AN EXPLICIT FORMULA FOR A BRANCHED COVERING

J.A. HILLMAN

Abstract. We give an explicit formula for a 2-fold branched covering from
\( \mathbb{C}P^2 \) to \( S^4 \), and relate it to other maps between quotients of \( S^2 \times S^2 \).

It is well known that the quotient of the complex projective plane \( \mathbb{C}P^2 \) by com-
plex conjugation is the 4-sphere \( [2, 4] \). (See also [1].) In the course of Massey’s
exposition he shows that \( \mathbb{C}P^2 \) is the quotient of \( S^2 \times S^2 \) by the involution which
exchanges the factors, and that \( \mathbb{R}P^2 \) is similarly a quotient of \( \mathbb{R}P^1 \times \mathbb{R}P^1 \). Lawson
has displayed very clearly the topology underlying these facts [3]. We shall use
harmonic coordinates for real and complex projective spaces to give explicit formu-
lae for some of these quotient maps. (We describe briefly the work of Massey and
Kuiper at the end of this note.)

A smooth map \( f : M \to N \) between closed \( n \)-manifolds is a 2-fold branched
covering if \( M \) has a codimension-2 submanifold \( B \) (the branch locus), such that
\( f|_{M\setminus B} \) is a 2-to-1 immersion, \( f|_B : B \to f(B) \) is a bijection onto a submanifold
\( f(B) \) (the branch set) and along \( B \) the map \( f \) looks like \((b, z) \mapsto (f(b), z^2)\) in local
coordinates, with \( b \in B \) and transverse complex coordinate \( z \in \mathbb{C} \).

Let \( A \) be the antipodal involution of \( S^2 \), and let \( \sigma \) and \( \tau \) be the diffeomorphisms
of \( S^2 \times S^2 \) given by \( \sigma(s, s') = (s', A(s)) \) and \( \tau(s, s') = (s', s) \), for \( s, s' \in S^2 \). Then
\( \sigma \) and \( \tau \) generate a dihedral group of order 8, since \( \sigma^4 = \tau^2 = 1 \) and \( \tau \sigma \tau = \sigma^{-1} \).

We shall view \( S^2 \) as the unit sphere in \( \mathbb{C} \times \mathbb{R} \). The stereographic projection
\( \gamma : S^2 \to \mathbb{C}P^1 \) is given by \( \gamma(z, t) = [z : 1 - t] \), for \((z, t) \in S^2 \), and its inverse is

\[
\gamma^{-1}([u : v]) = \left( \frac{2u \bar{v}}{|u|^2 + |v|^2}, \frac{|u|^2 - |v|^2}{|u|^2 + |v|^2} \right),
\]

for \([u : v] \in \mathbb{C}P^1 \). The action of the antipodal map on \( \mathbb{C}P^1 \) is given by

\[
\gamma A \gamma^{-1}([u : v]) = [\bar{v} : \bar{u}],
\]

for \([u : v] \in \mathbb{C}P^1 \).

Let \( \alpha, \beta \) and \( f : \mathbb{C}^2 \to \mathbb{C}^2 \) be the maps given by \( \alpha(w, z) = (\frac{w - z}{w^2 + 1}, \frac{w + z}{w^2 + 1}) \),
\( \beta(w, z) = (z, z^2 - 4w) \) and \( f(w, z) = (wz, w + z) \), for \((w, z) \in \mathbb{C}^2 \). Then \( \alpha \) and \( \beta \) are
diholomorphic, and \( \beta f \alpha(w, z) = (w, z^2) \). Therefore \( f \) is a 2-fold branched covering,
branched over \( \{(z_1, z_2) \in \mathbb{C}^2 \mid z_2^2 = 4z_1 \} \). The extension \( \hat{f} : \mathbb{C}P^1 \times \mathbb{C}P^1 \to \mathbb{C}P^2 \)
given by

\[
\hat{f}([u : v], [u' : v']) = [uu' : uv' + u'v : vv']
\]
is a 2-fold branched covering, branched over \( \hat{f}(\Delta) \), where \( \Delta \subset \mathbb{C}P^1 \times \mathbb{C}P^1 \) is the
diagonal. (Thus \( \hat{f}(\Delta) \) is the image of \( \mathbb{C}P^1 \) in \( \mathbb{C}P^2 \) under the Segre embedding.) The

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composite \( \lambda = \hat{f}(\gamma \times \gamma) \) is essentially Lawson’s map, giving
\[
S^2 \times S^2 / \langle \tau \rangle \cong \mathbb{CP}^1 \times \mathbb{CP}^1 / (\gamma \tau \gamma^{-1}) \cong \mathbb{CP}^2.
\]

Let \( c_n : \mathbb{CP}^n \to \mathbb{CP}^n \) be complex conjugation, for \( n \geq 1 \). Then \( \hat{f}(c_1 \times c_1) = c_2 \hat{f} \).
Lawson observed that if \( \theta : \mathbb{CP}^2 \to \mathbb{CP}^2 \) is the linear automorphism given by
\[
\theta([u : v : w]) = [(iu + w) : (1 - i)v : (u + iw)]
\]
then
\[
\lambda \sigma^2 = \theta^2 c_2 \lambda = \theta c_2 \theta^{-1} \lambda.
\]

Hence \( c_2 \) is conjugate to a map covered by the free involution \( \sigma^2 \). (Note that \( \theta^2([u : v : w]) = [w : -v : u] \), \( c_2 \theta c_2 = \theta^{-1} \) and \( \theta^4 = id_{\mathbb{CP}^2} \).

If we identify \( \mathbb{C} \) with \( \mathbb{R}^2 \) and use real harmonic coordinates we may instead extend \( f \) to a map \( g : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^4 \), given by
\[
g([r : s : t], [r' : s' : t']) = [rr' - ss', rs' + sr', rt' + tr', st' + ts', tt']
\]
This is a 2-fold branched covering, with branch locus the diagonal, and induces a diffeomorphism
\[
\mathbb{R}^2 \times \mathbb{R}^2 / (x, y) \sim (y, x) \cong \mathbb{R}^4.
\]

The map \( g \) has a lift \( \tilde{g} : S^2 \times S^2 \to S^4 \), given by
\[
\tilde{g}((r, s, t), (r', s', t')) = \nu(rr' - ss', rs' + sr', rt' + tr', st' + ts', tt'),
\]
where \( \nu : \mathbb{R}^5 \setminus \{0\} \to S^4 \) is the radial normalization. (If \( (r, s, t), (r', s', t') \in S^2 \)
then the norm of \( (rr' - ss', rs' + sr', rt' + tr', st' + ts', tt') \) is \( \sqrt{1 + 2tt(rr' + ss')} \).

We may also obtain \( \tilde{g} \) by normalizing the map
\[
((z, t), (z', t')) \mapsto (zz', zt' + ts', tt') \in \mathbb{C}^2 \times \mathbb{R} \setminus \{0\}, \quad \forall (z, t), (z', t') \in S^2.
\]

This map is invariant under \( \sigma^2 \) and \( \tau \), and is generically 4-to-1. Hence it factors through a map \( g^+ : S^2 \times S^2 / \langle \sigma^2 \rangle \to S^4 \), and induces maps \( G = \tilde{g} \lambda^{-1} : \mathbb{CP}^2 \to S^4 \) and \( h : S^2 \times S^2 / \langle \sigma \rangle \to \mathbb{R}^4 \). The latter three maps are each 2-fold branched coverings.

The five nontrivial subgroups of the group \( D_8 \) generated by \( \sigma \) and \( \tau \) that do not contain \( \tau \) (namely, \( \langle \sigma \tau \rangle, \langle \tau \sigma \rangle, \langle \sigma^2 \rangle, \langle \sigma \rangle \) and \( \langle \sigma \tau, \tau \sigma \rangle \)) each act freely. The lattice of quotients is:

![Diagram](https://via.placeholder.com/150)

The unlabelled maps are double coverings, and the diagram commutes. (The part of this diagram involving the vertices \( S^2 \times S^2 \), \( \mathbb{CP}^2 \), \( \mathbb{RP}^2 \times \mathbb{RP}^2 \), \( \mathbb{RP}^4 \) and \( S^4 \) is displayed in [4].)
On the affine piece $U_2 = \{ [z_0 : z_1 : z_2] \mid z_2 \neq 0 \} \cong \mathbb{C}^2$ we have

$$f^{-1}(\{p : q : 1\}) = (\frac{1}{2}(q \pm \sqrt{q^2 - 4p}) : 1), [\frac{1}{2}(q \mp \sqrt{q^2 - 4p}) : 1]).$$

Hence

$$G([p : q : 1]) = \nu(4p, 2pq - 2q, p\bar{p} + 1 - \frac{3}{2}q\bar{q} - \frac{1}{2}q^2 - 4p|)$$
on $U_2$. Homogenizing this formula gives

$$G([p : q : r]) = \nu(4p\bar{r}, 2p\bar{q} - 2q\bar{r}, q\bar{p} + r\bar{r} - \frac{3}{2}q\bar{q} - \frac{1}{2}q^2 - 4pq|).$$

The argument of $\nu$ is nonzero when $(p, q, r) \neq 0$, and its length is the square root of an homogeneous quartic polynomial in the real and imaginary parts of the harmonic coordinates of $\mathbb{C}P^2$. Thus $G$ is a real analytic function, and it is 2-to-1 on a dense open subset of its domain. Its essential structure is most easily seen after using $\theta$ to make a linear change of coordinates. Let $\delta = G\theta$. Then $\delta c_2 = \emptyset$, and $\delta$ is a 2-fold branched covering, with branch locus $\text{Re}(\mathbb{C}P^2) \cong \mathbb{R}P^2$, the set of real points of $\mathbb{C}P^2$. The complement of $\text{Re}(\mathbb{C}P^2)$ in $\mathbb{C}P^2$ is simply connected, and so $\pi_1(S^4 \setminus \delta(\mathbb{R}P^2)) = Z/2Z$, since $c_2$ acts freely on $\mathbb{C}P^2 \setminus \text{Re}(\mathbb{C}P^2)$. Thus $S^4$ is the quotient of $\mathbb{C}P^2$ by complex conjugation $[2, 4]$.

Remark: The main step in [4] used a result on fixed point sets of involutions of symmetric products to obtain a diffeomorphism $\mathbb{R}P^2 \times \mathbb{R}P^2 / (x, y) \sim (y, x) \cong \mathbb{R}P^4$. Our contribution has been the explicit branched covering $g : \mathbb{R}P^2 \times \mathbb{R}P^2 \to \mathbb{R}P^4$, and the subsequent formula for $G$. The argument in [2] was very different. Let $\eta : \mathbb{C}^3 \to \mathbb{R}^6$ be the function given by

$$\eta(z_1, z_2, z_3) = (|z_1|^2, |z_2|^2, |z_3|^2, \text{Re}(z_2\bar{z}_3), \text{Re}(z_3\bar{z}_1), \text{Re}(z_1\bar{z}_2)).$$

Then $\eta(\zeta v) = \eta(v)$ for all $v \in \mathbb{C}^3$ and $\zeta \in S^1$, so $\eta|_{S^5}$ factors through $\mathbb{C}P^2 = S^5 / S^1$. Moreover, $\eta(\bar{v}) = \eta(v)$ for all $v \in \mathbb{C}^3$, so $\eta|_{S^3}$ factors through $\mathbb{C}P^2 / \langle c_2 \rangle$. Kuiper then showed that $\eta(S^5)$ lies in the affine hyperplane defined by $x_1 + x_2 + x_3 = 1$, and is the boundary of the convex hull of the Veronese embedding of $\mathbb{R}P^2$ in $\mathbb{R}^5$. Hence $\eta$ induces a PL homeomorphism from $\mathbb{C}P^2 / \langle c_2 \rangle$ to $S^4$.

References


School of Mathematics and Statistics, University of Sydney, NSW 2006, Australia

E-mail address: jonathan.hillman@sydney.edu.au