in X$ .

We claim that the difference set  $A_x A_x^{-1}$  is *not* left syndetic. Indeed, assume that  $FA_x A_x^{-1} = G$  for some  $x \in X$ . Then, for all  $g \in G$ , we have

$$FA_x \cap g \cdot A_x = (FA \cap g \cdot A)_x \neq \emptyset.$$

In particular, the set  $FA \cap g \cdot A \subset S^1$  is non-empty for all  $g \in G$ . However, this contradicts Corollary 3.16.

#### 4. PROOF OF PROPOSITION 1.2

Let  $(G, \mu)$  be a countable measured group. A set  $\mathcal{C} \subset \mathcal{L}_\mu$  is *left ergodic* if every extremal element in  $\mathcal{C}$  corresponds to an *ergodic*  $\mu$ -stationary regular Borel probability measure on the Stone-Ćech compactification  $\beta G$  (with respect to the induced left action) via the Gelfand map

$$\tau(\varphi)(g) = \varphi(g \cdot \bar{e}), \quad g \in G,$$

By Corollary 3.14, the set  $\mathcal{F}_\mu \subset \mathcal{L}_\mu$  of Furstenberg-Poisson means on  $G$  is left ergodic, so Proposition 1.2 follows immediately from the following lemma.

**Lemma 4.1.** *Suppose  $\mathcal{C} \subset \mathcal{M}(G)$  is left ergodic and  $A \subset G$  is  $\mathcal{C}$ -large. Then,*

$$\sup d_{\mathcal{C}}^*(FA) = 1,$$

where the supremum is taken over all finite subsets  $F \subset G$ .

*Proof.* By Corollary 2.2, there exists an extremal  $\lambda \in \mathcal{C}$  such that

$$d_{\mathcal{C}}^*(A) = \lambda'(A) = \bar{\lambda}(\bar{A}) > 0.$$

Since  $\mathcal{C}$  is left ergodic,  $\bar{\lambda}$  is ergodic for the induced left action of  $G$  on  $\beta G$ . In particular, by the  $\sigma$ -additivity of  $\bar{\lambda}$ , we have

$$1 = \bar{\lambda}(G\bar{A}) = \sup \bar{\lambda}(F\bar{A}),$$

where the supremum is taken over all finite subsets  $F \subset G$ . Since  $F\bar{A}$  is clopen in  $\beta G$  for every finite  $F$ , we conclude that

$$1 = \sup \bar{\lambda}(F\bar{A}) = \sup \lambda'(FA) \leq \sup d_{\mathcal{C}}^*(FA),$$

which finishes the proof. □

## 5. PROOF OF PROPOSITION 1.3

Recall that a set  $C \subset G$  is *left syndetic* (or left relatively dense) if there exists a finite set  $F \subset G$  such that  $FC = G$ . By Lemma 2.11, this is equivalent to positivity of the lower  $\mathcal{L}_\mu$ -density.

**Proposition 5.1.** *Suppose  $C \subset G$  is left syndetic. Then*

$$d_{*\mu}^{\mathcal{L}}(A^{-1}C) \geq d_{\mathcal{F}_\mu}^*(A)$$

for all  $A \subset G$ .

This proposition can be used to give a quick proof of Proposition 1.3.

*Proof of Proposition 1.3.* Let  $(G, \mu)$  be a countable measured group, and assume for contradiction that  $A, B \subset G$  satisfy

$$d_{\mathcal{F}_\mu}^*(A) + d_{\mathcal{L}_\mu}^*(B) > 1,$$

but the product set  $AB$  is not right thick. Then, by Proposition 2.12, the complementary set  $C := (AB)^c$  is syndetic, and since  $A^{-1}C \subset B^c$ , we have

$$d_{*\mu}^{\mathcal{L}}(A^{-1}C) \leq d_{*\mu}^{\mathcal{L}}(B^c) = 1 - d_{\mathcal{L}_\mu}^*(B) < d_{\mathcal{F}_\mu}^*(A).$$

However, this contradicts Proposition 5.1. □

**5.1. Proof of Proposition 5.1.** Recall that if  $A$  is a subset of a countable group  $G$ , we denote by  $\overline{A}$  the corresponding "closure" set in the Stone-Ćech compactification  $\beta G$ . If  $q \in \beta G$ , we define

$$\overline{A}_q := \left\{ g \in G : g \cdot q \in \overline{A} \right\} \subset G.$$

In particular,  $A = \overline{A}_e$ .

**Lemma 5.2.** *Let  $A \subset G$  and  $\eta \in \mathcal{L}_\mu$ . For every Borel set  $D \subset \beta G$ , the function*

$$\psi(q) := \overline{\eta}(\overline{A}_q^{-1}D), \quad q \in \beta G,$$

*is  $\mu$ -harmonic and equal to the supremum of an increasing sequence of continuous functions on  $\beta G$ .*

*Proof.* Fix an increasing exhaustion  $(F_n)$  of  $G$  by finite sets, and define

$$\psi_n(q) := \overline{\eta}((\overline{A}_q \cap F_n)^{-1}D), \quad q \in \beta G,$$

for all  $n$ . By  $\sigma$ -additivity,  $\psi = \sup_n \psi_n$ , so it suffices to show that  $\psi_n$  is continuous for every  $n$ .

Fix  $q \in \beta G$  and  $n$ . By Lemma 2.6, the set of  $p \in \beta G$  such that

$$\overline{A}_p \cap F_n = \overline{A}_q \cap F_n$$

and thus  $\psi_n(p) = \psi_n(q)$ , is clopen. This clearly implies that  $\psi_n$  is continuous.

To show that  $\psi$  is  $\mu$ -harmonic, we first note that

$$\overline{A}_{g \cdot q} = \overline{A}_q \cdot g^{-1}, \quad \forall g \in G,$$

and thus,

$$\psi(g^{-1} \cdot q) = \overline{\eta}(g^{-1} \cdot \overline{A}_q^{-1}D) = \int_{\beta G} \chi_{\overline{A}_q^{-1}D}(g \cdot p) d\overline{\eta}(p).$$

Since  $\mu * \overline{\eta} = \overline{\eta}$ , we have  $\mu * \psi = \psi$  (remember that  $\mu$  is symmetric), which finishes the proof.  $\square$

**Lemma 5.3.** *Suppose  $A \subset G$ . If  $C \subset G$  is left syndetic, then*

$$\sup_{\eta \in \mathcal{F}_\mu} \int_{\beta G} \overline{\eta}(\overline{C}_q^{-1}\overline{A}) d\overline{\lambda}(q) \geq d_{\mathcal{F}_\mu}^*(A),$$

*for all  $\lambda \in \mathcal{L}_\mu$ .*

*Proof.* Fix  $A, C \subset G$ , with  $C$  left syndetic, and  $\lambda$  and  $\eta$  in  $\mathcal{L}_\mu$ . By Lemma 5.2, the function

$$\psi(q) := \overline{\eta}(\overline{C}_q^{-1}\overline{A}), \quad q \in \beta G,$$

is a lower semicontinuous  $\mu$ -harmonic function on  $\beta G$ . By Lemma 3.6, the set  $Z$  of minima for  $\psi$  is a non-empty, closed and  $G$ -invariant set, and since  $C$  is left syndetic,  $\overline{C}_q$  is non-empty for all  $q \in \beta G$  by Lemma 2.11.

In particular, there exists  $q_o \in Z$  such that  $e \in \overline{C}_{q_o}$ . Hence,

$$\overline{\eta}(\overline{A}) \leq \overline{\eta}(\overline{C}_{q_o}^{-1}\overline{A}) \leq \int_{\beta G} \overline{\eta}(\overline{C}_q^{-1}\overline{A}) d\overline{\lambda}(q).$$

Since  $\lambda, \eta \in \mathcal{L}_\mu$  are arbitrary, we are done.  $\square$

The following proposition is the main result of this subsection. Combined with Lemma 5.3 above, it immediately implies Proposition 5.4.

**Proposition 5.4.** *Suppose  $A, C \subset G$ . Then there exists an extremal  $\lambda \in \mathcal{L}_\mu$  such that*

$$d_*^{\mathcal{L}_\mu}(A^{-1}C) \geq \sup_{\eta \in \mathcal{F}_\mu} \int_{\beta G} \bar{\eta}(\overline{C}_q^{-1}\overline{A}) d\bar{\lambda}(q).$$

## 5.2. Proof of Proposition 5.4.

**Lemma 5.5.** *Suppose  $\psi$  is a bounded  $\mu$ -harmonic function on  $\beta G$ , equal to the supremum of an increasing sequence of continuous functions. Then,*

$$\psi(\bar{e}) \geq \sup_{\eta \in \mathcal{F}_\mu} \int_{\beta G} \psi d\bar{\eta}.$$

*Proof.* Note that the function

$$\varphi(g) := \psi(g \cdot \bar{e}), \quad g \in G,$$

belongs to  $\mathcal{H}_t^\infty(G, \mu)$ , and thus

$$\eta(\varphi) = \varphi(e) = \psi(\bar{e}),$$

for all  $\eta \in \mathcal{F}_\mu$ . By Lemma 2.3,

$$\eta(\varphi) \geq \int_{\beta G} \psi(q) d\bar{\eta}(q),$$

for all  $\eta \in \mathcal{F}_\mu$ , which finishes the proof.  $\square$

*Proof of Proposition 5.4.* Fix  $A, C \subset G$ . By Corollary 2.2, there exists an extremal  $\lambda \in \mathcal{L}_\mu$  such that

$$d_*^{\mathcal{L}_\mu}(A^{-1}C) = \lambda'(A^{-1}C) = \lambda'((A^{-1}\overline{C})_{\bar{e}}).$$

Since  $A^{-1}\overline{C}$  is a countable union of clopen sets in  $\beta G$ , Lemma 2.4 applies and

$$\lambda'((A^{-1}\overline{C})_{\bar{e}}) \geq \bar{\lambda}(A^{-1}\overline{C}).$$

Define the function

$$\psi(q) := \bar{\lambda}(\overline{A}_q^{-1}\overline{C}), \quad q \in \beta G.$$

By Lemma 5.2,  $\psi$  is  $\mu$ -harmonic and equal to the supremum of an increasing sequence of continuous functions. Hence, Lemma 5.5 applies, and

$$\psi(e) = \bar{\lambda}(\overline{A}_{\bar{e}}^{-1}\overline{C}) \geq \int_{\beta G} \bar{\lambda}(\overline{A}_q^{-1}\overline{C}) d\bar{\eta}(q),$$

for all  $\eta \in \mathcal{F}_\mu$ . By Fubini's Theorem, we have

$$\int_{\beta G} \bar{\lambda}(\overline{A}_q^{-1}\overline{C}) d\bar{\eta}(q) = \int_{\beta G} \int_{\beta G} \chi_{G(\overline{A} \times \overline{C})} d\bar{\lambda} \otimes \bar{\eta} = \int_{\beta G} \bar{\eta}(\overline{C}_q^{-1}\overline{A}) d\bar{\lambda}(q),$$

which finishes the proof.  $\square$

## 6. PROOF OF PROPOSITION 1.4

The proof of Proposition 1.4 will be done in several steps. Some of these steps are quite standard and well-known to experts, at least for amenable groups, but we still give the proofs for completeness.

**6.1. Reductions.** We shall try to re-cast Proposition 1.4 in a more  $C^*$ -algebraic language. This will simplify, and hopefully conceptualize, the approach.

Recall that if  $\mathcal{B}$  is a left  $G$ -invariant unital sub- $C^*$ -algebra of  $\ell^\infty(G)$ , then the Gelfand space  $\Delta(\mathcal{B})$  of  $\mathcal{B}$  is a compact hausdorff space, equipped with an action of  $G$  by homeomorphisms, and admitting a  $G$ -transitive point  $x_o$ . Furthermore, there is a left  $G$ -equivariant  $*$ -isomorphism

$$\tau_{\mathcal{B}} : C(\Delta(\mathcal{B})) \rightarrow \mathcal{B}$$

given by

$$\tau_{\mathcal{B}}(f)(g) := f(g \cdot x_o), \quad g \in G.$$

The composition of  $\tau_{\mathcal{B}}$  and the inclusion map of  $\mathcal{B}$  into  $\ell^\infty(G)$  gives rise to an left  $G$ -equivariant injective  $*$ -morphism and thus, by the Gelfand-Naimark correspondence, to a left  $G$ -equivariant surjection

$$\pi_{\mathcal{B}} : \beta G \rightarrow \Delta(\mathcal{B}),$$

where  $\beta G$  denotes the Stone-Ćech compactification of  $G$ . Recall that the set  $\mathcal{M}(G)$  of means on  $G$  (or equivalently, of states on  $\ell^\infty(G)$ ) corresponds to the set  $\mathcal{P}(\beta G)$  of regular Borel probability measures on  $\beta G$ . Note that the map  $\pi_{\mathcal{B}}$  also induces a continuous affine map

$$(\pi_{\mathcal{B}})_* : \mathcal{P}(\beta G) \rightarrow \mathcal{P}(\Delta(\mathcal{B})).$$

We denote the set of all positive elements in  $\mathcal{B}$  (i.e. the set of all non-negative functions on  $G$  which belong to  $\mathcal{B}$ ) by  $\mathcal{B}_+$ . Note that the transpose map

$$\pi_{\mathcal{B}}^* : C(\Delta(\mathcal{B})) \rightarrow C(\beta G)$$

is positive, i.e.  $\pi_{\mathcal{B}}^*\varphi$  is positive whenever  $\varphi$  belongs to  $\mathcal{B}_+$ .

**Proposition 6.1.** *Let  $\mathcal{B} \subset \ell^\infty(G)$  be a unital sub- $C^*$ -algebra and suppose  $C \subset G$ . For every  $\lambda \in \mathcal{P}(\beta G)$ , there exists a  $\lambda$ -measurable set  $C' \subset \Delta(\mathcal{B})$  with*

$$\overline{C} \subset \pi_{\mathcal{B}}^{-1}(C'),$$

modulo  $\lambda$ -null sets, such that whenever  $\varphi \in \mathcal{B}_+$  satisfies

$$\langle \pi_{\mathcal{B}}^*\varphi, \chi_{\overline{C}} \rangle_{L^2(\beta G, \lambda)} = 0,$$

then

$$\int_{C'} \varphi d(\pi_{\mathcal{B}})_*\lambda = 0.$$

*Proof.* Fix  $\lambda \in \mathcal{P}(\beta G)$  and consider the indicator function  $\chi_{\overline{C}}$  as an element in  $L^2(\beta G, \lambda)$ . Note that

$$V := L^2(\Delta(\mathcal{B}), (\pi_{\mathcal{B}})_*\lambda)^{\pi_{\mathcal{B}}} \subset L^2(\beta G, \lambda)$$

is a closed subspace, consisting of those elements in  $L^2(\beta G, \lambda)$  which are pull-backs of square-integrable elements on  $\Delta(\mathcal{B})$  under the map  $\pi_{\mathcal{B}}^*$ , so we can write

$$\chi_{\overline{C}} = \rho_{\mathcal{B}} + \rho_o$$

where  $\rho_{\mathcal{B}}$  denotes the orthogonal projection of  $\chi_{\overline{C}}$  onto  $V$ . Since this projection is positive, the function  $\rho_{\mathcal{B}}$  is non-negative and

$$\langle (\pi_{\mathcal{B}})^*\varphi, \chi_{\overline{C}} \rangle_{\lambda} = \langle \varphi, \rho_{\mathcal{B}} \rangle_{(\pi_{\mathcal{B}})_*\lambda} = 0.$$

Choose a  $\lambda$ -measurable representative of  $\rho_{\mathcal{B}}$  and define

$$C' := \left\{ x \in \Delta(\mathcal{B}) : \rho_{\mathcal{B}}(x) > 0 \right\} \subset \Delta(\mathcal{B}).$$

Clearly, the inclusion  $\overline{C} \subset \pi_{\mathcal{B}}^{-1}(C')$  holds modulo  $\lambda$ -null sets in  $\beta G$  and since  $\varphi$  is non-negative, we have

$$\int_{C'} \varphi d(\pi_{\mathcal{B}})_* \bar{\lambda} = 0,$$

which finishes the proof.  $\square$

**6.1.1. Left amenable sub-C\*-algebras.** We shall say that a left  $G$ -invariant sub-C\*-algebra  $\mathcal{B}$  of  $\ell^\infty(G)$  is *left amenable* if there exists a left  $G$ -invariant state on  $\mathcal{B}$ . Via the Gelfand-Naimark's correspondence, this is equivalent to saying that there exists a  $G$ -invariant regular Borel probability measure on the Gelfand space  $\Delta(\mathcal{B})$  of  $\mathcal{B}$ .

For an amenable sub-C\*-algebra  $\mathcal{B}$  of  $\ell^\infty(G)$ , we let  $\mathcal{L}_{\mathcal{B}}$  denote the (non-empty) weak\*-compact and convex set of all states on  $\ell^\infty(G)$  which restricts to a left invariant state on the algebra  $\mathcal{B}$ .

Let  $B \subset G$  and let  $\mathcal{B}$  denote the *Bebutov algebra* of  $B$ , i.e. the smallest left  $G$ -invariant unital sub-C\*-algebra of  $\ell^\infty(G)$  which contains the indicator function of  $B$ . In the introduction, we called a set  $B$  *strongly non-paradoxical* if the Bebutov algebra of  $B$  is amenable and there exists  $\lambda$  in  $\mathcal{L}_{\mathcal{B}}$  which gives positive measure to  $B$ .

Since  $\chi_B$  is idempotent in  $\mathcal{B}$ , there exists a clopen subset  $B' \subset \Delta(\mathcal{B})$  such that

$$\tau_{\mathcal{B}}(\chi_{B'}) = \chi_B.$$

Recall from Subsection 2.3 that we also introduced the notation  $\overline{B}$  for the clopen set which  $B$  corresponds to via the isomorphism between  $\ell^\infty(G)$  and  $C(\beta G)$ . From the discussions above, it should be clear that

$$\overline{B} = \pi_{\mathcal{B}}^{-1}(B').$$

Recall that the *non-paradoxical density* of  $B$  was defined earlier by

$$d_{np}^*(B) := d_{\mathcal{L}_{\mathcal{B}}}^*(B).$$

By Corollary 2.2, there always exists a state  $\lambda \in \mathcal{L}_{\mathcal{B}}$  which maximizes the right hand side. Let  $\bar{\lambda}$  denote the corresponding regular Borel probability measure on  $\beta G$  and set

$$\eta' := (\pi_{\mathcal{B}})_* \bar{\lambda}.$$

Note that  $\eta'$  is a  $G$ -invariant Borel probability measure on  $\Delta(\mathcal{B})$ . It is not hard to see that  $\eta'$  must be ergodic (since  $\lambda$  is extremal) and

$$d_{np}^*(B) = \eta'(B') = \sup_{\nu} \nu(B'),$$

where the supremum is taken over all  $G$ -invariant probability measures on  $\Delta(\mathcal{B})$ .

**Corollary 6.2.** *Let  $A, B \subset G$  and suppose  $B$  is strongly non-paradoxical. For every admissible  $\mu \in \mathcal{P}(G)$ , there exist  $\lambda \in \mathcal{P}_{\mu}(\beta G)$  such that the measure  $\eta' := (\pi_{\mathcal{B}})_* \lambda$  on  $\Delta(\mathcal{B})$  is  $G$ -invariant and ergodic with*

$$d_{np}^*(B) = \eta'(B')$$

and a Borel set  $C' \subset \Delta(\mathcal{B})$  with

$$\overline{AB} \supset \pi_{\mathcal{B}}^{-1}(C')^c$$

modulo  $\lambda$ -null sets in  $\beta G$ , such that

$$\eta'(a \cdot B' \cap C') = 0, \quad \forall a \in A.$$

*Proof.* From the discussion above, we know that there exists an ergodic and  $G$ -invariant regular Borel probability measure  $\eta'$  on  $\Delta(\mathcal{B}_B)$  with

$$d_{np}^*(B) = \eta'(B').$$

Fix  $\mu \in \mathcal{P}(G)$ . Since  $\eta'$  clearly belongs to  $\mathcal{P}_\mu(\Delta(\mathcal{B}))$ , Corollary 3.2 guarantees the existence of  $\lambda \in \mathcal{P}_\mu(\beta G)$  such that

$$(\pi_B)_*\lambda = \eta'.$$

Since  $\mathcal{B}_B$  is left  $G$ -invariant, the closed subspace

$$V := L^2(\Delta(\mathcal{B}), (\pi_B)_*\lambda)^{\pi_B} \subset L^2(\beta G, \lambda)$$

is  $G$ -invariant as well, so the function  $\pi_B^*\varphi_g$  belongs to  $V$  for all  $g \in G$ , where

$$\varphi_g := \chi_{B'}(g^{-1}\cdot).$$

Let  $C := (AB)^c$  and note that

$$\lambda(a \cdot \overline{B} \cap \overline{C}) = \langle \pi_B^*\varphi_a, \chi_{\overline{C}} \rangle_{L^2(\beta G, \lambda)} = 0,$$

for all  $a \in A$ . By Proposition 6.1, there exists a  $\lambda$ -measurable set  $C' \subset \Delta(\mathcal{B}_B)$  with

$$\overline{C} \subset \pi_B^{-1}(C')$$

modulo  $\lambda$ -null sets in  $\beta G$  such that

$$\int_{C'} \varphi_a d\eta' = \eta'(a \cdot B' \cap C') = 0$$

for all  $a \in A$ . □

Recall from Section 2 that the Fourier-Stiltjes algebra  $B(G)$  of a countable group  $G$  is defined as the  $*$ -algebra generated by all matrix coefficients of unitary representations of  $G$ . The uniform closure of  $B(G)$  in  $\ell^\infty(G)$  will here be referred to as the *Eberlein algebra* of  $G$  and we shall denote its (compact) Gelfand spectrum by  $eG$ . Let  $\lambda_o$  denote the unique left and right invariant state on  $E(G)$  and write

$$E(G) = AP(G) \oplus E_o(G),$$

where

$$E_o(G) := \left\{ \varphi \in E(G) : \lambda_o(|\varphi|^2) = 0 \right\}.$$

For a given  $\varphi \in E(G)$ , we shall write

$$\varphi = \varphi_{ap} + \varphi_o,$$

with  $\varphi_{ap} \in AP(G)$ .

Here,  $AP(G)$  denotes the  $C^*$ -algebra of almost periodic functions on  $G$ . Its Gelfand spectrum  $bG$  we shall refer to as the *Bohr compactification* of  $G$ . As was mentioned in Section 2, this compact hausdorff space can be endowed with a jointly continuous multiplication which gives  $bG$  the structure of a compact hausdorff group. In particular,  $bG$  admits a Haar probability measure  $m$ .

Because of the decomposition above, we see that

$$L^2(eG, \bar{\lambda}_o) \simeq L^2(bG, m)^\pi,$$

where  $\pi : eG \rightarrow bG$  is the  $G$ -equivariant map implementing the inclusion of  $\text{AP}(G)$  in  $\text{E}(G)$ . Indeed, the function  $\varphi_o$  above is equivalent to the null function in  $L^2(eG, \bar{\lambda}_o)$ .

Furthermore, since  $m$  is the unique  $G$ -invariant measure on  $bG$ , we have

$$\pi_* \bar{\lambda}_o = m.$$

In particular, if  $A \subset eG$  is a Borel set and  $\varphi \in C(eG)$  is non-negative, with the property that

$$\int_A \varphi d\bar{\lambda}_o = 0,$$

then there exists a Borel set  $A' \subset bG$  with  $A \subset \pi^{-1}(A')$  such that

$$\int_{A'} \varphi_{ap} dm = 0,$$

In particular, there exists a Borel set  $A'' \subset A'$  with  $m(A'') = m(A')$  on which  $\varphi_{ap}$  vanishes identically.

Recall that  $\mathcal{FS}$  denotes the weak\*-compact and convex set of means on  $G$  which restricts to  $\lambda_o$  on  $\text{B}(G)$  (and thus on  $\text{E}(G)$ ). The following corollary is now an almost immediate consequence of the discussion above.

**Corollary 6.3.** *Let  $A \subset G$  be  $\mathcal{FS}$ -large and suppose  $\varphi \in \text{B}(G)_+$  vanishes on  $A$ . Then there exists a Borel set  $A' \subset bG$  with*

$$m(A') \geq d_{\mathcal{FS}}^*(A)$$

*such that  $\varphi_{ap}$  vanishes on  $A'$ .*

*Proof.* By Corollary 2.2, there exists  $\lambda \in \mathcal{FS}$  such that

$$d_{\mathcal{FS}}^*(A) = \lambda'(A).$$

Let  $\bar{\varphi}$  denote the continuous function on  $\beta G$  corresponding to  $\varphi$ . Note that  $\bar{\varphi}$  is non-negative. Since  $\varphi \in \text{E}(G)$ , we can write

$$\bar{\varphi} = \pi_{\text{E}(G)}^* \varphi'$$

for some  $\varphi' \in C(eG)$ , which is again non-negative. By assumption,

$$\langle \pi_{\text{E}(G)}^* \varphi', \chi_{\bar{A}} \rangle_{L^2(\beta G, \bar{\lambda})} = 0,$$

and  $\bar{\lambda}_o = (\pi_{\text{E}(G)})_* \bar{\lambda}$ . Hence, by Proposition (6.1), there exists a measurable set  $A' \subset eG$  with

$$\bar{\lambda}_o(A') \geq \bar{\lambda}(\bar{A}) = d_{\mathcal{FS}}^*(A)$$

such that

$$\int_{A'} \varphi' d\bar{\lambda}_o = 0.$$

The corollary now follows from the discussion above.  $\square$

**6.2. Compact vectors and the Eberlein decomposition.** The aim of this subsection is to explicate the decomposition 2.2 for elements in  $B(G)$ . We stress that the material covered here is quite standard (at least for abelian, or more generally amenable, groups), and is certainly well-known to experts. We collect here a few basic results for completeness.

The basic setting can be described as follows. Let  $(\mathcal{H}, \rho)$  be a unitary representation on a Hilbert space  $\mathcal{H}$  (not necessarily separable). Let  $\mathcal{H}^\rho$  denote the closed subspace of  $\rho$ -fixed vectors and write  $P_\rho$  for the corresponding orthogonal projection.

Given  $x, y \in \mathcal{H}$ , we define the *matrix coefficient at  $(x, y)$*  by

$$\varphi_{x,y}^\rho(g) = \langle x, \rho(g)y \rangle, \quad g \in G.$$

Recall that  $\lambda_o$  denotes the unique left and right invariant mean on  $B(G)$ .

The following extension of the (weak) von Neumann's Ergodic Theorem will be quite useful.

**Lemma 6.4.** *For all  $x, y \in \mathcal{H}$ , we have*

$$\lambda_o(\varphi_{x,y}^\rho) = \langle P_\rho x, P_\rho y \rangle.$$

*Proof.* It suffices to establish the identity on a dense subspace of  $\mathcal{H}$ . Note that the identity is trivial if either  $x$  or  $y$  belongs to  $\mathcal{H}^\rho$ , and if  $x$  (or  $y$ ) is of the form

$$x = z - \rho(g_o)^{-1}z$$

for some  $z \in \mathcal{H}$  and  $g_o \in G$ , then

$$\varphi_{x,y}^\rho(g) = \langle z, \rho(g)y \rangle_{\mathcal{H}} - \langle z, \rho(g_o \cdot g)y \rangle_{\mathcal{H}}$$

for all  $g \in G$ . Since  $\lambda_o$  is left-invariant, we see that in this case,

$$\lambda_o(\varphi_{x,y}^\rho) = 0.$$

Hence it suffices to show that if  $x \in \mathcal{H}$  and

$$\langle x, z - \rho(g)^{-1}z \rangle_{\mathcal{H}} = 0$$

for all  $z \in \mathcal{H}$  and  $g \in G$ , then  $x \in \mathcal{H}^\rho$ . Equivalently, for all  $g \in G$  and  $z \in \mathcal{H}$ ,

$$\langle x - \rho(g)x, z \rangle_{\mathcal{H}} = 0,$$

which clearly implies that  $\rho(g)x = x$  for all  $g \in G$  and thus  $x \in \mathcal{H}^\rho$ . □

**Remark 6.1.** Note that the proof does not utilize the uniqueness of  $\lambda_o$ . In fact, this uniqueness is a consequence of the proof.

Recall that a vector  $x \in \mathcal{H}$  is *compact* (with respect to  $\rho$ ) if the orbit  $\rho(G)x \subset \mathcal{H}$  is pre-compact. Alternatively, the closure of the cyclic span of  $x$  can be decomposed into a direct sum of finite-dimensional representations (by the Peter-Weyl Theorem). We denote the *closed* subspace of  $\rho$ -compact vectors by  $\mathcal{H}_c$ .

The following lemma provides a convenient way of producing compact vectors for a general representation.

**Lemma 6.5.** *Suppose  $z \in (\mathcal{H} \otimes \mathcal{H})^{\rho \otimes \rho}$ . Then the operator  $K_z : \mathcal{H} \rightarrow \mathcal{H}$  defined by*

$$\langle K_z x, y \rangle = \langle z, x \otimes y \rangle$$

*is compact and  $\rho$ -intertwining. In particular,  $\text{Im}(K_z) \subset \mathcal{H}_c$ .*

*Proof.* Since  $\mathcal{H} \otimes \mathcal{H}$  is the Hilbert space completion of the algebraic tensor product, there exists a net  $(z_\alpha)$  of finite sums of basic tensors such that  $z_\alpha \rightarrow z$  in the Hilbert tensor norm. In particular,

$$\begin{aligned} \sup_{\|x\|_{\mathcal{H}}=1} \|K_{z_\alpha} x - K_z x\|_{\mathcal{H}} &= \sup_{\|x\|_{\mathcal{H}}=1} \left( \sup_{\|y\|_{\mathcal{H}}=1} |\langle z_\alpha - z, x \otimes y \rangle_{\mathcal{H} \otimes \mathcal{H}}| \right) \\ &\leq \sup_{\|x\|_{\mathcal{H}}=1} \left( \sup_{\|y\|_{\mathcal{H}}=1} \|z_\alpha - z\|_{\mathcal{H} \otimes \mathcal{H}} \cdot \|x \otimes y\|_{\mathcal{H} \otimes \mathcal{H}} \right) \\ &= \|z_\alpha - z\|_{\mathcal{H} \otimes \mathcal{H}} \rightarrow 0. \end{aligned}$$

Note that if

$$z_\alpha = \sum_{i \in F_\alpha} x_{\alpha,i} \otimes y_{\alpha,i}$$

for some finite set  $F_\alpha$  and  $x_{\alpha,i}, y_{\alpha,i} \in \mathcal{H}$ , then

$$K_{z_\alpha} x = \sum_{i \in F_\alpha} \langle x_{\alpha,i}, x \rangle_{\mathcal{H}} \cdot y_{\alpha,i},$$

so in particular  $K_{z_\alpha}$  has finite rank for all  $\alpha$ . Hence  $K_{z_\alpha}$  is compact for every  $\alpha$  and  $K_z$  is the norm limit of a net of such operators, and thus compact. Finally, if  $z \in (\mathcal{H} \otimes \mathcal{H})^{\rho \otimes \rho}$ , then

$$\begin{aligned} \langle K_z \rho(g)x, y \rangle_{\mathcal{H}} &= \langle z, \rho(g)x \otimes y \rangle_{\mathcal{H} \otimes \mathcal{H}} \\ &= \langle (\rho \otimes \rho)(g)z, \rho(g)x \otimes y \rangle_{\mathcal{H} \otimes \mathcal{H}} \\ &= \langle z, x \otimes \rho(g)^{-1}y \rangle_{\mathcal{H} \otimes \mathcal{H}} \\ &= \langle K_z x, \rho(g)^{-1}y \rangle_{\mathcal{H}} \\ &= \langle \rho(g)K_z x, y \rangle_{\mathcal{H}} \end{aligned}$$

for all  $g \in G$  and  $x, y \in \mathcal{H}$ . We conclude that  $K_z : \mathcal{H} \rightarrow \mathcal{H}$  is  $\rho$ -intertwining, which finishes the proof.  $\square$

We can now prove an explicit form of the decomposition 2.2 for elements in  $B(G)$ .

**Proposition 6.6.** *Suppose  $(\rho, \mathcal{H})$  is a unitary representation of  $G$ . If  $v, w \in \mathcal{H}$  and*

$$\varphi(g) = \langle \rho(g)v, w \rangle_{\mathcal{H}}, \quad g \in G,$$

*then  $\varphi \in B(G)$  and*

$$\varphi_{ap}(g) = \langle \rho(g)P_c v, P_c w \rangle_{\mathcal{H}}, \quad g \in G,$$

*where  $P_c$  denotes the orthogonal projection onto  $\mathcal{H}_c$ .*

*Proof.* We wish to show that if  $x, y \in \mathcal{H}$ , then the function defined by

$$\phi_o(g) := \langle x, \rho(g)y \rangle_{\mathcal{H}} - \langle P_c(x), \rho(g)P_c(y) \rangle_{\mathcal{H}}$$

belongs to  $B_o(G)$ . Since the function clearly belongs to  $B(G)$ , this amounts to show that

$$\lambda_o(|\phi_o|^2) = 0.$$

Define

$$x_o := x - P_c x \quad \text{and} \quad y_o := y - P_c y,$$

and note that

$$|\phi_o(g)|^2 = \langle x_o \otimes x_o, (\rho \otimes \rho)(g)y_o \otimes y_o \rangle_{\mathcal{H} \otimes \mathcal{H}}.$$

By Lemma 6.4, we have

$$\lambda_o(|\phi_o|^2) = \langle P_{\rho \otimes \rho} x_o \otimes x_o, P_{\rho \otimes \rho} y_o \otimes y_o \rangle.$$

Let  $z := P_{\rho \otimes \rho}(x_o \otimes x_o)$  and note that we can now write

$$\lambda_o(|\phi_o|^2) = \langle K_z y_o, y_o \rangle,$$

where the operator  $K_z : \mathcal{H} \rightarrow \mathcal{H}$  is defined as in Lemma 6.5. By the same lemma,  $K_z y_o \in \mathcal{H}_c$ , and since  $y_o$  is orthogonal to  $\mathcal{H}_c$ , we conclude that

$$\lambda_o(|\phi_o|^2) = \langle K_z y_o, y_o \rangle = 0,$$

which finishes the proof.  $\square$

**6.3. Applications to product sets.** Let us now connect all this to our previous discussions about product sets.

**6.3.1. Koopman representations.** Let  $X$  be a compact  $G$ -space and suppose  $\nu$  is a  $G$ -invariant and ergodic Borel probability measure on  $X$ . The  $G$ -action on  $X$  gives rise to a unitary representation  $\rho$  (the so called *Koopman representation*) of  $G$  on  $L^2(X, \nu)$  by

$$\rho(g)f = f(g^{-1}\cdot), \quad g \in G.$$

If  $B, C \subset X$  are  $\nu$ -measurable sets, then the function

$$\varphi(g) := \nu(g \cdot B \cap C) = \langle \rho(g)\chi_B, \chi_C \rangle_{L^2(X, \nu)}, \quad g \in G,$$

is an element in  $B(G)$  (and hence in  $E(G)$ ) and by Proposition we have

$$\varphi_{ap}(g) = \langle \rho(g)P_c \chi_B, P_c \chi_C \rangle_{L^2(X, \nu)},$$

where  $P_c$  denotes the projection onto the  $\rho$ -compact vectors in  $L^2(X, \nu)$ .

**6.3.2. Kronecker-Mackey factors.** We shall need the following classical result of Mackey.

**Proposition 6.7** ([20]). *There exists a  $\nu$ -conull set  $X' \subset X$ , a compactification  $(K, \iota_K)$  of  $G$ , a closed subgroup  $K_o < K$  and a  $G$ -equivariant measurable map  $\pi : X' \rightarrow K/K_o$  such that*

$$L^2(X, \mu)_c = L^2(K/K_o, m_o)^\pi,$$

where  $m_o$  denotes the Haar probability measure on  $K/K_o$ .

The compact homogeneous space  $K/K_o$  is often referred to as the *Kronecker* (or *Kronecker-Mackey*) factor of  $(X, \nu)$ .

We shall use Mackey's result in the following way. Let  $\psi_B$  and  $\psi_C$  denote  $m_o$ -measurable pointwise realizations of  $P_c \chi_B$  and  $P_c \chi_C$  respectively, and define

$$\tilde{B} := \left\{ k \in K/K_o : \psi_B(k) > 0 \right\} \quad \text{and} \quad \tilde{C} := \left\{ k \in K/K_o : \psi_C(k) > 0 \right\}.$$

Now assume that there is a set  $A \subset G$  on which  $\varphi_{ap}$  vanishes, i.e.

$$\int_{K/K_o} \psi_B(\iota_K(a)^{-1} \cdot k) \psi_C(k) dm_o(k) = 0, \quad \forall a \in A.$$

Since  $\psi_B$  and  $\psi_C$  are both non-negative functions, we must also have

$$m_o(\iota_K(a) \cdot \tilde{B} \cap \tilde{C}) = 0, \quad \forall a \in A.$$

Note that the function

$$\psi(g) := m(\iota_K(g) \tilde{B} \cap \tilde{C}), \quad g \in G,$$

is almost periodic.

6.3.3. *Connections to  $bG$ .* By the universal property of the Bohr compactification  $(bG, \iota_o)$ , there exists a continuous surjective homomorphism

$$\iota' : bG \rightarrow K$$

such that  $\iota_K = \iota' \circ \iota_o$ . Let  $p : K \rightarrow K/K_o$  denote the canonical quotient map and set  $p' := p \circ \iota'$ . Then  $p' : bG \rightarrow K/K_o$  is a continuous and  $G$ -equivariant surjection, so in particular  $p'_* m = m_o$ , and the identity above can be lifted to the identity

$$m(\iota_o(a) \cdot p'^{-1}(\tilde{B}) \cap p'^{-1}(\tilde{C})) = 0, \quad \forall a \in A.$$

Now suppose that  $A$  is a  $\mathcal{FS}$ -large set. Since the function

$$\varphi'(g) := m(\iota_o(g) \cdot p'^{-1}(\tilde{B}) \cap p'^{-1}(\tilde{C}))$$

is almost periodic function on  $G$ , Corollary 6.3 implies that there exists a Borel set  $A_o \subset bG$  with

$$m(A_o) \geq d_{\mathcal{FS}}^*(A)$$

such that  $\varphi'$  vanishes on  $A_o$ , so in particular,

$$\int_{bG} \int_{bG} \chi_{A_o}(a) \chi_{B_o}(a^{-1} \cdot k) \chi_{C_o}(k) dm(k) dm(a),$$

where  $B_o$  and  $C_o$  denote the sets  $p'^{-1}(\tilde{B})$  and  $p'^{-1}(\tilde{C})$  respectively. Note that

$$m(B_o) = m_o(\tilde{B}) \quad \text{and} \quad m(C_o) = m_o(\tilde{C}).$$

Upon changing the order of integration, we get

$$\int_{C_o} m(A_o \cap k \cdot B_o^{-1}) dm(k) = 0,$$

and thus  $C_o \subset U^c$ , modulo  $m$ -null sets in  $bG$ , where

$$U := \left\{ k \in bG : m(A_o \cap k \cdot B_o^{-1}) > 0 \right\}.$$

6.3.4. *Almost everywhere inclusions.* Let us summarize the discussion so far. Given a  $\mathcal{FS}$ -large set  $A \subset G$  and  $\nu$ -measurable sets  $B, C \subset X$ , we have constructed a  $m_o$ -measurable set  $\tilde{C} \subset K/K_o$  such that

$$C \subset \pi^{-1}(\tilde{C})$$

modulo  $\nu$ -null sets in  $X$  and an open set  $U \subset bG$  (of the explicit form above) such that

$$p'^{-1}(\tilde{C}) \subset U^c$$

modulo  $m$ -null sets in  $bG$ .

If  $Y$  is a  $G$ -space and  $C$  is a subset of  $Y$ , we define

$$C_y := \left\{ g \in G : g \cdot y \in C \right\}, \quad y \in Y.$$

The following lemma will be useful.

**Lemma 6.8.** *Let  $Y$  be a compact  $G$ -space and suppose  $\nu$  is a non-singular probability measure on  $Y$ . If  $A, B \subset Y$  are  $\nu$ -measurable sets with  $\nu(A \setminus B) = 0$ , then there exists a  $\nu$ -conull subset  $Y' \subset Y$  such that*

$$A_y \subset B_y, \quad \forall y \in Y'.$$

*Proof.* It suffices to prove that if  $N \subset Y$  is a  $\nu$ -null set, then  $N_y$  is empty for a conull set of  $y$  in  $Y$ . Note that

$$\left\{ y \in Y : N_y \neq \emptyset \right\} = G \cdot N.$$

Since  $G$  is countable and  $\nu$  is non-singular, we conclude that  $\nu(G \cdot N) = 0$ , which finishes the proof.  $\square$

The Bohr compactification  $(bG, \iota_o)$  can be viewed as a  $G$ -space in two ways (using the left or right action). In this paper, we shall use the left action, so if  $g \in G$ , we write

$$g \cdot k := \iota_o(g) \cdot k, \quad k \in bG,$$

so that if  $V \subset bG$  is any subset, then

$$V_k = \left\{ g \in G : g \cdot k \in V \right\} = \iota_o^{-1}(V \cdot k^{-1}).$$

The lemma above implies that there exists a  $\nu$ -conull subset  $X'' \subset X'$  such that

$$C_x \subset \pi^{-1}(\tilde{C})_x = \tilde{C}_{\pi(x)}, \quad \forall x \in X''$$

and a  $m$ -conull set  $Z \subset bG$  such that

$$\tilde{C}_{p'(k)} = p'^{-1}(\tilde{C})_k \subset U_k^c = \iota_o^{-1}(U^c \cdot k^{-1}), \quad \forall k \in Z.$$

In particular, for  $\nu$ -almost every  $x \in X$ , there exists  $k \in bG$  such that

$$C_x \subset \iota_o^{-1}(U^c \cdot k).$$

We summarize this observation in the following proposition.

**Proposition 6.9.** *Suppose  $A \subset G$  is  $\mathcal{FS}$ -large and  $B, C \subset X$  are  $\nu$ -measurable sets such that the intersection  $AB \cap C$  is a  $\nu$ -null set. Then there exist  $m$ -mesasurable sets  $A_o, B_o \subset bG$  with*

$$m(A_o) \geq d_{\mathcal{FS}}^*(A) \quad \text{and} \quad m(B_o) \geq \nu(B)$$

and a  $\nu$ -conull set  $X'' \subset X$  with the property that for all  $x \in X''$ , there exists  $k \in bG$  such that

$$C_x \subset \iota_o^{-1}(U^c \cdot k),$$

where

$$U := \left\{ k \in bG : m(A_o \cap k \cdot B_o^{-1}) > 0 \right\}.$$

**6.4. Proof of Proposition 1.4.** Let us first briefly recall the setting of Proposition 1.4. Let  $G$  be a countable group and fix an admissible probability measure  $\mu$  on  $G$  once and for all. Let  $A \subset G$  be a Fourier-Stiltjes large set and let  $B \subset G$  a strongly non-paradoxical set. Denote by  $\mathcal{B}_B$  the Bebutov algebra of  $B$  and by  $X$  the Gelfand spectrum of  $\mathcal{B}_B$ . Since  $\mathcal{B}$  is unital and separable,  $X$  is compact and second countable. Let  $B' \subset X$  denote the clopen set which corresponds to  $B$  under the Gelfand map.

**6.4.1. Reducing the measure-preserving spaces.** By Corollary 6.2 there exist  $\lambda \in \mathcal{P}_\mu(\beta G)$  such that the pushed forward probability measure  $\eta' := (\pi_B)_* \lambda$  on  $X$  is  $G$ -invariant and ergodic with

$$d_{np}^*(B) = \eta'(B')$$

and a  $\lambda$ -measurable set  $C' \subset X$  with

$$\overline{AB} \supset \pi_B^{-1}(C')^c$$

modulo  $\lambda$ -null sets in  $\beta G$ , so in particular, we have

$$\eta'(a \cdot B' \cap C') = 0, \quad \forall a \in A.$$

By Proposition 3.3,  $\lambda$  is non-singular and thus Lemma 6.8 guarantees that we can find a  $\lambda$ -conull set  $Z \subset \beta G$  such that

$$\overline{AB}_q \supset (\pi_B^{-1}(C')^c)_q = (C'^c)_x,$$

where  $x = \pi_B(q)$ , for all  $q \in Z$ .

**6.4.2. Reducing to  $bG$ .** By Proposition 6.9 applied with  $\nu = \eta'$ , there exist  $m$ -measurable sets  $\tilde{A}, \tilde{B} \subset bG$  with

$$m(\tilde{A}) \geq d_{\mathcal{F}\mathcal{S}}^*(A) \quad \text{and} \quad m(\tilde{B}) \geq \eta'(B')$$

and a  $\eta'$ -conull subset  $X' \subset X$  with the property that for all  $x \in X'$ , there is  $k \in bG$  such that

$$C'_x \subset \iota_o^{-1}(U^c \cdot k),$$

where

$$U := \left\{ k \in bG : m(\tilde{A} \cap k \cdot \tilde{B}^{-1}) > 0 \right\}.$$

Combining the two assertions above, we conclude that there exists a  $\lambda$ -conull set  $Z' \subset Z$  such that for all  $q \in Z'$ , there exists  $k \in bG$  such that

$$\overline{AB}_q \supset C'_x \supset \iota_o^{-1}(U \cdot k),$$

where  $x = \pi_B(q)$ , and this finishes the proof.

## 7. PROOF OF PROPOSITION 1.5

We shall now outline the proof of Proposition 1.5. After some preliminary work, we will see that it easily follows from Corollary 2.8.

7.0.3. *Almost automorphic points.* Let  $Y$  be a compact  $G$ -space. A point  $y \in Y$  is *almost automorphic* if whenever we have  $g_\alpha \cdot y \rightarrow y_o$  for some net  $(g_\alpha)$  in  $G$ , then  $g_\alpha^{-1} \cdot y_o \rightarrow y$ .

We stress that this is a very special kind of point and such points do not always exist. However, it is easy to see that if  $(K, \iota_o)$  is a compactification of  $G$  and  $Y := K/K_o$  for some closed subgroup  $K_o < K$ , where  $G$  acts by

$$g \cdot kK_o = gkK_o, \quad kK_o \in K/K_o,$$

then *every* point in  $Y$  is almost automorphic.

7.0.4. *Constructing joint extensions.* Now let  $X$  be another compact  $G$ -space, equipped with a  $G$ -transitive point  $x_o$ . If  $Y$  admits an almost automorphic point, we shall see that it is easy to construct joint extensions of the two systems.

**Lemma 7.1.** *Suppose  $y \in Y$  is an almost automorphic point. Then, for all  $x \in X$ , there exists  $y_o \in Y$  such that*

$$(x, y) \in \overline{G(x_o, y_o)}.$$

*Proof.* Choose a net  $(g_\alpha)$  in  $G$  such that  $g_\alpha \cdot x_o \rightarrow x$  and extract a subnet  $(g_\beta)$  with the property that  $g_\beta^{-1} \cdot y$  converges to a point  $y_o \in Y$ . Since  $y$  is almost automorphic, we have  $g_\beta \cdot y_o \rightarrow y$ , and thus

$$g_\beta \cdot (x_o, y_o) \rightarrow (x, y),$$

along this subnet. □

Fix  $x \in X$  and an almost automorphic point  $y \in Y$  and let

$$Y' := \overline{G \cdot y_o} \quad \text{and} \quad Z := \overline{G(x_o, y_o)},$$

where  $y_o$  is the point guaranteed by the previous lemma. In particular,  $z := (x, y) \in Z$ .

Let  $z_o := (x_o, y_o)$  and denote by  $\pi_X$  and  $\pi_{Y'}$  the canonical projections of  $Z$  onto  $X$  and  $Y'$  respectively. Let  $\pi : \beta G \rightarrow Z$  be the (left)  $G$ -equivariant map with  $\pi(\bar{e}) = z_o$  induced from the inclusion  $C(Z) \hookrightarrow C(\beta G)$ .

7.0.5. *Small boundary property.* Fix an admissible probability measure  $\mu$  on  $G$  and suppose  $U \subset Y'$  is Jordan measurable with respect to all  $\mu$ -stationary probability measures on  $Y'$ . In particular, if  $Y'$  is a compact homogeneous space as above, then this condition reduces to  $U$  being Jordan measurable with respect to the Haar measure (since every  $\mu$ -stationary Borel probability measure on a compact homogenous space is invariant - see e.g. [10]).

Let  $B \subset X$  be a clopen set and suppose

$$B_x \supset U_y.$$

Define

$$B' := (\pi_X \circ \pi)^{-1}(B) \quad \text{and} \quad U' := (\pi_{Y'} \circ \pi)^{-1}(U)$$

and choose  $q \in \beta G$  such that  $\pi(q) = z$ . Then

$$B'_q \supset U'_q.$$

Note that  $B' \subset \beta G$  is clopen and since  $U'$  is the pull-back under a  $G$ -equivariant continuous map from  $\beta G$  onto  $Z$ , Lemma 2.7 implies that  $U'$  is Jordan measurable with respect all elements in  $\mathcal{L}_\mu$ .

In particular, by Corollary 2.8 (recall that  $\mathcal{L}_\mu$  is left saturated by Lemma 3.4), there exists a  $\mathcal{L}_\mu$ -conull set  $T \subset G$  such that

$$B'_\varepsilon \supset U'_\varepsilon \cap T.$$

Since  $T$  is right thick by Corollary 3.5, we have proved Proposition 1.5.

## 8. PROOF OF PROPOSITION 1.6

Recall that if  $K$  is a compact hausdorff group and  $U \subset K$  is an open set, then there exists a finite set  $F \subset K$  such that  $FU = K$ . We define the *syndeticity index*  $s_K(U)$  as the minimal cardinality of such a finite set.

**8.1. Syndeticity index of product sets.** Let  $K$  be a compact hausdorff group. By Steinhaus Lemma, whenever  $A, B \subset K$  are Borel sets with positive Haar measures, then the product set  $AB$  has non-empty interior. We wish to bound the syndeticity index of this interior in terms of the Haar measures of  $A$  and  $B$ .

We begin with a few simple lemmata.

**Lemma 8.1.** *Suppose  $C, D \subset K$  are Borel sets. Then there exists  $k_o \in K$  such that*

$$m(C \cap k_o \cdot D) \geq m(C) \cdot m(D).$$

*Proof.* This is immediate from the identity

$$\int_K m(C \cap k \cdot D) dm(k) = m(C) \cdot m(D).$$

□

**Lemma 8.2.** *Suppose  $E \subset K$  is a Borel set with positive Haar measure and define the open set*

$$U := \left\{ k \in K : m(E \cap k \cdot E) > 0 \right\} \subset K.$$

*Then,*

$$s_K(U) \leq \left\lfloor \frac{1}{m(E)} \right\rfloor.$$

*Proof.* If  $U = K$ , there is nothing to prove, so assume there exists  $k_1 \notin U$ . For  $n \geq 2$ , we choose

$$k_n \notin U \cup \left( \bigcup_{j=1}^{n-1} k_j \cdot U \right),$$

provided the right hand side does not equal  $K$ . Note that  $k_i^{-1} \cdot k_j \notin U$  for all  $i < j$ , so in particular,

$$m(k_i \cdot E \cap k_j \cdot E) = m(E \cap k_i^{-1} \cdot k_j \cdot E) = 0,$$

for all  $i \neq j$ . Hence

$$m\left(E \cup \left( \bigcup_{j=1}^{n-1} k_j E \right)\right) = m(E) + \sum_{j=1}^{n-1} m(k_j \cdot E) = n \cdot m(E),$$

for all  $n$ , provided that

$$K \neq U \cup \left( \bigcup_{j=1}^{n-1} k_j \cdot U \right).$$

Since  $E \subset K$  has positive measure, there exists  $n \geq 1$ , such that

$$n \cdot m(E) \leq 1 \quad \text{and} \quad (n+1) \cdot m(E) > 1.$$

For this  $n$ , the set

$$F := \{e, k_1, \dots, k_{n-1}\} \subset K$$

must satisfy  $F \cdot U = K$ , and hence

$$s_K(U) \leq n \leq \frac{1}{m(E)},$$

which finishes the proof.  $\square$

**Proposition 8.3.** *Suppose  $C, D \subset K$  are Borel sets with positive Haar measures and define the open set*

$$U := \left\{ k \in K : m(C \cap k \cdot D) > 0 \right\} \subset K.$$

Then,

$$s_K(U) \leq \left\lfloor \frac{1}{m(C) \cdot m(D)} \right\rfloor.$$

*Proof.* By Lemma 8.1 there exists  $k_o \in K$  such that the set

$$E := C \cap k_o \cdot D$$

has measure at least  $m(C) \cdot m(D)$ . Hence,

$$m(C \cap k \cdot D) \geq m(E \cap k \cdot k_o^{-1} \cdot E)$$

for all  $k \in K$ . Define the open set

$$U' := \left\{ k \in K : m(E \cap k \cdot E) > 0 \right\} \subset K.$$

By Lemma 8.2 there exists a finite set  $F \subset K$  with

$$|F| \leq \left\lfloor \frac{1}{m(E)} \right\rfloor \leq \left\lfloor \frac{1}{m(C) \cdot m(D)} \right\rfloor.$$

such that  $FU' = K$ . Since

$$U \supset U' \cdot k_o,$$

we conclude that

$$s_K(U) \leq \left\lfloor \frac{1}{m(C) \cdot m(D)} \right\rfloor.$$

$\square$

**8.2. Proof of Proposition 1.6.** There is no reason to expect that the set  $U$  in Proposition 8.3 is Jordan measurable. However, as we shall see in this subsection, it always contains an open Jordan measurable subset of the same syndeticity index. This will follow from the following proposition, which for compact and *metrizable* groups is quite standard. Although the setting can be adapted so that only this weaker version of the proposition needs to be applied, we shall here give the proof of the same statement for *any* compact hausdorff group.

**Proposition 8.4.** *Suppose  $C \subset K$  is a closed set. Then  $C$  equals the intersection of a decreasing net  $(C_\alpha)$  of closed Jordan measurable sets in  $K$ .*

Before we turn to the proof of this proposition, we give an easy corollary.

**Corollary 8.5.** *Suppose  $U \subset K$  is an open set with the property that there exists a finite set  $F \subset K$  such that  $FU = K$ . Then there exists an open Jordan measurable subset  $U' \subset U$  such that  $FU' = K$ .*

*Proof.* By Proposition 8.4, there exists an increasing net  $(V_\alpha)$  of open Jordan measurable sets with

$$U = \bigcup_{\alpha} V_{\alpha}.$$

In particular,

$$K = FU = \bigcup_{\alpha} FV_{\alpha}.$$

Hence,  $(FV_{\alpha})$  is an open covering of  $K$ , so we may extract a finite sub-covering. Since  $(V_{\alpha})$  is nested, there must exist  $\alpha$  such that

$$K = FV_{\alpha},$$

which finishes the proof. □

The proof of Proposition 1.6 is now immediate.

*Proof of Proposition 1.6.* Suppose  $A, B \subset K$  are Borel sets with positive  $m$ -measures and define the open set

$$U := \left\{ k \in K : m(A \cap k \cdot B) > 0 \right\} \subset K.$$

By Lemma 8.3,

$$s_K(U) \leq \left\lfloor \frac{1}{m(A) \cdot m(B)} \right\rfloor,$$

and by Corollary 8.5, we can find a Jordan measurable set  $U' \subset U$  with

$$s_K(U') \leq \left\lfloor \frac{1}{m(A) \cdot m(B)} \right\rfloor,$$

which finishes the proof. □

**8.3. Proof of Proposition 8.4.** Recall (see Chapter 2 in [22] for the proof) that if  $X$  is any (locally) compact hausdorff space and  $K \subset X$  is compact and  $U \subset X$  is an open set containing  $K$ , then there exists an open subset  $V$  with compact closure such that

$$K \subset V \subset U.$$

This simple property of locally compact hausdorff spaces is the only thing required to give a proof of the following statement, which clearly implies Proposition 8.4.

**Proposition 8.6.** *Suppose  $K \subset X$  is compact. Then, for every open subset  $U \supset K$ , there exists a  $\mu$ -Jordan measurable set  $K \subset C \subset U$ .*

*Proof.* Fix  $\epsilon > 0$  and a decreasing sequence  $(\epsilon_n)$  of positive real numbers, converging to zero, with  $\epsilon_0 = \epsilon$ .

By regularity of  $\mu$ , we can find an open set  $U_0 \subset U$  with

$$K \subset U_0$$

and

$$\mu(U_0) \leq \mu(K) + \epsilon_0.$$

Since  $X$  is a locally compact hausdorff space, there exists an open set  $V_{0,1} \subset X$  with compact closure such that

$$K \subset V_{0,1} \subset \overline{V_{0,1}} \subset U_0.$$

By regularity of  $\mu$ , we choose an open set  $U_1 \subset X$  such that

$$\mu(U_1) \leq \mu(\overline{V_{0,1}}) + \epsilon_1$$

and since  $X$  is a locally compact hausdorff space we can find open sets  $V_{1,0}, V_{1,1} \subset X$  such that

$$K \subset \overline{V_{0,1}} \subset V_{1,0} \subset \overline{V_{1,0}} \subset V_{1,1} \subset \overline{V_{1,1}} \subset U_1.$$

Again, by regularity of  $\mu$ , we can find an open set  $U_2 \subset V_{1,1}$  with

$$\mu(U_2) \leq \mu(\overline{V_{1,0}}) + \epsilon_2$$

and open sets  $V_{2,0}, V_{2,1} \subset X$  such that

$$K \subset \overline{V_{1,0}} \subset V_{2,0} \subset \overline{V_{2,0}} \subset V_{2,1} \subset \overline{V_{2,1}} \subset U_2 \subset V_{1,1}.$$

For  $n > 2$ , we can find an open set  $U_{n+1} \subset V_{n,1}$  with

$$\mu(U_{n+1}) \leq \mu(\overline{V_{n,0}}) + \epsilon_{n+1}$$

and open sets  $V_{n+1,0}, V_{n+1,1} \subset X$  such that

$$K \subset \overline{V_{n,0}} \subset V_{n+1,0} \subset \overline{V_{n+1,0}} \subset V_{n+1,1} \subset \overline{V_{n+1,1}} \subset U_{n+1} \subset V_{n,1}.$$

By construction

$$V_{n,0} \subset V_{n+1,0} \quad \text{and} \quad V_{n,1} \supset V_{n+1,1},$$

for all  $n$  and hence, if we define

$$V := \bigcup_n V_{n,0} \quad \text{and} \quad C := \bigcap_n \overline{V_{n,1}},$$

then  $V \subset X$  is open and  $C \subset X$  is closed, and moreover

$$\mu(V) = \lim_n \mu(V_{n+1,0}) \quad \text{and} \quad \mu(C) = \lim_n \mu(\overline{V_{n+1,1}}).$$

Since

$$\mu(\overline{V_{n+1,1}}) \leq \mu(U_{n+1}) \leq \mu(\overline{V_{n,0}}) + \epsilon_{n+1} \leq \mu(V_{n+1,0}) + \epsilon_{n+1},$$

for all  $n$  and  $\epsilon_n \rightarrow 0$ , we have

$$\mu(C) = \lim_n \mu(\overline{V_{n+1,1}}) \leq \lim_n (\mu(V_{n+1,0}) + \epsilon_{n+1}) = \mu(V).$$

Since  $V \subset C$ , we conclude that  $C$  is  $\mu$ -Jordan measurable, and

$$\mu(C) \leq \mu(U_o) \leq \mu(K) + \epsilon.$$

To prove the second assertion, we first note that the set  $K \subset V_o$  is arbitrary, and thus we can always ensure that there exists a closed  $\mu$ -Jordan measurable set

$$K \subset C \subset V_o.$$

Since  $K$  is compact, there exists a decreasing net  $(V_\alpha)$  of open sets with

$$K = \bigcap V_\alpha,$$

and from the arguments above, we know that, for every  $\alpha$ , there exists a  $\mu$ -Jordan measurable set such that

$$K \subset C_\alpha \subset V_\alpha,$$

which finishes the proof. □

## REFERENCES

1. Aliprantis, C. D.; Border, K.C. *Infinite dimensional analysis. A hitchhiker's guide*. Third edition. Springer, Berlin, 2006. xxii+703 pp. ISBN: 978-3-540-32696-0
2. Babillot, M. *An introduction to Poisson boundaries of Lie groups*. Probability measures on groups: recent directions and trends, 1C90, Tata Inst. Fund. Res., Mumbai, 2006.
3. Beiglböck, M; Bergelson, V. Fish, A. *Sumset phenomenon in countable amenable groups*, Adv. Math. 223:2 (2010), 416172.
4. Bergelson. V; Furstenberg H. ; Weiss, B. *Piecewise-Bohr sets of integers and combinatorial number theory*, in Topics in Discrete Mathematics, Algorithms and Combinatorics 26, Springer, Berlin, 2006, pp. 1317.
5. Bergelson, V.; Hindman, N.; McCutcheon, R. *Notions of size and combinatorial properties of quotient sets in semigroups*. Proceedings of the 1998 Topology and Dynamics Conference (Fairfax, VA). Topology Proc. 23 (1998), Spring, 2317.
6. Bogolyubov, N.N.: *Some algebraical properties of almost periods*, Zap. Kafedry Mat. Fiziki Kiev 4 (1939), 185–194.
7. Chou, C. *On topologically invariant means on a locally compact group*. Trans. Amer. Math. Soc. 151 1970 443176.
8. Di. Nasso, M.; Lupini, M. *Product sets and Delta-sets in amenable group*. Pre-print. Arxiv. <http://arxiv.org/abs/1211.4208>
9. Følner, E. *Generalization of a theorem of Bogoliouboff to topological abelian groups*. With an appendix on Banach mean values in non-abelian groups, Math. Scand. 2 (1954), 517.
10. Furstenberg, H.; Glasner, E. *Stationary dynamical systems*. Dynamical numbersinterplay between dynamical systems and number theory, 117, Contemp. Math., 532, Amer. Math. Soc., Providence, RI, 2010.
11. Gamelin, T. W. *Uniform algebras*. Prentice-Hall, Inc., Englewood Cliffs, N. J., 1969. xiii+257 pp.
12. Glasner, E.: *Ergodic theory via joinings*. AMS Mathematical surveys and monographs, Volume 101, 2003.
13. Griesmer, J. T. *Sumsets of dense sets and sparse sets*. Israel J. Math. 190 (2012), 229172.
14. Jin, R. *The sumset phenomenon*. Proc. Amer. Math. Soc. 130 (2002), no. 3, 855171

15. Johnstone, P. T. *Stone spaces*. Cambridge Studies in Advanced Mathematics, 3. Cambridge University Press, Cambridge, 1982. xxi+370 pp. ISBN: 0-521-23893-5
16. Kaimanovich, V. A.; Vershik, A. M. *Random walks on discrete groups: boundary and entropy*. Ann. Probab. 11 (1983), no. 3, 457-170.
17. Kaniuth, E.; *A course in commutative Banach algebras*. Springer Graduate Texts in Mathematics. ISBN: 978-0-387-72475-1.
18. Krengel, Ulrich *Ergodic theorems. With a supplement by Antoine Brunel*. de Gruyter Studies in Mathematics, 6. Walter de Gruyter Co., Berlin, 1985. viii+357 pp. ISBN: 3-11-008478-3
19. Kriz, I.; *Large independent sets in shift-invariant graphs: solution of Bergelson's problem*. Graphs Combin. 3 (1987), no. 2, 145-178.
20. Mackey, G.W.: *Ergodic transformation groups with a pure point spectrum*. Illinois J. Math. **8** 1964, pp. 593-600.
21. Rosenblatt, J.; *Ergodic and mixing random walks on locally compact groups*. Math. Ann. 257 (1981), no. 1, 311-7.
22. Rudin, W. *Real and complex analysis*. Third edition. McGraw-Hill Book Co., New York, 1987. xiv+416 pp. ISBN: 0-07-054234-1
23. Steinhaus, H.: *Sur les distances des points de mesure positive*, Fund. Math. **1**(1920), 93-104.

DEPARTMENT OF MATHEMATICS , ETH ZÜRICH, ZÜRICH, SWITZERLAND  
*Current address:* Department of Mathematics, Chalmers, Gothenburg, Sweden  
*E-mail address:* micbjo@chalmers.se

SCHOOL OF MATHEMATICS AND STATISTICS, UNIVERSITY OF SYDNEY, AUSTRALIA  
*E-mail address:* alexander.fish@sydney.edu.au