

KOSZUL ALGEBRAS AND FLOW LATTICES

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ABSTRACT. We provide a homological algebraic realization of the lattices of integer cuts and integer flows of graphs. To a finite 2-edge-connected graph Γ with a spanning tree T , we associate a finite dimensional Koszul algebra $A_{\Gamma,T}$. Under the construction, planar dual graphs with dual spanning trees are associated Koszul dual algebras. The Grothendieck group of the category of finitely-generated $A_{\Gamma,T}$ modules is isomorphic to the Euclidean lattice $\mathbb{Z}^{E(\Gamma)}$, and we describe the sublattices of integer cuts and integer flows on Γ in terms of the representation theory of $A_{\Gamma,T}$. The grading on $A_{\Gamma,T}$ gives rise to q -analogs of the lattices of integer cuts and flows; these q -lattices depend non-trivially on the choice of spanning tree. We give a q -analog of the matrix-tree theorem, and prove that the q -flow lattice of (Γ_1, T_1) is isomorphic to the q -flow lattice of (Γ_2, T_2) if and only if there is a cycle preserving bijection from the edges of Γ_1 to the edges of Γ_2 taking the spanning tree T_1 to the spanning tree T_2 . This gives a q -analog of a classical theorem of Caporaso-Viviani and Su-Wagner.

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1. INTRODUCTION

1.1. Lattices and Grothendieck groups. An *integer lattice* is a finitely generated free Abelian group with a symmetric, \mathbb{Z} -valued bilinear form, sometimes required to be non-degenerate. Integer lattices appear naturally in the representation theory of finite dimensional algebras: if A is a finite dimensional algebra, the Grothendieck group of the additive category of finitely-generated projective left A modules, denoted G_0 , is a free abelian group with basis the distinct isomorphism classes of indecomposable projective modules. The pairing $([P], [Q]) = \dim \text{Hom}(P, Q)$ is bilinear, and though it is not always symmetric, it sometime is – e.g. when A is a symmetric algebra. In this way many important lattices in Lie theory arise in geometric representation theory as Grothendieck groups of additive categories.

It is interesting to explore the relationship between properties of lattices and structures in homological algebra. To give an example, denote by K_0 the Grothendieck group of the abelian category of all finite-dimensional A -modules; K_0 is also a free abelian group, which by the Jordan-Holder theorem is spanned by the classes of the simple modules. The Hom pairing identifies K_0 as the dual of G_0 , at least as an abelian group. If G_0 is a lattice, it is tempting to try to identify K_0 as the *dual lattice*, after endowing K_0 with the *Euler form*:

$$\langle [M], [N] \rangle := \sum_{i=0}^{\infty} (-1)^i \dim \text{Ext}_{A(B)}^i(M, N).$$

In general this alternating sum need not converge; however, if A has finite homological dimension, the Euler form gives a bilinear pairing on K_0 . Moreover, since in that case every simple module has a finite length projective resolution, the natural map $G_0 \rightarrow K_0$ is an isomorphism of lattices, so that G_0 is a *unimodular lattice*. Thus, the lattice-theoretic notion of unimodularity is captured by a categorical notion of homological finiteness.

In lattice theory, unimodular lattices are often constructed by *gluing* non-unimodular lattices of smaller rank (see [CS, Chapters 4.3, 16]). Perhaps the simplest example, which is the only one we consider in this paper, is when the unimodular lattice is the Euclidean lattice, and it is obtained by gluing together two smaller rank lattices. This example is of interest in graph theory: given a 2-edge-connected graph Γ with edge set $E(\Gamma)$, the lattice of integer flows $\mathcal{F}(\Gamma)$ and the lattice of integer cuts $\mathcal{C}(\Gamma)$ glue to form the Euclidean lattice $\mathbb{Z}^{E(\Gamma)}$; in particular, $\mathcal{F}(\Gamma)$ and $\mathcal{C}(\Gamma)$ appear as mutual orthogonal complements inside the Euclidean lattice $\mathbb{Z}^{E(\Gamma)}$.

The main construction of the current paper gives a homological algebra lift of this particular occurrence of lattice gluing. More specifically, given a 2-connected graph Γ with a choice of a spanning tree T , we construct a *bipartite algebra* $A_{\Gamma, T}$, as the quotient of a path algebra. The first basic result is that $A_{\Gamma, T}$ is standard Koszul; it follows from this that the Grothendieck groups $K_0 \cong G_0$ equipped with the Euler/Hom pairing are isomorphic to the Euclidean lattice $\mathbb{Z}^{E(G)}$.

Moreover, the subcategory of $A_{\Gamma, T}\text{-mod}$ generated by a distinguished subset of projective modules descends in the Grothendieck group to the lattice of integer flows $\mathcal{F}(\Gamma)$; similarly, the lattice of integer cuts $\mathcal{C}(G)$ is realized in the Grothendieck group as the span of a complementary collection of simple modules.

When Γ is planar, the pair (Γ, T) has a planar dual $(\Gamma^!, T^!)$, and it is immediate from the construction that the algebra assigned to $(\Gamma^!, T^!)$ is the Koszul dual of that assigned

to (Γ, T) . This is reminiscent of other combinatorial shadows of Koszul duality, such as the Gale duality/Koszul duality phenomena appearing in work of the second author with Braden–Proudfoot–Webster [BLPW] on symplectic duality for hypertoric varieties.

From this point of view, the structure of the Euclidean lattice is categorified by that of a finite dimensional standard Koszul algebra. It would be interesting to see whether or not other unimodular lattices, such as the Leech lattice and Niemeier lattices, appear as “homologically finite” structures in classical or p -dg homological algebra. Although we don’t address it in the current document, this question is a large part of our motivation for writing this paper.

1.2. q -lattices and graph theory. One interesting piece of structure enjoyed by the bipartite algebras $A_{\Gamma, T}$ is a non-negative (Koszul) grading. We may therefore consider categories of graded $A_{\Gamma, T}$ modules, so that the Grothendieck groups of these categories are free $\mathbb{Z}[q, q^{-1}]$ modules endowed with $\mathbb{Z}[q, q^{-1}]$ -valued bilinear forms; these free modules are interesting in their own right.

We define a q -lattice to be a finitely generated free $\mathbb{Z}[q, q^{-1}]$ -module L with a non-degenerate sesqui-linear (that is, linear in the second argument and q -anti-linear in the first argument) form

$$\langle \bullet, \bullet \rangle : L \times L \longrightarrow \mathbb{Z}[q, q^{-1}],$$

together with a q -anti-linear involution $d : L \longrightarrow L$ such that for all $x, y \in L$,

$$\langle x, y \rangle = \langle d(y), d(x) \rangle,$$

and such that d becomes the identity map after setting $q = 1$. The Grothendieck groups of categories of graded $A_{\Gamma, T}$ modules are then q -lattices. We consider the analogs of basic lattice-theoretic notions such as duality, determinants, unimodularity, and gluing for q -lattices (Section 3). We define the q -cut and q -flow lattices of graphs, and, in Section 4, prove q -analogues of two famous classical results in graph theory.

We note here that are many generalizations of integer lattices in the literature of lattices to rings other than \mathbb{Z} , however, most focus on the theory of lattices over principal ideal domains. Thus, as far as we are aware, there doesn’t currently exist a well-developed theory of q -lattices, to which our q -cut and q -flow lattices belong. However, $\mathbb{Z}[q, q^{-1}]$ shares some features with principal ideal domains: for example, projective modules over $\mathbb{Z}[q, q^{-1}]$ are always free [Sw].

It is easy to see from their construction that the isomorphism class of the q -cut and q -flow lattices of a graph-and-spanning-tree pair (Γ, T) depends non-trivially on the spanning tree T , in contrast to the theory of classical cut and flow lattices. We now briefly explain the graph-theoretic information that is captured by including the parameter q .

A theorem of Su–Wagner and Caporaso–Viviani [SW, CV] states that for two (two-edge-connected) graphs Γ_1 and Γ_2 , there exists a lattice isomorphism between the lattices of integer flows $\mathcal{F}(\Gamma_1) \cong \mathcal{F}(\Gamma_2)$, if and only if there exists a bijection $F : E(\Gamma_1) \rightarrow E(\Gamma_2)$, which preserves cycles. The q -analogue of this theorem, Theorem 4.5 of this paper, illustrates that the q -cut and q -flow lattices we define essentially determine the spanning tree T :

Theorem. *For two graphs Γ_1 and Γ_2 with respective spanning trees T_1 and T_2 , there exists a q -lattice isomorphism $\mathcal{F}_q(\Gamma_1, T_1) \cong \mathcal{F}_q(\Gamma_2, T_2)$, if and only if there exists a cycle-preserving bijection of edges $F : E(\Gamma_1) \rightarrow E(\Gamma_2)$ such that $F(T_1) = T_2$.*

The classical Matrix-Tree Theorem (see for example [Bi, Chapter 6]) of graph theory, also known as the Kirchoff Theorem, states that the determinant of $\mathcal{C}(\Gamma)$ – that is, the determinant of the pairing matrix of the bilinear form on $\mathcal{C}(\Gamma)$ – counts the number of spanning trees of Γ . In Theorem 4.6 we prove the following q -analogue, which further illustrates how the q -cut and q -flow lattices retain the spanning tree information:

Theorem. *Let Γ be a graph with chosen spanning tree T . Then the determinant of $\mathcal{C}_q(\Gamma, T)$ is a polynomial $\sum_{i=0}^r c_i q^{2i}$ with $c_0 = 1$, and $r = |E(T)| = |V(G)| - 1$. The coefficient c_i is equal to the number of spanning trees of Γ which differ from T in exactly i edges.*

Finally, we note that the constructions and results of this paper generalise in a straightforward way from graphs to regular matroids. Regular matroids possess a notion of duality (also known as Gale duality), which generalizes planar graph duality, and exchanges the lattices of integer cuts and integer flows. It is this more general matroid duality that really corresponds to Koszul duality at the level of bipartite algebras in our main construction. While we predominantly use the language of graphs for simplicity in the body of the paper, we will take short detours into the world of regular matroids to indicate how our constructions apply in that setting. A more matroid-centred approach can be found in the University of Sydney undergraduate honours thesis of Leo Jiang [Ji].

1.3. Plan. The paper is structured as follows. In Section 2 we introduce bipartite algebras and establish basic properties of their module categories. In Section 3 we define q -lattices, and basic notions such as Gram matrices and lattice gluing. Our guiding examples are Grothendieck groups of graded module categories over bipartite algebras. In Section 4, the most substantial section of this paper, we construct bipartite algebras from graphs with a choice of a spanning tree. We use their module categories to categorify the classical cut and flow lattices, study the q -cut and q -flow lattices obtained from this construction, and prove the graph theoretic theorems mentioned above.

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2. BIPARTITE ALGEBRAS

We start by introducing *bipartite algebras*, associated to bipartite graphs via a path algebra construction. In Section 4 we will discuss how bipartite algebras arise naturally from (not necessarily bipartite) graphs along with a choice of a spanning tree, and in this way they give rise to a homological algebraic realization of the lattices of integer cuts and flows associated to graphs.

Definition 2.1. Let B be a bipartite graph, with vertices $V(B) = V_0(B) \cup V_1(B)$. Let $Q(B)$ denote the double quiver of B , that is, the directed graph in which each edge of B is replaced by two arrows, one in each orientation. We associate two \mathbb{C} -algebras to B :

- $A(B) := \text{Path}(Q(B))/\{\text{length two paths which start and end in } V_0(B)\},$

- $A^!(B) := \text{Path}(Q(B))/\{\text{length two paths which start and end in } V_1(B)\}$.

Here $\text{Path}(Q(B))$ denotes the *path algebra* of $Q(B)$, whose underlying vector space is spanned by the oriented paths in the quiver $Q(B)$. Multiplication is given by concatenation of paths whenever the last vertex of the first path agrees with the first vertex of the second, and defined to be zero otherwise.

We call $A(B)$ the *bipartite algebra* associated to B . Both $A(B)$ and $A^!(B)$ are \mathbb{Z} -graded by path length. We note also that it is immediate from the definitions that any path consisting of three or more edges is zero in both $A(B)$ and $A^!(B)$. In what follows we denote by e_i the constant path at vertex i : the $\{e_i\}$ are pairwise orthogonal idempotents, with $1 = \sum_{i \in V(B)} e_i$.

2.1. Finitely generated graded $A(B)$ -modules. Let $A(B)\text{-mod}$ denote the category of finitely generated *graded* left $A(B)$ -modules. Classifying simple and indecomposable projective $A(B)$ -modules is straightforward from the quiver description of $A(B)$. We state the basic results below, leaving proofs to the interested reader or to [GG, Ji].

Proposition 2.1. *Isomorphism classes of ungraded simple $A(B)$ -modules are in bijection with the bipartite vertex set $V(B)$. Denote the simple module corresponding to $e_i \in V(B)$ by L_i : as a vector space L_i is one dimensional, e_i acts as the identity, and all other e_j as well as the positively graded subalgebra of $A(B)$, act by zero.*

As graded modules, the L_i are declared to be contained in degree zero. All graded simple modules are shifts of the L_i . Since $A(B)$ is Artinian, indecomposable projective modules are in bijection with simple modules. Hence, we obtain the following:

Corollary 2.2. *The indecomposable projective $A(B)$ -modules are given by the projective covers of the simple modules. The projective cover L_i is $P_i := A(B) \cdot e_i$, which has a as a vector space by paths which end at i .*

The modules P_i are naturally graded by path length, and all graded indecomposable projective $A(B)$ -modules are shifts of the P_i . Morphisms between indecomposable projective modules are given as follows:

Proposition 2.3. *The set of A -homomorphisms $\text{Hom}(P_i, P_j)$ has a basis consisting of paths from i to j . That is, $\text{Hom}(P_i, P_j) \cong e_i A(B) e_j$, as graded vector spaces. The map which sends a path α to the reverse path $\bar{\alpha}$ extends linearly to a vector space isomorphism $\text{Hom}(P_i, P_j) \cong \text{Hom}(P_j, P_i)$.*

Next, we show that $A(B)$ is a standard Koszul algebra. In general, a graded algebra A is *Koszul* if every graded simple left module admits a finite linear projective resolution. Let $\{L_i : i \in I\}$ denote the simple left A -modules, and let $\{P_i : i \in I\}$ be their respective projective covers. We recall the definition of a standard Koszul algebra:

Definition 2.2. The Koszul algebra A is called *standard Koszul* if there exists a set of *standard modules* $\{V_i : i \in I\}$ with surjections $P_i \xrightarrow{p_i} V_i \xrightarrow{\pi_i} L_i$, such that for some partial order $<$ on I ,

- (1) For all $i \in I$ the module $\ker(\pi_i)$ has a filtration where each sub-quotient is isomorphic to L_j for some $j < i$, and
- (2) For all $i \in I$ the module $\ker(p_i)$ has a filtration where each sub-quotient is isomorphic to V_k for some $k > i$.

In this case, the category $A(B)\text{-mod}$ is called a *highest weight category*. For a more detailed introduction to highest weight categories and Koszul algebras, see Section 5 of [BLPW]. Algebras whose module categories are highest weight are also called *quasi-hereditary*.

Proposition 2.4. *Each simple graded left $A(B)$ -module L_i has a finite linear projective resolution.*

Proof. Let N_i denote the neighbourhood of the vertex i in B , that is, the set of vertices adjacent to i . Let x_{ij} denote the length one path $i \rightarrow j$, where i and j are adjacent vertices. We'll denote the degree shift in the path length grading by $\{\cdot\}$.

First assume $i \in V_1(B)$. Then the linear projective resolution for L_i is

$$\bigoplus_{j \in N_i} P_j\{1\} \xrightarrow{\sum \cdot x_{ji}} P_i \rightarrow L_i.$$

Similarly, if $i \in V_0(B)$, then L_i has linear projective resolution

$$\bigoplus_{j \in N_i} \bigoplus_{k \in N_j} P_k\{2\} \xrightarrow{\sum \cdot x_{kj}} \bigoplus_{j \in N_i} P_j\{1\} \xrightarrow{\sum \cdot x_{ji}} P_i \rightarrow L_i.$$

Note that in the sum $\bigoplus_{j \in N_i} \bigoplus_{k \in N_j} P_k\{2\}$, each indecomposable projective module may appear multiple times. \square

Proposition 2.5. *With respect to the partial order given by $V(1) < V(0)$, the standard modules over $A(B)$ are $V_i = P_i$ when $i \in V(0)$, and $V_i = L_i$ when $i \in V(1)$.*

Proof. We need to check that the conditions (1) and (2) of Definition 2.2 are satisfied. Assume that $i \in V(0)$. In this case $V_i = P_i$, so condition (2) is vacuous. As for condition (1), a basis for $\ker(\pi_i)$ consists of the length one paths x_{ji} for $j \in N(i)$. Let $N(i) = \{j_1, \dots, j_r\}$. Then the filtration $A(B)x_{j_1 i} \subseteq A(B)x_{j_1 i} \oplus A(B)x_{j_2 i} \subseteq \dots \subseteq A(B)x_{j_1 i} \oplus A(B)x_{j_2 i} \oplus \dots \oplus A(B)x_{j_r i}$ satisfies condition (1). The proof is similar for $i \in V(1)$. \square

A Koszul algebra A is always quadratic, and in this case the quadratic dual algebra is also called the Koszul dual, and denoted by $A^!$.

Proposition 2.6. *The algebra $A^!(B)$ of Definition 2.1 is the Koszul dual of $A(B)$.*

Proof. Let $R = \mathbb{C}\{e_i : i \in V(B)\}$ be the ring generated by the idempotents e_i , and $M = \mathbb{C}\{x_{ij} : x_{ij} \in E(Q(B))\}$ be the R -bimodule spanned by the edges of $Q(B)$. Then the tensor algebra $T := T_R(M)$ is the path algebra of $Q(B)$.

We have $A(B) = T/TWT$, where $W \subseteq M \otimes_R M$ is the R -bimodule of length two paths which start and end in $V(0)$, as in Definition 2.1. The annihilator $W^\perp \subseteq M^* \otimes M^*$ is easily identified with the set of length two paths which start and end in $V(1)$, which shows that the quadratic dual $T_R(M^*)/T_R(M^*)W^\perp T_R(M^*)$ is isomorphic to $A^!(B)$. \square

Let L_i be a simple $A(B)$ -module and P_i its projective cover. The *injective hull* I_i of L_i , as a vector space, is isomorphic to the linear dual of P_i , with the negative path length grading. This is naturally an $A(B)^{op}$ -module. There is an isomorphism $A(B) \xrightarrow{\cong} A(B)^{op}$ sending a path x to the reverse path \bar{x} . This defines the action of $A(B)$ on $I_i = P_i^*$.

Similarly, costandard modules Λ_i are the linear duals of the corresponding standard modules V_i , with the action of $A(B)$ defined via the bar isomorphism as above. This in particular means that $\Lambda_i \cong L_i$ for $e_i \in V_1(B)$, and $\Lambda_i \cong I_i$ for $e_i \in V_0(B)$.

Remark 2.1. Note that “acting on the linear dual via the bar isomorphism” is a contravariant auto-equivalence of *ungraded* finitely generated $A(B)$ -modules, denoted d , with the properties that $dL_i = L_i$, $dP_i = I_i$ and $dV_i = \Lambda_i$. However, d reverses gradings and grading shifts, for instance $d(M\{1\}) = (dM)\{-1\}$, and similarly for morphisms. \square

2.2. The Grothendieck group. The category $A(B)$ -mod of finite dimensional graded $A(B)$ -modules is an abelian category. The *graded Grothendieck group* $K_0 := K_0(A(B)$ -mod) is, by definition, the $\mathbb{Z}[q, q^{-1}]$ -module generated by the isomorphism classes of modules, modulo the relations $[A] - [B] + [C] = 0$ for any short exact sequence of modules $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$; and $q[A] = [A\{1\}]$, where $\{\cdot\}$ denotes the grading shift.

Definition 2.3. The *graded Euler form* on K_0 is defined by

$$\langle [M], [N] \rangle := \sum_{i=0}^{\infty} (-1)^i \text{qdim Ext}_{A(B)}^i(M, N).$$

Here qdim denotes the graded dimension, that is, $\text{qdim Ext}_{A(B)}^i(M, N)$ is the Laurent polynomial in which the coefficient of q^k is the dimension of the degree k piece of $\text{Ext}_{A(B)}^i(M, N)$.

The graded Euler form is a non-degenerate q -sesqui-linear form on $K_0(A(B)$ -mod). Here q -sesqui-linear means that for any $f \in \mathbb{Z}[q, q^{-1}]$ and $M, N \in A(B)$ -mod, with $\bar{f}(q) := f(q^{-1})$,

$$\langle \bar{f}[M], [N] \rangle = f \langle [M], [N] \rangle = \langle [M], f[N] \rangle.$$

Sesquilinearity is a reflection of the behaviour of the Hom functor with respect to grading shifts:

$$\text{Hom}(M\{-1\}, N) = \text{Hom}\{1\}(M, N) = \text{Hom}(M, N\{1\}).$$

Proposition 2.7. K_0 is a free $\mathbb{Z}[q, q^{-1}]$ -module of rank $n = |V(B)|$. The classes of simple modules $\{[L_i]\}_{i \in V(B)}$, indecomposable projectives $\{[P_i]\}$, indecomposable injectives $\{[I_i]\}$, standard modules $\{[V_i]\}$, and costandard modules $\{[\Lambda_i]\}$ all form bases for K_0 .

Proof. The existence of a Jordan-Hölder filtration implies that the classes of simple modules form a spanning set for K_0 . The uniqueness of the Jordan-Hölder composition factors means that the classes of simple modules indeed form a basis, and the rank of K_0 is n . Every simple module has a finite projective resolution, hence the isomorphism classes of indecomposable projective modules also span, and since there is n of them they form a basis.

For any quasi-hereditary algebra A , costandard modules form a right dual set to standard modules in A -mod [CPS, 3.11] meaning that

$$\text{Ext}^i(V_k, \Lambda_l) = \begin{cases} \mathbb{C} & \text{if } k = l \text{ and } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, $\langle [V_k], [\Lambda_l] \rangle = \delta_{kl}$. This implies that standard and costandard modules are both independent sets. To see this, assume for example that there is a linear dependence $\sum_{k=1}^n f_k(q)[V_k] = 0$, where $f_k(q) \in \mathbb{Z}[q, q^{-1}]$. Then

$$0 = \left\langle \sum_k f_k(q)[V_k], \sum_{l=1}^n f_l(q)[\Lambda_l] \right\rangle = \sum_{k=1}^n f_k(q^{-1})f_k(q).$$

Note that $f_k(q^{-1})f_k(q)$ has a non-negative constant term, which is zero only if $f_k = 0$. Hence, $f_k = 0$ for all k , a contradiction. Therefore, standard and costandard modules each form bases.

The involution d of Remark 2.1 induces a q -antilinear map on K_0 with $d[L_i] = [L_i]$. So, if $[L_k] = \sum_{i=1}^n c_i(q)[P_i]$ with $c_i(q) \in \mathbb{Z}[q, q^{-1}]$, then $[L_k] = d[L_k] = \sum_{i=1}^n c_i(q^{-1})[I_i]$. Hence the indecomposable injective modules $\{[I_i]\}$ also span and hence form a basis. \square

Remark 2.2. The involution d satisfies a symmetry property with respect to the graded Euler form: $\langle [M], [N] \rangle = \langle d[N], d[M] \rangle$. To check this, write $[M]$ and $[N]$ as $\mathbb{Z}[q, q^{-1}]$ -linear combinations of simple modules.

3. Q-LATTICES

Recall that a *lattice* is a finitely generated free abelian group with a symmetric bilinear form, typically required to be non-degenerate. The following definition lifts lattices to free modules over the ring of Laurent polynomials: the guiding example is the graded Grothendieck group of Section 2. In this section we establish the analogues of some basic notions and facts of lattice theory.

Definition 3.1. A *q -lattice* is a finitely generated free $\mathbb{Z}[q, q^{-1}]$ -module L with a non-degenerate sesquilinear¹ form

$$\langle \bullet, \bullet \rangle : L \times L \longrightarrow \mathbb{Z}[q, q^{-1}];$$

and equipped with a q -anti-linear involution $d : L \longrightarrow L$ such that for all $x, y \in L$

$$\langle x, y \rangle = \langle d(y), d(x) \rangle,$$

and $d_{q=1} : L \otimes_{\mathbb{Z}[q, q^{-1}]} \mathbb{Z} \longrightarrow L \otimes_{\mathbb{Z}[q, q^{-1}]} \mathbb{Z}$ is the identity map.

After setting $q = 1$, a q -lattice becomes an ordinary lattice, although note that it is possible for a non-degenerate q -pairing to become degenerate at $q = 1$. The involution d deforming the identity is used to give the appropriate q -analog for the symmetry of the form.

Since the sesquilinear form is not symmetric, one has to distinguish between various “left” and “right” notions: left and right duals, orthogonality and complements will be defined later in this section. The involution d can be used to move between opposite side notions.

Example 3.1. In this language, the Grothendieck group K_0 of the category of finite-dimensional graded $A(B)$ -modules is a q -lattice under the graded Euler form, with d induced by the duality map defined in Remark 2.1. On K_0 , d is the unique q -anti-linear map which fixes the classes $[L_i]$ of simple modules.

3.1. Gram matrices and change of basis. Recall that given a basis $\mathcal{B} = \{b_1, \dots, b_n\}$ for a classical \mathbb{Z} -lattice L , the bilinear form is encoded in the *Gram matrix* $A = A_{\mathcal{B}}$, where $A_{ij} = \langle b_i, b_j \rangle \in \mathbb{Z}$. Given any two elements $x, y \in L$, let $x_{\mathcal{B}}$ and $y_{\mathcal{B}}$ denote the column vectors expressing x and y , respectively, in the basis \mathcal{B} . Then $\langle x, y \rangle = x_{\mathcal{B}}^t A y_{\mathcal{B}}$, where the superscript t denotes the matrix transpose.

Let $\mathcal{B}' = \{b'_1, \dots, b'_n\}$ be a different basis for L , and $T = T_{\mathcal{B}}^{\mathcal{B}'}$ the change of basis matrix. That is, the i -th column of T is the vector $(b'_i)_{\mathcal{B}}$. Then the Gram matrix $A_{\mathcal{B}'}$ is $T^t A_{\mathcal{B}} T$. The

¹That is, q -anti-linear in the first argument, and linear in the second argument, as described after Definition 2.3.

Gram matrix determines the lattice up to isomorphism, and two lattices with Gram matrices A and A' are isomorphic if and only if there exists an integer matrix T , invertible over \mathbb{Z} , such that $A' = T^t AT$.

Similarly, given a choice of a basis $\mathcal{B} = \{b_1, \dots, b_n\}$ for a q -lattice L , the sesquilinear form is again encoded in the Gram matrix $A_{\mathcal{B}}$, where $A_{ij} = \langle b_i, b_j \rangle \in \mathbb{Z}[q, q^{-1}]$. Given elements $x, y \in L$, and $x_{\mathcal{B}}$ and $y_{\mathcal{B}}$ the column vectors expressing x and y in the basis \mathcal{B} , we have $\langle x, y \rangle = x_{\mathcal{B}}^* A y_{\mathcal{B}}$, where the superscript $*$ denotes the matrix transpose composed with the involution $q \mapsto q^{-1}$.

Let $\mathcal{B}' = \{b'_1, \dots, b'_n\}$ be another basis of L , and again let T denote the change of basis matrix, where the i -th column of T is the vector $(b'_i)_{\mathcal{B}}$. Note that T is an invertible matrix over $\mathbb{Z}[q, q^{-1}]$, so that $\det T = \pm q^k$ for some $k \in \mathbb{Z}$. Then the Gram matrix with respect to the basis \mathcal{B}' is $T^* A T$. It is still true that the Gram matrix determines the q -lattice up to isomorphism, and two q -lattices given by Gram matrices A and A' are isomorphic if and only if there exists an invertible matrix T over $\mathbb{Z}[q, q^{-1}]$ such that $A' = T^* A T$.

Note in particular that the determinant of the Gram matrix is an isomorphism invariant of both classical lattices and q -lattices, that is, it is independent of the choice of basis in which the Gram matrix is written. Hence, it is also called the *determinant of the lattice*.

3.2. Dual q -lattices. In classical lattice theory, given an integer lattice L , the *lattice dual* is defined as $L^\vee := \{x \in L \otimes_{\mathbb{Z}} \mathbb{Q} \mid \langle x, v \rangle \in \mathbb{Z} \quad \forall v \in L\}$. The lattice dual is typically not an integer lattice, as elements may pair non-integrally with each other. The lattice dual is a free abelian group of the same rank as L , with a \mathbb{Q} -valued symmetric bilinear form, and includes L .

Given a basis $\mathcal{B} = \{b_1, \dots, b_n\}$ of the classical lattice L , the dual basis in $L \otimes_{\mathbb{Z}} \mathbb{Q}$ is the unique basis $\mathcal{B}^\vee = \{b_1^\vee, \dots, b_n^\vee\}$ with the property that $\langle b_i, b_j^\vee \rangle = \delta_{ij}$ for all $1 \leq i, j \leq n$. The dual basis \mathcal{B}^\vee is a basis for L^\vee .

If $L = L^\vee$, the lattice is called *unimodular*. A lattice is unimodular if and only if its determinant, the determinant of the Gram matrix, is a unit of \mathbb{Z} , that is, ± 1 . The Gram matrix $A_{\mathcal{B}}$ also arises as the change of basis matrix $T_{\mathcal{B}^\vee}^{\mathcal{B}}$, or in other words the matrix of the inclusion map $L \hookrightarrow L^\vee$, with respect to the bases \mathcal{B} and \mathcal{B}^\vee .

We now describe the analogous duality notions for q -lattices.

Definition 3.2. Given a q -lattice L , we define the *right dual* L^\vee of L to be

$$L^\vee := \{x \in L \otimes_{\mathbb{Z}[q, q^{-1}]} \mathbb{Q}(q) \mid \langle v, x \rangle \in \mathbb{Z}[q, q^{-1}] \quad \forall v \in L\}.$$

Given a basis $\mathcal{B} = \{b_1, \dots, b_n\}$ of L , the *right dual basis* in $L \otimes_{\mathbb{Z}[q, q^{-1}]} \mathbb{Q}(q)$ is the unique basis $\mathcal{B}^\vee = \{b_1^\vee, \dots, b_n^\vee\}$ with the property that $\langle b_i, b_j^\vee \rangle = \delta_{ij}$ for all $1 \leq i, j \leq n$.

Similarly, the *left dual* of L is ${}^\vee L := \{x \in L \otimes_{\mathbb{Z}[q, q^{-1}]} \mathbb{Q}(q) \mid \langle x, v \rangle \in \mathbb{Z}[q, q^{-1}] \quad \forall v \in L\}$, and the *left dual basis* of \mathcal{B} is the unique basis ${}^\vee \mathcal{B} = \{{}^\vee b_1, \dots, {}^\vee b_n\}$ with the property that $\langle {}^\vee b_i, b_j \rangle = \delta_{ij}$ for all $1 \leq i, j \leq n$. Note that d intertwines left/right duality in the sense that $(d(\mathcal{B}))^\vee = d({}^\vee \mathcal{B})$.

Remark 3.2. The left and right dual q -lattices are indeed free $\mathbb{Z}[q, q^{-1}]$ -modules, with basis given by the left and right dual bases, respectively. They are not necessarily q -lattices, as the pairing on them might not be valued in the Laurent polynomial ring $\mathbb{Z}[q, q^{-1}]$. As for the existence and uniqueness of the dual bases, observe that given a non-degenerate q -sesquilinear pairing, one can orthogonalise bases in a $\mathbb{Q}(q)$ -vector space using the Gram-Schmidt process

(without normalisation). This implies that left and right orthogonal complements exist and are unique. Given a basis vector $b_i \in \mathcal{B}$, the dual basis vector ${}^\vee b_i$ (or b_i^\vee) lies in the one-dimensional left (or right) orthogonal complement of the subspace spanned by $\mathcal{B} \setminus \{b_i\}$, and can be chosen so that $\langle {}^\vee b_i, b_i \rangle = 1$, (or $\langle b_i, b_i^\vee \rangle = 1$).

For example, in $K_0 := K_0(A(B)\text{-mod})$, costandard modules form a right dual basis to standard modules, indecomposable projectives are left dual to the simple modules, and indecomposable injectives are right dual to the simple modules. This, as well as the asymmetry of the orthogonality relation, is illustrated in the following example.

Example 3.3. Let B denote the complete bipartite graph on three vertices with $V_0(B) = \{e_1, e_2\}$ and $V_1(B) = e_3$. Then the Grothendieck group K_0 of finitely generated graded $A(B)$ -modules is a rank three q -lattice. The isomorphism classes of indecomposable projective modules $[P_i]$, for $i = 1, 2, 3$, form a basis, and with respect to this basis the Gram matrix is symmetric:

$$\begin{bmatrix} 1 & 0 & q \\ 0 & 1 & q \\ q & q & 1 + 2q^2 \end{bmatrix}$$

The basis $\{[P_1], [P_2], [P_3]\}$ is left dual to the basis given by the simple modules $\{[L_1], [L_2], [L_3]\}$, so that $\langle [P_i], [L_j] \rangle = \delta_{ij}$. However, $\langle [L_j], [P_i] \rangle$ is not necessarily equal to δ_{ji} : from Proposition 2.4 we see that

$$\begin{aligned} [L_1] &= (1 + q^2)[P_1] + q^2[P_2] - q[P_3], \\ [L_2] &= q^2[P_1] + (1 + q^2)[P_2] - q[P_3], \\ [L_3] &= -q[P_1] - q[P_2] + [P_3]. \end{aligned}$$

Hence, for example, $\langle [L_1], [P_1] \rangle = q^{-2}$, and $\langle [L_3], [P_1] \rangle = q - q^{-1}$. Note that at the value $q = 1$ one recovers the symmetric ungraded Euler form, so that $\langle [L_j], [P_i] \rangle_{q=1} = \delta_{ji}$.

The standard modules in this example are $V_1 = P_1$, $V_2 = P_2$, and $V_3 = L_3$; the costandard modules are $\Lambda_1 = I_1$, $\Lambda_2 = I_2$ and $\Lambda_3 = L_3$. The classes of the costandard modules form a right dual basis to the classes of the standard modules. Note that in the ungraded case, when $q = 1$, the involution d is the identity on K_0 , and hence $[V_i]_{q=1} = [\Lambda_i]_{q=1}$. That is, in the ungraded case both the standard and the costandard modules form orthonormal bases for K_0 at $q = 1$. For general q , however, the Gram matrix in the basis $\{V_1, V_2, V_3\}$ is:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ q - q^{-1} & q - q^{-1} & 1 \end{bmatrix}$$

This pattern holds more generally for the pairings between graded standard modules for bipartite algebras: $\langle [V_i], [V_j] \rangle = \delta_{ij}$ unless $i \in V_1(B)$, $j \in V_0(B)$ and i is adjacent to j in B , in which case $\langle [V_i], [V_j] \rangle = q - q^{-1}$.

Proposition 3.1. *If L is a q -lattice with \mathcal{B} a basis, then the Gram matrix $A_{\mathcal{B}}$ coincides with the matrix of the embedding $L \hookrightarrow L^\vee$ with respect to the bases \mathcal{B} and \mathcal{B}^\vee . The matrix of the embedding $L \hookrightarrow {}^\vee L$ with respect to the bases \mathcal{B} and ${}^\vee \mathcal{B}$ coincides with the conjugate $\overline{A}_{\mathcal{B}}$, that is, the involution $q \mapsto q^{-1}$ applied to each matrix entry of $A_{\mathcal{B}}$.*

Proof. Like the classical case, this is elementary linear algebra. We need to show that $b_i = \sum_{j=1}^n \langle b_j, b_i \rangle b_j^\vee$, and $b_i = \sum_{j=1}^n \overline{\langle b_i, b_j \rangle} b_j$, for each $i = 1, \dots, n$. For the first equality, take the pairing on the left with each basis vector b_k ; for the second, do the same but on the right. \square

Example 3.4. As before, let K_0 denote the Grothendieck group of finitely generated graded $A(B)$ -modules, a q -lattice. Let G denote the Gram matrix in the basis given by the classes of indecomposable projective modules $\{[P_i]\}$. Simple modules form a right dual basis to this, so Proposition 3.1 means that the i -th column of G is the vector $[P_i]$ written in the basis $\{[L_i]\}$. Applying d , one obtains that the matrix whose i -th column is the class of the injective module $[I_i]$, written in the basis $\{[L_i]\}$, is \overline{G} , where the bar denotes the involution $q \mapsto q^{-1}$. In turn, via changes of bases, this implies that the Gram matrix in the basis $\{[L_i]\}$ is \overline{G}^{-1} , and the Gram matrix in the basis $\{[I_i]\}$ coincides with G .

Definition 3.3. A q -lattice is *unimodular* if $L^\vee = L$.

Luckily, we don't need to distinguish between left and right unimodularity: even without the assumption of the existence of the involution d , it is true that $L^\vee = L$ if and only if ${}^\vee L = L$. To see this, observe that if $L^\vee = L$ for a q -lattice L with basis \mathcal{B} , then \mathcal{B}^\vee is also a basis for L . Since, by definition, ${}^\vee(\mathcal{B}^\vee) = \mathcal{B}$, it follows that ${}^\vee L = L$. Observe also that a q -lattice is unimodular if and only if the determinant (of the Gram matrix) is a unit in $\mathbb{Z}[q, q^{-1}]$, that is, $\det L = \pm q^k$.

3.3. q -Lattice gluing. In classical lattice theory, *lattice gluing* is an important construction which allows for producing larger, indecomposable lattices from smaller components, see for example [CS, Chapter 4.3]. To glue two lattices L_1 and L_2 , one takes their direct sum, and then adjoins carefully selected elements of $L_1^\vee \oplus L_2^\vee$ to obtain an integer lattice. Of course the construction can be generalised to more than two glued components.

Lattice gluing is of particular interest when the resulting lattice is *unimodular*, and used in the classification of unimodular lattices of small rank [CS, Chapter 16]. It is therefore important to understand when it may be possible to glue two lattices together so that the end result is unimodular. In this case, the original lattices are embedded as mutual orthogonal complements in the unimodular lattice. A theorem (this formulation due to [BHN], also in [CS]) provides an important necessary condition for when this is possible:

Theorem 3.2. [BHN] *Let L be a unimodular lattice, and L_1 and L_2 sub-lattices which are mutual orthogonal complements of each other within L . Then*

- (1) *The images of the orthogonal projections of L onto $L_1 \otimes_{\mathbb{Z}} \mathbb{Q}$ and $L_2 \otimes_{\mathbb{Z}} \mathbb{Q}$ are L_1^\vee and L_2^\vee , respectively.*
- (2) *The glue groups of L_1 and L_2 are isomorphic, that is, $L_1^\vee/L_1 \cong L_2^\vee/L_2$.*
- (3) *The determinants of L_1 and L_2 are equal.*

We end this section with the q -analogue of this theorem:

Theorem 3.3. *Let L be a unimodular q -lattice and let L_1 and L_2 be sub-lattices such that L/L_1 and L/L_2 are free $\mathbb{Z}[q, q^{-1}]$ -modules. Assume further that $L_1^\perp = L_2$ (L_2 is the right orthogonal complement of L_1 in L), and $L_1 = {}^\perp L_2$ (L_1 is the left orthogonal complement of L_2 in L). Then:*

- (1) Any $x \in L$ can be uniquely written as a sum $x = x_1 + x_2$ where $x_1 \in L_1 \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q)$ and $x_2 \in L_2 \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q)$. This defines left/right orthogonal projections

$$\pi_1 : L \rightarrow L_1 \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q), \quad \pi_1(x) = x_1$$

and

$$\pi_2 : L \rightarrow L_2 \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q), \quad \pi_2(x) = x_2.$$

Then the image of π_1 is L_1^\vee and the image of π_2 is ${}^\vee L_2$.

- (2) $L_1^\vee/L_1 \cong {}^\vee L_2/L_2$.
(3) The determinants of L_1 and L_2 are equal up to units.

Proof. We prove the first statement for the projection π_1 . Since L/L_1 is free, every basis $\mathcal{B}_1 = \{b_1, \dots, b_r\}$ of L_1 can be extended to a basis $\mathcal{B} = \{b_1, \dots, b_r, c_{r+1}, \dots, c_n\}$ of L .

Since L is unimodular, the right dual basis $B^\vee = \{b_1^\vee, \dots, b_r^\vee, c_{r+1}^\vee, \dots, c_n^\vee\}$ is also a basis for L . Note that for $j > r$, $c_j^\vee \in (L_1 \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q))^\perp = L_2 \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q)$. Hence, $\pi_1(c_j^\vee) = 0$, and therefore $\{\pi_1(b_j^\vee) : j \leq r\}$ spans the image of π_1 .

For $j \leq r$, $b_j^\vee = \pi_1(b_j^\vee) + \pi_2(b_j^\vee)$. Then, for any $k \leq j$, $\langle b_k, \pi_2(b_j^\vee) \rangle = 0$, since $\pi_2(b_j^\vee) \in (L_1 \otimes \mathbb{Q}(q))^\perp$. So $\langle b_k, \pi_1(b_j^\vee) \rangle = \langle b_k, b_j^\vee \rangle = \delta_{jk}$, so $\{\pi_1(b_j^\vee) : j \leq r\}$ is a right dual basis to \mathcal{B}_1 . This completes the proof of statement (1) for π_1 , and the case of π_2 is similar.

For the second statement, we will prove that $L_1^\vee/L_1 \cong L/(L_1 \oplus L_2) \cong {}^\vee L_2/L_2$. Consider the map of $\mathbb{Z}[q, q^{-1}]$ -modules $\bar{\pi}_1 : L/(L_1 \oplus L_2) \rightarrow L_1^\vee/L_1$, given by $[x] \mapsto [\pi_1(x)]$, where $x \in L$, and square brackets denote cosets. The map $\bar{\pi}_1$ is well defined since $\pi_1(L_1 \oplus L_2) = L_1$.

It is also injective: if $[\pi_1(x)] = 0$, then $\pi_1(x) \in L_1$. Now, for all $x \in L$, $x = \pi_1(x) + \pi_2(x)$, with $\pi_2(x) \in {}^\vee L_2$. But L_2 is the right orthogonal complement of L_1 in L , so if $x \in L$ and $\pi_1(x) \in L_1$, then $\pi_2(x) \in L_2$. Thus $[\pi_1(x)] = 0$ implies $x \in L_1 \oplus L_2$, hence $\bar{\pi}_1$ is injective. In addition, $\bar{\pi}_1$ is surjective because the image of π_1 is L_1^\vee . This completes the proof of the first isomorphism, and the second is similar.

For statement (3), choose bases $\mathcal{B}_1 = \{b_1, \dots, b_r\}$ for L_1 and $\mathcal{B}_2 = \{b_{r+1}, \dots, b_n\}$ for L_2 . By part (1), the image of π_1 is L_1^\vee , and the kernel of π_1 is L_2 . Furthermore, by assumption L/L_2 is free. So, given the dual basis $\mathcal{B}_1^\vee = \{b_1^\vee, \dots, b_r^\vee\}$, we can obtain a basis of L by choosing arbitrary vectors $\pi_1^{-1}(b_j^\vee)$ in the preimage of b_j^\vee and taking the union $\pi_1^{-1}(\mathcal{B}_1^\vee) \cup \mathcal{B}_2$, where $\pi_1^{-1}(\mathcal{B}_1^\vee) = \{\pi_1^{-1}(b_j^\vee) : j \leq r\}$. Similarly, after choosing vectors $\pi_2^{-1}({}^\vee b_k)$, we have that $\mathcal{B}_1 \cup \pi_2^{-1}({}^\vee \mathcal{B}_2)$ also forms a basis of L , where $\pi_2^{-1}({}^\vee \mathcal{B}_2) = \{\pi_2^{-1}({}^\vee b_k) : r < k \leq n\}$.

Consider the q -lattice dilations D_1 and D_2 ,

$$\begin{array}{ccc} L & \xrightarrow{D_1} & L_1 \oplus L_2 \\ & \uparrow D_2 & \\ & L & \end{array}$$

where D_1 is defined by taking the ordered basis vectors $\pi_1^{-1}(\mathcal{B}_1^\vee) \cup \mathcal{B}_2$ to the ordered basis $\mathcal{B}_1 \cup \mathcal{B}_2$; similarly D_2 maps $\mathcal{B}_1 \cup \pi_2^{-1}({}^\vee \mathcal{B}_2)$ to $\mathcal{B}_1 \cup \mathcal{B}_2$. After extending scalars to $\mathbb{Q}(q)$, note that now $L \otimes \mathbb{Q}(q) = L_1 \otimes \mathbb{Q}(q) \oplus L_2 \otimes \mathbb{Q}(q)$, and $D_1 \otimes \text{id}$ and $D_2 \otimes \text{id}$ are vector space

automorphisms. (We abuse notation and write $D_1 \otimes \text{id}$ and $D_2 \otimes \text{id}$ as simply D_1 and D_2):

$$\begin{array}{ccc} L \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q) & \xrightarrow{D_1} & (L_1 \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q)) \oplus (L_2 \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q)) \\ & \swarrow D_1^{-1} \circ D_2 & \uparrow D_2 \\ & & L \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q) \end{array}$$

In particular, by composing D_2 with D_1^{-1} we obtain an automorphism of $L \otimes \mathbb{Q}(q)$, which restricts to a lattice automorphism of L . The determinant of a q -lattice automorphism is a unit in $\mathbb{Z}[q, q^{-1}]$, that is, the determinant of $D_1^{-1} \circ D_2$ is equal to $\pm q^k$ for some integer k . Therefore, the determinants of D_1 and D_2 agree up to units.

What remains is to compute these determinants. We accomplish this by writing D_1 , as a matrix in the basis $\mathcal{B}_1^\vee \cup \mathcal{B}_2 = \{b_1^\vee, \dots, b_r^\vee, b_{r+1}, \dots, b_n\}$. We know that D_1 is the identity when restricted to $L_2 \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q)$. Furthermore, for $1 \leq i \leq r$, we have $b_i^\vee = \pi_1^{-1}(b_i^\vee) + \beta_i$, for some $\beta_i \in L_2 \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q)$, and so $D_1(b_i^\vee) = b_i + \beta_i$.

Proposition 3.1 implies that the matrix of the linear map

$$\begin{aligned} L_1 \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q) &\rightarrow L_1 \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Q}(q) \\ b_i^\vee &\mapsto b_i, \end{aligned}$$

written in the basis \mathcal{B}_1^\vee , is the Gram matrix A_{L_1, \mathcal{B}_1} of L_1 . So the matrix of D_1 is a block diagonal matrix $\left[\begin{array}{c|c} A_{L_1, \mathcal{B}_1} & 0 \\ \hline * & \text{id} \end{array} \right]$.

Therefore, the determinant of D_1 is equal to the determinant of L_1 . Similarly, the determinant of D_2 is equal to the determinant of L_2 , completing the proof. \square

Remark 3.5. We note here that the proof of the above theorem did not use the existence of the involution $d : L \longrightarrow L$ deforming the identity. Note also that in the classical Theorem 3.2, statement (3) is almost immediate from (2): the glue groups are finite abelian groups, and the determinant is the size of the glue group by a geometric counting argument. In Theorem 3.3, the statements remain true but the proof is more involved, as the left and right glue groups are torsion $\mathbb{Z}[q, q^{-1}]$ -modules.

In Theorem 3.2 the mutual orthogonal complement condition implies that the abelian group L/L_i is torsion free for $i = 1, 2$, which in turn implies that these abelian groups are free. In Theorem 3.3, the mutual orthogonal complement assumption implies that the $\mathbb{Z}[q, q^{-1}]$ -modules L/L_i are torsion free, but it does not follow formally from this that they are free modules. This is why we assume they are free in the statement of Theorem 3.3 but not in Theorem 3.2.

4. THE q -LATTICES OF INTEGER CUTS AND FLOWS ASSOCIATED TO A SPANNING TREE

4.1. Classical cut and flow lattices. Let Γ be a finite graph, with loop edges and multiple edges allowed. For simplicity, we assume that Γ is 2-edge-connected, that is, Γ is connected and remains connected after the removal of any one edge. In this paper we'll abbreviate this and say that Γ is 2-connected.

Let $V = V(\Gamma)$ and $E = E(\Gamma)$ denote the vertex set and the edge set of Γ , respectively. Choose an arbitrary orientation ω for Γ (that is, a direction for each edge): this makes Γ

into a 1-dimensional CW-complex with cellular chain complex

$$0 \rightarrow C_1(\Gamma, \mathbb{Q}) \xrightarrow{\partial} C_0(\Gamma, \mathbb{Q}) \rightarrow 0,$$

where for an edge e that begins at vertex v and ends at w , $\partial(e) = w - v$. We equip $C_1(\Gamma, \mathbb{Q})$ and $C_0(\Gamma, \mathbb{Q})$ with inner products $\langle \cdot, \cdot \rangle$ by declaring E and V to be orthonormal bases. Then the adjoint of the boundary map, $\partial^* : C_0(\Gamma, \mathbb{Q}) \rightarrow C_1(\Gamma, \mathbb{Q})$ can be expressed by the formula $\partial^*(v) = \sum_{e \in E} \langle \partial(e), v \rangle e$.

If $|E| = n$, then $C_1(\Gamma, \mathbb{Q}) \cong \mathbb{Q}^n$ has an integer lattice $C_1(\Gamma, \mathbb{Z}) \cong \mathbb{Z}^n$ inside it. In $C_1(\Gamma, \mathbb{Z})$, the sublattices $\{\text{im}(\partial^*) \cap C_1(\Gamma, \mathbb{Z})\}$ and $\{\ker(\partial) \cap C_1(\Gamma, \mathbb{Z})\}$ are mutual orthogonal complements. They are called the *lattice of integer cuts* (the *cut lattice*) and the *lattice of integer flows* (the *flow lattice*), respectively, and each inherits the inner product restricted from $C_1(\Gamma, \mathbb{Z}) \cong \mathbb{Z}^n$. We denote the cut lattice and the flow lattice by $\mathcal{C}(\Gamma)$ and $\mathcal{F}(\Gamma)$, respectively. We denote the inner products by $\langle \cdot, \cdot \rangle_{\mathcal{C}(\Gamma)}$ and $\langle \cdot, \cdot \rangle_{\mathcal{F}(\Gamma)}$ respectively, dropping the subscripts whenever it does not cause confusion.

Note that even though the definition of $\mathcal{C}(\Gamma)$ and $\mathcal{F}(\Gamma)$ depends on the choice of an orientation, changing the orientation does not change the isomorphism type of the lattices. Let ω' be another orientation of Γ that differs from ω by switching the direction of a single edge e , and let $\mathcal{C}'(\Gamma)$ and $\mathcal{F}'(\Gamma)$ be the cut- and flow lattices corresponding to the orientation ω' . The isomorphism $C_1(\Gamma) \xrightarrow{\cong} C_1(\Gamma)$, which sends e to $-e$, and all other edges to themselves induces isomorphisms $\mathcal{C}(\Gamma) \cong \mathcal{C}'(\Gamma)$ and $\mathcal{F}(\Gamma) \cong \mathcal{F}'(\Gamma)$.

In combinatorial terms, the cut lattice is generated by the *cuts* of Γ , as follows. Given a partition $V = V_0 \cup V_1$, the corresponding *cut* is the signed sum of the edges connecting V_0 to V_1 , where each edge oriented from V_0 towards V_1 participates with positive sign, and edges oriented from V_1 towards V_0 participate with negative sign.

The flow lattice, in turn, is generated by oriented *cycles* (that is, closed walks) in Γ . Each oriented cycle gives rise to an element of $\mathcal{F}(\Gamma)$: a signed sum of the edges of the cycle, in which edges participate with positive sign if their orientation agrees with the orientation of the cycle, and with negative sign otherwise.

This combinatorial description leads to a (well-known) construction of bases for $\mathcal{C}(\Gamma)$ and $\mathcal{F}(\Gamma)$. Fix a spanning tree T of Γ . Removing an edge $e \in T$ splits T into two connected components. This defines a vertex partition $V = V_{0,e} \cup V_{1,e}$, where e is directed from $V_{0,e}$ towards $V_{1,e}$. Denote the corresponding cut by

$$K_e = \sum_{e' \in E_{V_0 \rightarrow V_1}} e' - \sum_{e'' \in E_{V_1 \rightarrow V_0}} e''.$$

This is called the *fundamental cut* corresponding to the edge $e \in T$. Note that the only spanning tree edge appearing in K_e is e and it always appears with positive sign. The set of fundamental cuts $\{K_e : e \in T\}$ forms a basis for $\mathcal{C}(\Gamma)$.

As for the flow lattice, every edge $f \notin T$ creates a single cycle when added to T , called the *fundamental cycle* of f , and denoted C_f . The corresponding basis element of $\mathcal{F}(\Gamma)$ is

$$C_f = \sum_{f' \in C_f} \pm f'.$$

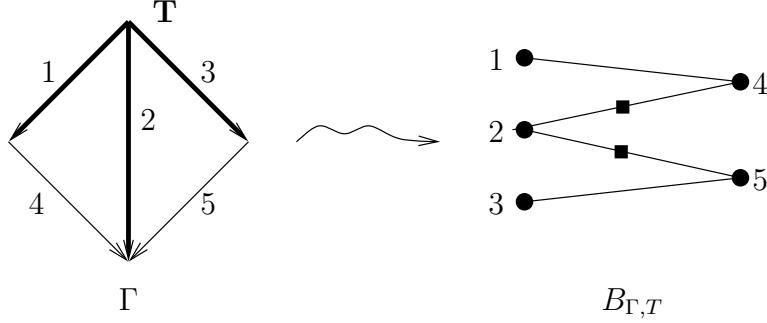


FIGURE 1. The bipartite graph $B_{\Gamma,T}$ associated to a graph. Spanning tree edges of Γ are drawn bold, negative edges of $B_{\Gamma,T}$ are marked with a square.

Here, the signs in the sum are assigned as follows: the orientation of f in ω induces an orientation of the cycle C_f , and each edge $f' \in C_f$ appears with a positive sign if its orientation agrees with the orientation of C_f , and with a negative sign otherwise. The set of fundamental cycles $\{C_f : f \in T^c = E \setminus T\}$ forms a basis of $\mathcal{F}(\Gamma)$.

4.2. Bipartite algebras from graphs. Let Γ be a 2-connected graph with numbered edge set $E = \{1, 2, \dots, n\}$ and orientation ω , and fix a spanning tree $T \subset E$. Define a bipartite graph $B_{\Gamma,T}$ whose *vertex set* is the edge set of Γ , with the bipartition

$$V_0(B_{\Gamma,T}) = T, \quad V_1(B_{\Gamma,T}) = E \setminus T.$$

The edges of $B_{\Gamma,T}$ are defined as follows. Connect the vertex $j \in V_1(B_{\Gamma,T})$ to exactly those vertices $i \in V_0(B_{\Gamma,T})$ which occur in the fundamental cycle C_j of j , as shown on an example in Figure 1.

Next, we decorate the edges of $B_{\Gamma,T}$ with signs arising from the orientation ω . Consider an edge f in the bipartite graph $B_{\Gamma,T}$. This edge corresponds to a pair of edges (i, j) in Γ , with $i \in T$, $j \notin T$. The edge i participates in C_j with a positive or negative sign: this is the edge sign associated to f . In Figure 1 we denote negative edges of $B_{\Gamma,T}$ by a small square placed on the edge.

Note that the bases for $\mathcal{C}(\Gamma)$ and $\mathcal{F}(\Gamma)$ associated to the spanning tree T can be read off $B_{\Gamma,T}$. It is a short combinatorial exercise to show that the edge i appears in the fundamental cycle C_j , if and only if j appears in the fundamental cut K_i . Furthermore, their signs are always opposite: i appears in C_j with positive sign if and only if j appears in K_i with negative sign.

While changing the orientation ω of Γ does not change the isomorphism types of $\mathcal{C}(\Gamma)$ and $\mathcal{F}(\Gamma)$, it has an effect on the edge signs of $B_{\Gamma,T}$. Namely, changing the orientation of an edge i of Γ flips the sign of all the edges of $B_{\Gamma,T}$ incident to the vertex i , but it does not otherwise change the bipartite graph.

Set $A_{\Gamma,T} := A(B_{\Gamma,T})$, as defined in Section 2. The edge signs of $B_{\Gamma,T}$ turn the double quiver $Q(B_{\Gamma,T})$ into a signed quiver (both directions of an edge inherit the same sign). This induces a \mathbb{Z}_2 -grading on the path algebra: the \mathbb{Z}_2 -degree of a path is the parity of the number of negative edges in it. Since the relations are homogeneous, there is an induced \mathbb{Z}_2 -grading on $A_{\Gamma,T}$, and on finitely generated $A_{\Gamma,T}$ -modules. The Groethendieck group $K_0(A_{\Gamma,T}\text{-mod})$ is now a free module over the ring $\mathbb{Z}[q, q^{-1}, t]/(t^2 = 1)$, and the graded Euler form is valued in $\mathbb{Z}[q, q^{-1}, t]/(t^2 = 1)$, and is t -bilinear and q -sesquilinear. The substitutions

$t = 1$ and $t = -1$ both turn the Grothendieck group into a q -lattice. From here on we will also refer to the \mathbb{Z}_2 -grading as the t -grading.

The statements of Section 2 hold with $t = 1$, and otherwise are easily modified to include the t -grading. The simple modules L_i are contained in t -degree 0. The indecomposable projective modules P_i are naturally t -graded, as their elements can be viewed as paths. Homomorphisms between indecomposable projective modules are spanned by paths as in Proposition 2.3, and t -graded accordingly. In Proposition 2.4, projective resolutions can be adjusted to account for the t -grading by inserting t -shifts whenever the edge “leading to” an indecomposable projective module has a negative sign. For instance, in the example of Figure 1, the projective resolution of L_4 is $P_2\{1\}\langle 1 \rangle \oplus P_1\{1\} \rightarrow P_4$, where $\langle \cdot \rangle$ denotes the t -grading shift. The functor d respects the t -grading, and descends to a t -linear, q -anti-linear map on the Grothendieck group.

Proposition 4.1. *For edges $i, j \notin T$ the pairing $\langle [P_i], [P_j] \rangle|_{q=1, t=-1}$ agrees with the pairing $\langle C_i, C_j \rangle_{\mathcal{F}(\Gamma)}$ of the corresponding fundamental cycles in the flow lattice of Γ .*

Proof. Since P_i and P_j are projective modules, the Ext in the definition of the graded Euler pairing reduces to Hom. So we need to show that when $i, j \notin T$,

$$q\text{tdim Hom}(P_i, P_j)|_{q=1, t=-1} = \langle C_i, C_j \rangle_{\mathcal{F}(\Gamma)}.$$

As shown in Proposition 2.3, a basis for $\text{Hom}(P_i, P_j)$ is given by paths from e_i to e_j in $B_{\Gamma, T}$ all of which are of length 2, except for the length zero path e_i in the case $i = 1$. The length 2 paths, in turn, correspond to T -edges in common between C_i and C_j . The substitution $t = -1$ gives these T -edges signs according to whether an edge participates in C_i and C_j with the same sign or opposite signs. This is exactly how the pairing $\langle C_i, C_j \rangle_{\mathcal{F}(\Gamma)}$ is defined. \square

Remark 4.1. In light of Example 3.4, we may substitute indecomposable injective modules for the indecomposable projective modules in the statement of Proposition 4.1. Specifically, for any signed bipartite graph B and modules $M, N \in A(B)\text{-mod}$, in $K_0(A(B)\text{-mod})$ we have $\langle [M], [N] \rangle(q, t) = \langle [dN], [dM] \rangle(q^{-1}, t)$. Hence, for $e_i, e_j \notin T$, and indecomposable injective $A_{\Gamma, T}$ -modules I_i and I_j , $\langle [I_i], [I_j] \rangle|_{q=1, t=-1} = \langle C_i, C_j \rangle_{\mathcal{F}(\Gamma)}$.

The following is a dual statement to Proposition 4.1, which we will prove in Section 4.3:

Proposition 4.2. *For edges $i, j \in T$ the pairing $\langle [L_i], [L_j] \rangle|_{q=1, t=-1}$ agrees with the pairing $\langle K_i, K_j \rangle_{\mathcal{C}(\Gamma)}$ of the corresponding fundamental cuts in the cut lattice of Γ .*

4.3. Koszul duality and matroid duality. Rather than proving Proposition 4.2 directly we explain how it is dual to the statement of Proposition 4.1. Proposition 2.5 implies that if $B^!$ denotes the bipartite graph obtained from B by switching the bipartition, then $A(B^!) \cong A^!(B)$, where $A^!(B)$ is the Koszul dual of $A(B)$.

If B is a signed bipartite graph, we set the convention that $B^!$ is the bipartite graph obtained from B by switching the bipartition and *all edge signs*. This induces a t -grading on $A(B^!) \cong A^!(B)$.

4.3.1. Koszul duality and Grothendieck groups. By Theorem 1.2.6 of [BGS], there is a Koszul duality derived equivalence between *derived categories of graded modules*

$$K : \mathcal{D}^b(A(B)\text{-mod}) \rightarrow \mathcal{D}^b(A(B^!)\text{-mod}).$$

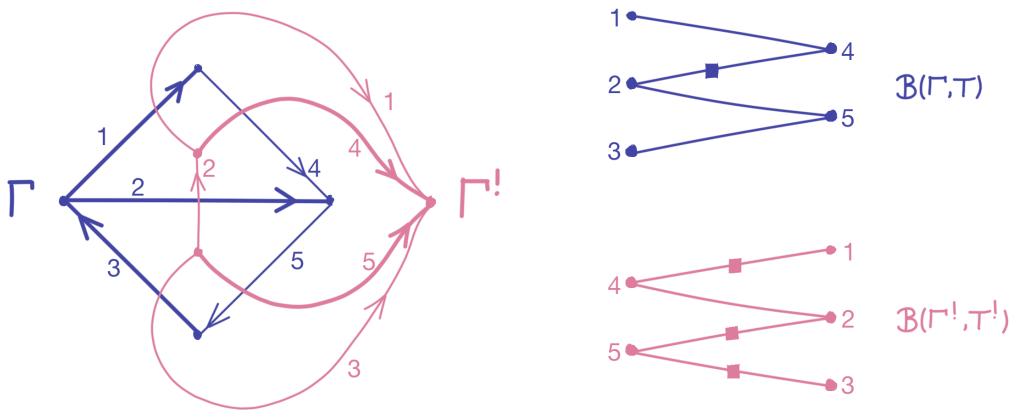


FIGURE 2. On the left, we show an oriented graph Γ with spanning tree T shown in thick edges, and its oriented planar dual $\Gamma^!$, with spanning tree $T^!$ complementary to T . The corresponding signed bipartite graphs are shown on the right: $B_{\Gamma^!, T^!}$ is obtained from $B_{\Gamma, T}$ by flipping the bipartition and all edge signs. As an example, the fundamental cycle $C_2^!$ and the fundamental cut K_2 are both given by “ $2 + 4 - 5$ ”.

If M is a module over $A(B)$, we write $M \in \mathcal{D}^b(A(B)\text{-mod})$ and mean the complex whose homological degree 0 chain module is M , and whose other chain modules are zero; and similarly for modules over $A(B^!)$.

If $M \in A(B)\text{-mod}$, $\{\cdot\}$ denotes the internal grading shift, and $[\cdot]$ denotes the homological grading shift, then $K(M\{n\}) = (KM)[-n]\{-n\}$. On the other hand, K commutes with t -shifts.

For $i = 1, \dots, n$, K sends the simple module L_i over $A_{\Gamma, T}$ to the indecomposable projective module $P_i^!$ over the Koszul dual $A_{\Gamma, T}^!$.

The Grothendieck group of $\mathcal{D}^b(A(B)\text{-mod})$ coincides with $K_0(A(B)\text{-mod})$ via the graded Euler characteristic, that is, the alternating sum of graded dimensions of chain modules. Koszul duality induces a \mathbb{Z} -linear map

$$K : K_0(A(B)\text{-mod}) \rightarrow K_0(A(B^!)\text{-mod}),$$

determined by $K[L_i] = [P_i^!]$, $K(q) = -q^{-1}$, and $K(t) = t$. Furthermore, we have

$$\langle [M], [N] \rangle = \langle K(M), K(N) \rangle|_{q \rightarrow -q^{-1}}.$$

4.3.2. Planar Graph Duality and Matroid Duality. If Γ is an oriented planar graph with spanning tree T , and $\Gamma^!$ its oriented planar dual, then there is a canonical bijection $\varphi : E(\Gamma) \rightarrow E(\Gamma^!)$, and $E(\Gamma^!) \setminus \varphi(T)$ forms a spanning tree for $\Gamma^!$. It is a short exercise to check that the bipartite graph $B_{\Gamma^!, E(\Gamma) \setminus T}$ coincides with $(B_{\Gamma, T})^!$. See Figure 2 for an example.

Furthermore, $\mathcal{F}(\Gamma) \cong \mathcal{C}(\Gamma^!)$ canonically, and vice versa. For $i \in T$, $K_i = C_i^!$, where K_i denotes the fundamental cut for i in $\mathcal{C}(\Gamma)$, and $C_i^!$ denotes the fundamental cycle for i in $\mathcal{F}(\Gamma^!)$. For $i \notin T$, $C_i = K_i^!$.

Proof of Proposition 4.2 for planar graphs. From Proposition 4.1 for the dual graph $\Gamma^!$, we have that for Γ -edges $i, j \in T$,

$$\langle [P_i^!], [P_j^!] \rangle|_{q=1, t=-1} = \langle C_i^!, C_j^! \rangle_{\mathcal{F}(\Gamma^!)}. \quad 17$$

On the left, Koszul duality gives

$$\langle [L_i], [L_j] \rangle|_{q=1, t=-1} = \langle [P_i^!], [P_j^!] \rangle|_{q=-1, t=-1} = \langle [P_i^!], [P_j^!] \rangle|_{q=1, t=-1},$$

since pairings between the classes of “same side” indecomposable projective modules are functions of q^2 .

On the right, planar graph duality gives

$$\langle C_i^!, C_j^! \rangle_{\mathcal{F}(\Gamma^!)} = \langle K_i, K_j \rangle_{\mathcal{C}(\Gamma)},$$

and combining the two sides one obtains the statement of Proposition 4.2. \square

If Γ is not planar, there is no dual graph, though there is still a Koszul dual bipartite algebra, and a dual (flipped) bipartite graph. In this case these dualities correspond to the *matroid dual* of the oriented graphical matroid associated to Γ . Thus in fact the most natural language for describing the combinatorial origin for Koszul duality of bipartite algebras is to assign a bipartite graph and bipartite algebra to any *oriented regular matroid* with a matroid basis, rather than just to a graph with a spanning tree. To simplify language in this paper, we will not present the details of the construction for regular matroids, and instead define the lattices of integer cuts and flows at the bipartite graph level. However, for the reader who is interested in matroid statements, we give a brief summary of the basic points here.

Each oriented regular matroid has a regular *oriented matroid dual*. Each oriented graph has an associated oriented *graphic matroid*, which is always regular, and for planar graphs the dual matroid of the graphic matroid agrees with the graphic matroid of the planar dual graph. For non-planar graphs, the dual of the graphic matroid is a regular, non-graphic matroid.

Regular matroids have associated lattices of integer cuts and flows, and everything in this paper can be extended to that context without difficulty. In particular, the analogue of a spanning tree is a *basis* of the matroid, and the analogue of a spanning tree basis for $\mathcal{C}(\Gamma)$ or $\mathcal{F}(\Gamma)$ is called the *fundamental basis* of the cut or flow lattice *associated to a basis of the matroid*. The complement of a basis for an oriented regular matroid forms a basis for the dual matroid, and matroid duality exchanges cut and flow lattices. Hence, the planar graph statements above also apply to regular matroids.

4.3.3. Cuts and Flows for Bipartite Graphs. Given a signed bipartite graph B with numbered vertex set $E = E_0 \cup E_1$, let \mathbb{Z}^E denote the Euclidean lattice generated by elements of E (ie, elements of E form an orthonormal basis). The flow lattice $\mathcal{F}(B)$ of B is the sublattice of \mathbb{Z}^E generated by fundamental cycles C_i associated to each $i \in E_1$ as follows:

$$C_i = i + \sum_{j \in N_i} \epsilon_{ij} j,$$

where N_i denotes the set of neighbours of i and ϵ_{ij} is the sign of the edge $i-j$ in B . Similarly, the cut lattice $\mathcal{C}(B)$ of B is the sublattice of \mathbb{Z}^E spanned by fundamental cuts K_i for $i \in E_0$, where

$$K_i = i - \sum_{j \in N_i} \epsilon_{ij} j.$$

It is clear that for a bipartite graph arising from a graph with a spanning tree, $\mathcal{F}(B_{\Gamma,T}) \cong \mathcal{F}(\Gamma)$ and $\mathcal{C}(B_{\Gamma,T}) \cong \mathcal{C}(\Gamma)$, and the isomorphisms preserve fundamental cuts and flows.

Furthermore, $\mathcal{F}(B) \cong \mathcal{C}(B^!)$ and $\mathcal{C}(B) \cong \mathcal{F}(B^!)$. Proposition 4.1 and its proof remains true for flow lattices of signed bipartite graphs.

Proof of Proposition 4.2. The proof for planar graphs generalises directly, replacing T with E_0 , and planar graph duality with bipartite graph duality. \square

4.3.4. Bipartite graphs and classes of matroids. For those interested in matroid aspects, we clarify the relationship between graphs, signed bipartite graphs, and various classes of matroids.

From an oriented graph Γ with a choice of spanning tree T , we constructed a signed bipartite graph. From the pair (Γ, T) , one also obtains an oriented graphic matroid $\mathcal{M}(G)$ with a chosen basis \mathcal{B} . It is possible to construct the signed bipartite graph $B_{\Gamma, T}$ from $\mathcal{M}(G)$: the vertices correspond to the base set E of the matroid, partitioned into the basis and non-basis elements. Signed edges incident to a non-basis element are drawn according to the fundamental signed circuit in the fundamental basis of $\mathcal{F}(\mathcal{M}(G))$ corresponding to \mathcal{B} . In fact, this construction of signed bipartite graphs from matroids works not only for graphic matroids but for any oriented regular matroid with a chosen basis.

Conversely, for any signed bipartite graph B , we can construct an (oriented) matroid with a chosen basis. The base set of this matroid is the vertex set E of B , the basis is the E_1 side of the vertex partition, and circuits are generated by the fundamental circuits corresponding to elements of E_1 . This matroid is always \mathbb{Q} -representable, but not necessarily regular. For example, the complete bipartite graph on $2 + 2$ vertices, with one negative edge sign, gives rise to a matroid not representable over \mathbb{F}_2 .

In general, a \mathbb{Q} -representable matroid with a chosen basis would give rise to a bipartite graph with \mathbb{Z} -weighted edges as opposed to a signed bipartite graph. So signed bipartite graphs can be seen as a class of matroids between regular matroids and \mathbb{Q} -representable matroids.

4.4. The q -cut and q -flow lattices. For a signed bipartite graph B with vertex set $E = E_0 \cup E_1$, the Groethendieck group $K_0(A(B)\text{-mod})|_{t=-1}$ is a q -lattice in the sense of Section 3, with the graded Euler form at $t = -1$, and the involution d of Remark 2.1. We define the q -cut and q -flow lattices as the appropriate q -sublattices of the Groethendieck group. After giving the definition, we will see that it is also possible to define these objects combinatorially, without any reference to the bipartite algebra $A(B)$ or its representation category. The rest of this Section focuses on the intricate combinatorial properties of these new graph invariants.

Definition 4.1. For a signed bipartite graph B , with vertex set $E = E_0 \cup E_1$, the q -flow lattice $\mathcal{F}_q(B)$ is the $\mathbb{Z}[q, q^{-1}]$ -submodule of $K_0 = K_0(A_B\text{-mod})|_{t=-1}$ generated by the classes of projective modules $\{[P_i] : i \in E_1\}$, with the inherited sesquilinear form. The involution d is defined to be the \mathbb{Z} -linear map which fixes the classes $\{[P_i]\}$, and for which $d(q) = q^{-1}$.

Similarly, the q -cut lattice $\mathcal{C}_{q(B)}$ is generated by the classes of simple modules $\{[L_i] : i \in E_0\}$, with the inherited form, and $d([L_i]) = L_i$.

If $B = B_{\Gamma, T}$ for some graph Γ with chosen spanning tree T , then we call B *graphical*, and denote $\mathcal{F}_q(B_{\Gamma, T}) = \mathcal{F}_q(\Gamma, T)$, and $\mathcal{C}_q(B_{\Gamma, T}) = \mathcal{C}_q(\Gamma, T)$.

Note that in the case of the q -flow lattice, the involution d is the composition of the involution d on K_0 with the canonical pairing-preserving isomorphism between the $\mathbb{Z}[q, q^{-1}]$ -submodule generated by $\{[P_i] : i \in E_1\}$, and that generated by the classes of indecomposable injective modules $\{[I_i] : i \in E_1\}$.

In the classical case, for a graph (or regular matroid) Γ , $\mathcal{F}(\Gamma)$ and $\mathcal{C}(\Gamma)$ are mutual orthogonal complements in the Euclidean lattice $\mathbb{Z}^{E(\Gamma)}$. The q -analogue of this statement is the following:

Proposition 4.3. *In K_0 , $\mathcal{C}_q(B)$ is the right orthogonal complement to $\mathcal{F}_q(B)$, and $\mathcal{F}_q(B)$ is the left orthogonal complement of $\mathcal{C}_q(B)$.*

Proof. Recall that simple modules form a right dual basis to indecomposable projective modules with respect to the graded Euler pairing. That is, $\langle [P_i], [L_j] \rangle = \delta_{ij}$ for all $1 \leq i, j \leq n$. This implies that the right orthogonal complement $[P_i]^\perp$ of $[P_i]$ is the span of $\{[L_j] : j \neq i\}$, since the simple modules form a basis, and pairing with $[P_i]$ on the left picks out the coefficient of $[L_i]$. The right orthogonal complement of $\mathcal{F}_q(B)$ is the intersection $\bigcap_{e_i \in E_1} [P_i]^\perp$, that is, the span of $\{[L_j] : e_j \in E_0\}$, which is by definition $\mathcal{C}_q(B)$.

The same argument, using the fact that indecomposable projective modules also form a basis, proves that the left orthogonal complement of $\mathcal{C}_q(B)$ is $\mathcal{F}_q(B)$. \square

Remark 4.2. Given a signed bipartite graph B , with vertex set $E = E_0 \cup E_1$ it is possible to define the q -cut and q -flow lattices without reference to the category $A(B)\text{-mod}$. Namely, in the basis $\{[P_i]\}_{i \in E}$, the q -Gram matrix of the “Euclidean q -lattice” K_0 is of the form

$$G_{K_0, \{[P_i]\}} = \left[\begin{array}{c|c} I & C \\ \hline C^t & G_{\mathcal{F}} \end{array} \right].$$

Here I is the identity matrix of size $|E_0|$, and $C = qM_B$, where M_B denotes the signed adjacency matrix of B . That is, if $i \in E_0$ and $j \in E_1$ then C_{ij} is zero if i and j are not adjacent, q if they are adjacent via a positive edge, and $-q$ if adjacent via a negative edge of B . C^t is the transpose of C .

The matrix $G_{\mathcal{F}}$ is the *Gram matrix of the q -flow lattice* in the basis $\{[P_i]\}_{i \in E_1}$, namely, for $e_i, e_j \in E_1$ and $\langle \cdot, \cdot \rangle_{\mathcal{F}(B)}$ denoting the pairing in the classical flow lattice of B ,

$$(G_{\mathcal{F}})_{ij} = \begin{cases} 1 + (\langle C_i, C_i \rangle_{\mathcal{F}(B)} - 1)q^2 & \text{when } i = j \\ \langle C_i, C_j \rangle_{\mathcal{F}(B)}q^2 & \text{when } i \neq j. \end{cases}$$

The classes $[L_i]$ ($i \in E$) written in the basis $\{[P_i]\}_{i \in E}$ are the columns of $G_{K_0, \{[P_i]\}}^{-1}$. The involution d on K_0 is defined by q -anti-linearity and fixing the elements $\{[L_i] : i \in E\}$.

The q -flow lattice can be defined as the submodule spanned by $\{[P_i] : e_i \in E_1\}$, with the inherited form and the involution given by $d([P_i]) = [P_i]$, $d(q) = q^{-1}$.

The q -cut lattice is the right orthogonal complement of the q -flow lattice, or equivalently, the q -sublattice generated by $\{[L_i] : i \in E_0\}$. The Gram matrix of $\mathcal{C}_q(B)$ is given in the basis $\{[L_i] : i \in E_0\}$ is given by

$$(G_{\mathcal{C}})_{ij} = \begin{cases} 1 + (\langle K_i, K_i \rangle_{\mathcal{C}(B)} - 1)q^2 & \text{when } i = j \\ \langle K_i, K_j \rangle_{\mathcal{F}(B)}q^2 & \text{when } i \neq j. \end{cases}$$

Remark 4.3. It was an arbitrary choice to define the q -flow lattice via the basis of indecomposable projectives, and one equally reasonable choice would have been to use indecomposable injective modules (cf Remark 4.1). Then $\mathcal{C}_q(B)$ would be the *left* orthogonal complement of $\mathcal{F}_q(B)$ in K_0 . The Gram matrix would remain the same (but in the basis $\{[I_i]\}_{i \in E_1}$), and d would fix the elements $[I_i]$.

In classical lattice theory, if an orthonormal basis of a \mathbb{Z} -lattice exists, then this basis is unique up to signs and permutation of the basis vectors. Furthermore, if v is an element of norm ± 1 , then one of $\{v, -v\}$ is an element of this basis. There is a similar rigidity statement for a special class of q -lattices which have a basis satisfying certain conditions, but which need not be orthonormal. This includes in particular q -cut and q -flow lattices:

Lemma 4.4. *Suppose a q -lattice L of rank n has a basis $\{B_i : i = 1, \dots, n\}$ and that there exists $k \in \mathbb{Z}$ such that for all $i, j = 1, \dots, n$:*

- $\langle B_i, B_i \rangle = 1 + c_{ii}q^k$ with $c_{ii} \in \mathbb{Z}$, and
- $\langle B_i, B_j \rangle = \langle B_j, B_i \rangle = c_{ij}q^k$ for $i \neq j$ with $c_{ij} \in \mathbb{Z}$.

Then such a basis is unique up to permutation and signs: if for any $B \in L$, $\langle B, B \rangle = 1 + cq^k$ for some $c \in \mathbb{Z}$, then $B = \pm B_i$ for some i .

Proof. Assume that for some $B \in L$, $\langle B, B \rangle = 1 + cq^k$, and $B = \sum_{i=1}^n \alpha_i B_i$ for some $\alpha_i \in \mathbb{Z}[q, q^{-1}]$. Then

$$\langle B, B \rangle = \sum_{i,j=1}^n \bar{\alpha}_i \alpha_j \langle B_i, B_j \rangle = \left(\sum_{i=1}^n \bar{\alpha}_i \alpha_i \right) + q^k \left(\sum_{i=1}^n c_{ii} \bar{\alpha}_i \alpha_i + \sum_{i < j} c_{ij} (\bar{\alpha}_i \alpha_j + \bar{\alpha}_j \alpha_i) \right),$$

where $\bar{\alpha}_i(q) = \alpha_i(q^{-1})$. Denote

$$X(q) = \sum_{i=1}^n \bar{\alpha}_i \alpha_i, \quad \text{and} \quad Y(q) = \sum_{i=1}^n c_{ii} \bar{\alpha}_i \alpha_i + \sum_{i < j} c_{ij} (\bar{\alpha}_i \alpha_j + \bar{\alpha}_j \alpha_i).$$

Notice that X and Y are symmetric Laurent polynomials: $X(q) = X(q^{-1})$ and $Y(q) = Y(q^{-1})$. So we have

$$1 + cq^k = X + Yq^k.$$

Substituting q^{-1} for q we obtain

$$1 + cq^{-k} = X + Yq^{-k}.$$

Subtract the second equation from the first to get

$$c(q^k - q^{-k}) = Y(q^k - q^{-k}),$$

therefore $Y = c$ and $X = 1$.

Going back to the definition of X , note that the constant term of $\bar{\alpha}_i \alpha_i$ is a non-negative integer, and it is 0 if and only if $\alpha_i = 0$. Since $1 = X(q) = \sum_{i=1}^n \bar{\alpha}_i \alpha_i$, it must be that $\alpha_i = 0$ for all but one i , and for the one exception $\alpha_i = \pm 1$. Hence, $B = \pm B_i$ for some i , and as a result such a basis is unique up to permutation and signs. \square

Definition 4.2. Let L_q be a q -lattice with a given basis $\{B_i : i = 1 \dots n\}$. If for any element $x \in L_q$ the norm $\langle x, x \rangle$ determines whether $x = \pm B_i$ for some i , then we call L_q *Gram-rigid*.

In other words, Gram-rigid q -lattices have a “canonical basis” producing a Gram matrix of a certain form. The q -flow lattice of a signed bipartite graph is Gram-rigid with the basis $\{[P_i]\}_{i \in E_1}$, and the q -cut lattice is Gram rigid with the basis $\{[L_j]\}_{j \in E_0}$.

4.5. The q -cut and q -flow lattices and 2-isomorphisms of graphs. A theorem of Su–Wagner [SW] and Caporaso–Viviani [CV] states that the classical lattice of integer flows is a *complete 2-isomorphism invariant* of 2-edge-connected graphs. That is, $\mathcal{F}(\Gamma_1) \cong \mathcal{F}(\Gamma_2)$ if and only if there exists a cycle-preserving bijection between $E(\Gamma_1)$ and $E(\Gamma_2)$. Such a bijection is called a *two-isomorphism* of graphs. Dually, the lattice of integer cuts is a complete 2-isomorphism invariant of graphs without loops.

The Su–Wagner result is stated in the context of regular matroids, where matroid isomorphism replaces 2-isomorphisms of graphs, and 2-edge-connected graphs translate to matroids without co-loops. This result is useful not only in graph theory and matroid theory but also in low-dimensional topology [Gr].

The following q -analogue of this theorem shows that $\mathcal{F}_q(\Gamma, T)$ and $\mathcal{C}_q(\Gamma, T)$ are complete invariants of the two-isomorphism type of the pair (Γ, T) , in other words, the q -lattices “remember” the spanning tree:

Theorem 4.5. *For 2-edge-connected loopless graphs Γ_1 and Γ_2 with respective spanning trees T_1 and T_2 , the following are equivalent:*

- (1) $\mathcal{F}_q(\Gamma_1, T_1) \cong \mathcal{F}_q(\Gamma_2, T_2)$;
- (2) *There exists a cycle-preserving bijection $F : E(\Gamma_1) \rightarrow E(\Gamma_2)$ for which $F(T_1) = T_2$;*
- (3) $\mathcal{C}_q(\Gamma_1, T_1) \cong \mathcal{C}_q(\Gamma_2, T_2)$.

Proof. We first prove (2) \Rightarrow (1). It is a classical fact that the map F lifts to a map of lattices $\tilde{F} : \mathbb{Z}^{E(\Gamma_1)} \rightarrow \mathbb{Z}^{E(\Gamma_2)}$, where $\tilde{F}(e) = \pm F(e)$, such that \tilde{F} restricts to an isomorphism $\bar{\varphi} : \mathcal{F}(\Gamma_1) \xrightarrow{\cong} \mathcal{F}(\Gamma_2)$. Since $F(T_1) = T_2$, $\bar{\varphi}$ sends fundamental cycles corresponding to T_1 to fundamental cycles corresponding to T_2 , and therefore it lifts to an isomorphism $\varphi : \mathcal{F}_q(\Gamma_1, T_1) \xrightarrow{\cong} \mathcal{F}_q(\Gamma_2, T_2)$.

To prove (1) \Rightarrow (2), given an isomorphism $\varphi : \mathcal{F}_q(\Gamma_1, T_1) \xrightarrow{\cong} \mathcal{F}_q(\Gamma_2, T_2)$, set $q = 1$ to obtain an isomorphism $\bar{\varphi} : \mathcal{F}(\Gamma_1) \xrightarrow{\cong} \mathcal{F}(\Gamma_2)$. By a strong version of the Su–Wagner–Caporaso–Viviani theorem (see proof of [Gr, Thm.3.8]), $\bar{\varphi}$ extends to an isomorphism $F' : \mathbb{Z}^{E(\Gamma_1)} \rightarrow \mathbb{Z}^{E(\Gamma_2)}$, which sends each edge of Γ_1 to a signed edge of Γ_2 .

Forgetting signs, we obtain a cycle-preserving bijection $F' : E(\Gamma_1) \rightarrow E(\Gamma_2)$, which in particular sends T_1 -fundamental cycles to T_2 -fundamental cycles. Note that this does not imply that F' sends T_1 to T_2 . If an edge e of Γ_1 participates in at least two T_1 -fundamental cycles, then e is an edge of T_1 , and $F'(e)$ an edge of T_2 . On the other hand, if an edge $e \in T_1$ participates in a unique fundamental cycle, say C_i , then $F'(e)$ may be a non- T_2 edge of $F'(C_i)$.

However, if two edges $e_1, e_2 \in E(\Gamma_1)$ both only appear in the fundamental cycle C_i , then e_1 and e_2 appear together in *any cycle* of Γ_1 , and so the transposition of e_1 and e_2 is a cycle-preserving automorphism of $E(\Gamma_1)$. Hence, F' can always be composed with such transpositions to obtain a cycle preserving bijection $F : E(\Gamma_1) \rightarrow E(\Gamma_2)$ which sends T_1 to T_2 .

To prove (2) \Leftrightarrow (3), similar arguments can be made using fundamental cuts. \square

Note that, building on the Su–Wagner result, Theorem 4.5 can be generalised to regular matroids with a chosen basis, and then a duality argument implies (2) \Leftrightarrow (3).

The Su–Wagner/Caporaso–Viviani Theorem implies in particular that for 2-edge-connected loopless graphs the isomorphism class of $\mathcal{F}(\Gamma)$ determines the isomorphism class of $\mathcal{C}(\Gamma)$, and

vice versa. Similarly, Theorem 4.5 says that for these graphs with a choice of spanning tree, the isomorphism type of $\mathcal{F}_q(\Gamma, T)$ determines the isomorphism type of $\mathcal{C}_q(\Gamma, T)$ and vice versa. This, however, is not true in general for the q -cut and q -flow lattices associated to signed bipartite graphs whose associated matroid is non-regular: an example is shown in Figure 3. Note that for a bipartite graph “2-edge-connected” means there is no isolated vertex in $V(0)$, and “loopless” means there’s no isolated vertex in $V(1)$.

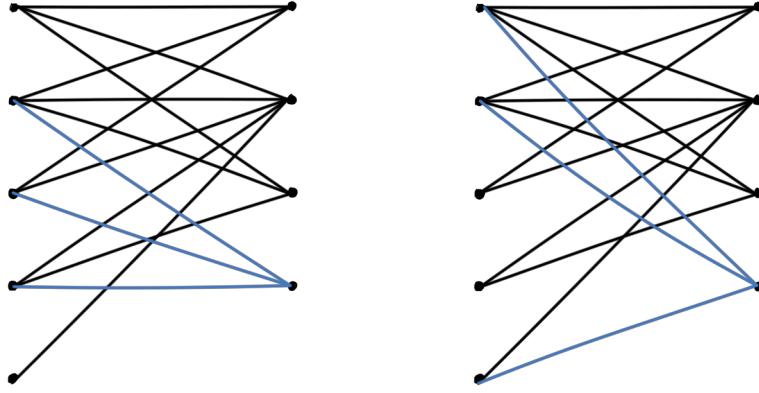


FIGURE 3. The isomorphism class of $\mathcal{F}_q(B)$ does not necessarily determine the isomorphism class of $\mathcal{C}_q(B)$ for signed bipartite graphs whose associated matroid is not regular: the bipartite graphs above have identical q -flow lattices, but non-isomorphic q -cut lattices (non-isomorphism is easily seen for Gram-rigid q -lattices). The blue edges highlight the difference between the two graphs.

4.6. A q -Matrix-Tree theorem. The classical integer cut and flow lattices *glue* together to form the Euclidean lattice \mathbb{Z}^E ; this implies through Theorem 3.2 that for any signed bipartite graph B , $\det \mathcal{F}(B) = \det \mathcal{C}(B)$. Analogously, the q -cut and q -flow lattices glue to form a unimodular q -lattice: they embed as mutual one-sided orthogonal complements in a unimodular q -lattice satisfying the conditions of Theorem 3.3. This implies that $\det \mathcal{F}_q(B) = \det \mathcal{C}_q(B)$, up to units.

In graph theory, the famous Matrix-Tree Theorem states that the determinant of the classical cut and flow lattices counts the number of spanning trees for a graph. (It also applies to regular matroids, where the determinant equals the number of bases.) Our final result is a q -analogue of this theorem, which further illustrates how the q -cut and q -flow lattices encode information about the choice of spanning tree. This theorem also applies to regular matroids, with the same proof; here we state and prove it for graphs.

Theorem 4.6. *Given a graph Γ and a spanning tree $T \subseteq E(\Gamma)$, set $r := |T|$. Then*

$$\det \mathcal{C}_q(\Gamma, T) = \sum_{i=0}^r c_i q^{2i},$$

where c_i is the number of spanning trees T' of Γ with $|T' \cap T| = r - i$. (In particular, $c_0 = 1$.)

This proof is based on the proof of the classical Matrix-Tree Theorem presented in [Bi, Chapter 6].

Proof. The plan for the proof is to define a matrix Q_0 , and prove that on one hand, $\det Q_0 = \sum_{i=0}^r c_i q^{2i}$, and on the other hand, that Q_0 is a q -Gram matrix for $\mathcal{C}_q(\Gamma, T)$.

Let D denote the signed “ q -incidence matrix” for (Γ, T) : the rows of D are indexed by $V(\Gamma)$, and the columns are indexed by $E(G)$, with T -columns preceding non- T columns. The entries of D are defined as follows:

$$D_{ij} = \begin{cases} 1 & \text{if the edge } j \in T \text{ ends at the vertex } i, \\ -1 & \text{if the edge } j \in T \text{ begins at the vertex } i, \\ q & \text{if the edge } j \notin T \text{ ends at the vertex } i, \\ -q & \text{if the edge } j \notin T \text{ begins at the vertex } i. \end{cases}$$

Define $Q := DD^t$. Q is a symmetric matrix whose rows and columns are indexed by $V(\Gamma)$. By a simple computation, $Q_{ii} = \alpha_i + \beta_i q^2$, where α_i is the number of T -edges incident to vertex i , and β_i is the number of non- T edges incident to vertex i . When $i \neq j$, let γ_{ij} denote the number of non- T edges between the vertices i and j , in either direction. If the vertices i and j are also connected by a single T -edge in either direction, then $Q_{ij} = -1 - \gamma_{ij} q^2$. Otherwise $Q_{ij} = -\gamma_{ij} q^2$.

Note that the rank of D is $r = |T|$, since the T -columns are linearly independent, and there are $r+1$ rows summing to zero. Hence, Q is a symmetric $(r+1) \times (r+1)$ matrix of rank r , so all cofactors of Q are the same. (A cofactor is the determinant of any $r \times r$ submatrix.) Let D_0 denote D with the last row deleted. Let $Q_0 := D_0 D_0^t$.

For any subset $J \subseteq E(\Gamma)$, $|J| = r$, let D_J denote the minor of D_0 which contains the columns in J . By the Binet-Cauchy Theorem, $\det Q_0 = \sum_J \det D_J \cdot \det D_J^t$. Observe that $\det D_J = 0$ if and only if J contains a cycle.

If J does not contain a cycle, then, since $|J| = r$, J is a spanning tree, and $\det D_J = \pm q^k$, where k is the number of non- T edges in J . To see this, observe that there is at least one vertex l which is a leaf of J and does not correspond to the deleted last row of D . The row l has a single non-zero entry in D_J , which is ± 1 if the single J -edge incident to it is also in T , and $\pm q$ otherwise. Use the cofactor expansion of $\det D_J$ according to this row, which therefore has a single non-zero component, where the cofactor corresponds to a spanning tree for $\Gamma \setminus l$. Hence, by repeating this process, we obtain the determinant $\pm q^k$.

Thus, we have shown that $\det Q_0 = \sum_{i=0}^r c_i q^{2i}$. It remains to prove that Q_0 is a Gram matrix for $\mathcal{C}_q(\Gamma, T)$, by exhibiting a change of basis matrix T such that the Gram matrix $(G_C)_{ij}$ defined in Remark 4.2 equals $T^t Q_0 T$.

For intuition, we note that the (classical) lattice of integer cuts has a basis corresponding to any set of r vertices (all but one of the vertices). The basis element corresponding to a vertex v is the cut given by the partition $\{v\} \cup (V(\Gamma) \setminus v)$. As an element of $\mathbb{Z}^{E(\Gamma)}$, this cut is a linear combination of the edges incident to v , where incoming edges appear with $+1$ coefficient and the outgoing edges with -1 coefficient. The basis consists of cuts corresponding to all but one of the vertices since the cut corresponding to the last vertex is the negative sum of all the others. The lattice Q_0 is the quantised version of the Gram matrix in this vertex basis, with the last vertex omitted. The change of basis matrix T is the matrix whose columns are the fundamental cuts K_i written in the vertex basis.

With the above in mind, we define the matrix T as follows. Let the edges of T be numbered $1, \dots, r$. Label the vertices of Γ corresponding to the rows of D_0 with v_1, \dots, v_r , in the order of their appearance as rows in D_0 . Let v_{r+1} denote the vertex corresponding to the omitted

last row of D . The fundamental cut $K_i = V_{0,i} \cup V_{1,i}$ includes v_{r+1} on one side of the partition. The change of basis matrix T is the $r \times r$ matrix given by:

$$T_{ij} = \begin{cases} 1 & \text{if } v_i \in V_{0,j} \text{ and } v_{r+1} \in V_{1,j}; \\ -1 & \text{if } v_i \in V_{1,j} \text{ and } v_{r+1} \in V_{0,j}; \\ 0 & \text{otherwise.} \end{cases}$$

It is then a straightforward check that $\mathcal{C}_q(\Gamma, T) = T^t Q_0 T = (T^t D_0)(D_0^t T)$. \square

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