Finite Gelfand Pairs and Cracking Points of the Symmetric Groups

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December 2018

1 Introduction

Let G be a finite group and K a subgroup of G. Denote by $L(G)$ the set of complex-valued functions on G. This is an algebra under the convolution product

$$f \star g(x) = \frac{1}{|G|} \sum_{y \in G} f(xy^{-1}) g(y).$$

The pair (G, K) is said to be a *Gelfand pair* if the subalgebra $L(K \backslash G / K)$ of K-biinvariant functions in $L(G)$ is commutative.

Gelfand pairs are perhaps most well-known in the context of Lie groups, where there is an analogous definition in terms of the algebra of integrable K-biinvariant functions on the group [3]. In the Lie group setting, the Gelfand pair structure can be used to construct irreducible unitary representations of the larger group G. These techniques played a pivotal role in describing the representation theory of semi-simple Lie groups [8, 6]. In the finite group setting, the representation theory of the larger group G is already understood, so constructing representations is not the primary use of finite Gelfand pairs. However, finite Gelfand pairs arise naturally in various areas outside of group theory including statistics, experimental design, and combinatorics, so the development of the theory of finite Gelfand pairs has utility in a diverse range of mathematics [5, 2]. More details of these connections can be found in Section 2.

In this paper, we study a certain family of finite Gelfand pairs associated to wreath products of symmetric groups, following the setup in [11]. Given a finite group Γ, the symmetric group S_n acts on Γ^n by permuting the factors, allowing us to form the wreath product $G_n := \Gamma^n \rtimes S_n$. Let Δ_n be the diagonal subgroup of Γ^n; that is, the subgroup consisting of elements of the form $(g, \ldots, g)$ for some $g \in \Gamma$. Then $K_n := \Delta_n \times S_n$ is a subgroup of G_n, and we may consider pairs of the form (G_n, K_n). We wish to understand for which n the (G_n, K_n) are Gelfand pairs.

For abelian Γ, (G_n, K_n) is always a Gelfand pair [11]. For non-abelian Γ, the following two results are established in [11]:

1. $(G_{|\Gamma|}, K_{|\Gamma|})$ is not a Gelfand pair.

2. There is some integer $N(\Gamma)$ with $3 \leq N(\Gamma) \leq |\Gamma|$ such that (G_n, K_n) is a Gelfand pair for $n < N(\Gamma)$, and is not a Gelfand pair for $n \geq N(\Gamma)$. The integer $N(\Gamma)$ is called the **cracking point** of Γ (and Γ is said to "crack" at $N(\Gamma)$).

It is well known that any group-subgroup pair (G, K) is a Gelfand pair if and only if the G-representation $L(G/K)$ is multiplicity free. The authors of [11] show that (G_n, K_n) is a Gelfand pair whenever (Γ^n, Δ_n) is a Gelfand pair and they give decompositions of the spaces $L(G_n/K_n)$ and $L(\Gamma^n/\Delta_n)$ in terms of the irreducible representations of Γ and S_n.

This REU project is a continuation of our project with Faith Pearson from the Summer 2018 semester. The main result of our previous project was the establishment of a new upper bound on the cracking points of groups with irreducible representations of a particular form. The precise statement of this result and its proof are given in Lemma 4.1. We also made progress in determining the cracking points of the Symmetric groups S_k, using Lemma 4.1 to show that $N(S_k) \leq 4$ for $k > 7$, and showing directly that $S(S_4) = 4$ and $N(S_5) = N(S_6) = N(S_7) = 3$.

This paper begins with a discussion of the applications of finite Gelfand pairs outside of group theory. Afterward, we describe in more detail the above mentioned decompositions given in [11] and we introduce the main result of this project, an important observation that simplifies the computations of cracking points. We then apply this result to prove that $N(S_k) = 3$ for $k \geq 5$.

2 Applications

In this section, we will discuss three contexts in which finite Gelfand pairs appear naturally. We begin with a description of how finite Gelfand paris have been used to determine the rate at which certain Markov chains approach equilibrium. We then discuss how association schemes can be constructed using finite Gelfand pairs and how they relate to experimental design. Finally, we will see how Gelfand pairs arise from the study of parking functions, which have applications in combinatorics.

Under certain conditions, group actions on sets can be used to describe Markov chains [5]. This is illustrated by the following example.

Example 2.1. Suppose we are given two urns, one containing r red balls and the other containing b black balls, with $r + b = n$. Consider a process wherein at each step we randomly pick one ball from each urn and switch the two. We can describe this process in terms of group actions. In particular, let X be the set of r element subsets of a set with n elements, representing the r balls contained in the first urn. Let our initial configuration (when the balls are completely separated by color) be represented by $x_0 = \{1, 2, \ldots, r\} \subset \{1, 2, \ldots, n\}$. Then S_n acts on X where an r-element subset $x \in X$ is sent to its image under the natural action of S_n on $\{1, \ldots, n\}$. For example, if $\sigma \in S_n$ is the transposition $(r\ n)$, then $\sigma x_0 = \{1, 2, \ldots, r-1, n\}$. The stabilizer N of x_0 under this action is isomorphic to $S_r \times S_b$, permutations which do not move any balls between the urns.

In general, let G be a finite group acting transitively on a set X and let N be the stabilizer of a point $x_0 \in X$. Then there is a natural identification of X with G/N. If P is a probability measure on G, then picking a sequence of elements $g_1, g_2, \dots$ with probability P gives rise to a random walk on X, namely $x_0, g_1 x_0, g_2 g_1 x_0, \dots$. In the above example, P would be uniform on all permutations that switch exactly one ball between the urns, and 0 for all other permutations.

There is an advantage to describing a Markov chain in terms of a group action on a set, especially in the case when (G, N) is a Gelfand pair. Indeed, if (G, N) is a Gelfand pair and the probability measure P is N-bi-invariant, then the long-term behavior of such a process can be better understood in terms of the Fourier transform of P. In the above example $(S_n, S_r \times S_b)$ is a Gelfand pair, and this can be used to determine the rate at which the resulting Markov chain converges to the uniform distribution [5]. In contrast, suppose we add a third urn with w white balls so that $r + b + w = n$ and consider the process wherein we pick two urns at random and switch a random pair of balls as before. The corresponding pair resulting from this set up is $(S_n, S_r \times S_b \times S_w)$, which is not a Gelfand pair. As a result, the long-term behavior of this system is more complicated, and the number of steps required to reach equilibrium is not known.

Finite Gelfand pairs have also been useful in constructing association schemes, which are of interest in coding theory and experimental design[4, 2]. An association scheme is a finite set X together with a collection of binary relations $R_0, R_1 \dots, R_d$ satisfying certain axioms. A convenient way of stating the axioms is to view each R_i as a $|X| \times |X|$ matrix with entries indexed by the elements of X. In particular, for $x, y \in X$ we let R_i be the matrix with a 1 in the position (x, y) if $(x, y) \in R_i$, and 0 elsewhere. The axioms for association schemes can be formulated in terms of the matrices as follows:

1. $R_0 = \text{id}$.
2. $\sum_{i=0}^{d} R_i = J$, the matrix with a 1 in every spot. In other words, $\{R_i\}_{i=0}^{d}$ is a partition of $X \times X$.
3. For every i, there is a j such that $R_i^T = R_j$.
4. $R_i R_j = \sum_{k=0}^{d} p_{ij}^k R_k$ where p_{ij}^k are non-negative integers. If $R_i R_j = R_j R_i$, the association scheme is said to be commutative.

As an example, let (G, K) be a Gelfand pair. An element $g \in G$ acts on $G/K \times G/K$ by $g(x, y) = (gx, gy)$ where $x, y \in G/K$. This defines an association scheme where the set X is G/K, and the relations on X are the orbits of the action of G on $X \times X$. Finite Gelfand pairs can also be used to construct generalizations of the Johnson and Hamming schemes, which are of importance in coding theory [4].

Association schemes are useful in determining optimal incomplete block designs for experiments [2]. Block design refers to the practice of separating experimental units into blocks with similar characteristics in order to account for unwanted variation. For example, when testing the efficacy of various medical drugs a researcher might block patients by age in order to account for the variation it causes. In some cases, it may only be possible to expose each block to a subset of the treatments being tested. Such a design is called an incomplete block design. The machinery of association schemes is useful in determining the best way of choosing subsets of the treatments to allocate to blocks, and in analyzing the data [2].

Finite Gelfand pairs, and in particular the construction of (G_n, K_n) with which this paper is concerned, have also arisen naturally in combinatorics in relation to parking functions [1]. A parking function of length n is a non-decreasing sequence $(a_1, \ldots, a_n)$ of positive integers such that $a_i \leq i$. Parking functions appear in various settings in algebraic combinatorics including hyperplane arrangements, labelled trees, and diagonal harmonics [1, 10, 9]. There is a one-to-one correspondence of the set of parking functions of length n and the group $\mathbb{Z}_{n+1}^n/\Delta\mathbb{Z}_{n+1}^n$, where $\mathbb{Z}_{n+1}$ is the cyclic group of order $n+1$ and $\Delta\mathbb{Z}_{n+1}^n$ is the diagonal of $\mathbb{Z}_{n+1}^n$ [1]. This correspondence sends the term a_i in the sequence to $a_i - 1 \mod n+1$. Studying parking functions in the context of the group $\mathbb{Z}_{n+1}^n/\Delta\mathbb{Z}_{n+1}^n$ led the authors of [1] to construction of (G_n, K_n) in the case $\Gamma = \mathbb{Z}_{n+1}$. They were also the first to suggest that for non-abelian Γ, (G_n, K_n) is not a Gelfand pair for sufficiently large n, which is the motivation of this project.

3 New Results

In this section we present our main result, Prop. 3.1, but first we describe the decompositions of $L(\Gamma^n/\Delta_n)$ and $L(G_n/K_n)$ given in [11]. Let $\{\pi_\ell\}_{\ell \in S}$ be the finite collection of irreducible representations of Γ, where S is an indexing set in bijection with the conjugacy classes of Γ. The space $L(\Gamma^n/\Delta_n)$ decomposes into irreducible Γ^n-representations as follows:

$$L(\Gamma^n/\Delta_n) = \sum_{\pi_{\ell_1},\ldots,\pi_{\ell_{n-1}} \in \hat{\Gamma}} \sum_{\pi_{\ell_n} \in \hat{\Gamma}} m(\pi_{\ell_1}, \ldots, \pi_{\ell_{n-1}} | \pi_{\ell_n}) \pi_{\ell_1} \hat{\otimes} \cdots \hat{\otimes} \pi_{\ell_{n-1}} \hat{\otimes} \pi_{\ell_n}^*,$$

where $m(\pi_{\ell_1}, \ldots, \pi_{\ell_{n-1}} | \pi_{\ell_n})$ is the multiplicity of π_{ℓ_n} in $\pi_{\ell_1} \otimes \cdots \otimes \pi_{\ell_{n-1}}$, $\pi_{\ell_n}^*$ is the dual representation of π_{ℓ_n}, and $\hat{\otimes}$ denotes the exterior tensor product. The multiplicity $m(\pi_{\ell_1}, \ldots, \pi_{\ell_{n-1}} | \pi_{\ell_n})$ can be computed via the inner product on the space of complex valued class functions on Γ; that is,

$$m(\pi_{\ell_1}, \ldots, \pi_{\ell_{n-1}} | \pi_{\ell_n}) = \frac{1}{|\Gamma|} \sum_{C} (\prod_{i=1}^{n-1} \chi_{\pi_{\ell_i}}(C)) \overline{\chi}_{\pi_{\ell_n}}(C) |C| \tag{3.1}$$

where C runs over the conjugacy classes of Γ. Here $\chi_{\pi_{\ell_i}}$ is the character of the representation π_{ℓ_i}. Note that (Γ^n, Δ_n) is a Gelfand pair when $L(\Gamma^n/\Delta_n)$ is multiplicity free, which happens exactly when $m(\pi_{\ell_1}, \ldots, \pi_{\ell_{n-1}} | \pi_{\ell_n}) \leq 1$ for all choices of $\pi_{\ell_1}, \ldots, \pi_{\ell_n}$.

On the other hand, Benson and Ratcliff also describe the decomposition of $L(G_n/K_n)$. Retaining the same definitions from above, define a representation π of Γ^n by $\pi := \pi_{\ell_1} \hat{\otimes} \cdots \hat{\otimes} \pi_{\ell_{n-1}} \hat{\otimes} \pi_{\ell_n}$. Then S_n acts on π by permuting the factors, and we denote by S_π the stabilizer of π in S_n. Furthermore, let ω be the intertwining representation of S_π and let ρ be a representation in $\widehat{S}_\pi$. From π and ρ one can construct the induced representation $R_{\pi,\rho} = ind_{\Gamma^n \rtimes S_\pi}^{G_n}((\pi \circ \omega) \hat{\otimes} \rho)$, which is irreducible. Moreover, all irreducible representations of G_n are of this form [11, 4]. By [11, Lem. 3.3], the number of K_n-fixed vectors in $R_{\pi,\rho}$ (which is equal to the multiplicity of $R_{\pi,\rho}$ in $L(G_n/K_n)$) is equal to the number of ($K_\pi := \Delta_n \times S_\pi$)-fixed vectors in $(\pi \circ \omega)\hat{\otimes}\rho$. This can be calculated by taking the inner product with the trivial character on K_π:

$$\frac{1}{|\Delta_n \times S_\pi|} \sum_{(\delta,\sigma)\in K_\pi} \chi_{\pi\circ\omega}(\delta,\sigma)\chi_\rho(\sigma) = \frac{1}{|S_\pi|} \sum_{\sigma\in S_\pi} \left(\frac{1}{|\Delta_n|} \sum_{\delta\in\Delta_n} \chi_{\pi\circ\omega}(\delta,\sigma) \right) \chi_\rho(\sigma). \tag{3.2}$$

Now the middle sum on the right hand side of (3.2) is a class function on S_π. This class function plays an important role in our story so we give it a name:

$$M_\pi(\sigma) := \frac{1}{|\Delta_n|} \sum_{\delta\in\Delta_n} \chi_{\pi\circ\omega}(\delta,\sigma). \tag{3.3}$$

Equation (3.2) determines the coefficient of χ_ρ in the decomposition of M_π into irreducible characters of S_π. From this we see that (G_n, K_n) is a Gelfand pair if and only if for each choice of π, the coefficient of χ_ρ in M_π is less than or equal to 1 for all $\rho \in \widehat{S}_\pi$.

Now we wish to highlight a key observation concerning $M_\pi(e)$, the value of M_π on the identity $e \in S_\pi$. First, note that

$$\chi_{\pi\circ\omega}(\delta, e) = \prod_{i=1}^{n} \chi_{\pi_{\ell_i}}(\delta).$$

Thus, substituting this into (3.3), we have

$$M_\pi(e) = \frac{1}{|\Delta_n|} \sum_{\delta\in\Delta_n} \Big(\prod_{i=1}^{n} \chi_{\pi_{\ell_i}}(\delta) \Big) = \frac{1}{|\Gamma|} \sum_{C} \Big(\prod_{i=1}^{n} \chi_{\pi_{\ell_i}}(C) \Big) |C|, \tag{3.4}$$

where again C runs over the conjugacy classes of Γ. Note that $\Delta_n \simeq \Gamma$. Now equation (3.4) is equal to the equation given in (3.1) whenever $\chi_{\pi_{\ell_n}}$ is real-valued. In summary, we have proven the following result.

Proposition 3.1. *Let Γ be a finite group, and let $\{\pi_\ell\}_{\ell\in S}$ be the irreucible representations of Γ. Then for $\pi = \pi_{\ell_1}\hat{\otimes}\cdots\hat{\otimes}\pi_{\ell_n}$,*

$$M_\pi(e) = m(\pi_{\ell_1}, \dots, \pi_{\ell_{n-1}} | \pi_{\ell_n})$$

if the character $\chi_{\pi_{\ell_n}}$ is real-valued.

With this, we can simplify calculations used in computing cracking points, and in some cases circumvent the necessity for complete character tables. An example of such utility is given below.

4 Cracking Points of S_k

In this section we use Prop 3.1 to compute the cracking points of the symmetric groups. First, we introduce a lemma, which is a result from our Summer 2018 REU project.

Lemma 4.1. *Let S_π be the stabilizer of $\pi = \pi_{\ell_1} \hat{\otimes} \cdots \hat{\otimes} \pi_{\ell_n}$ in S_n. If $M_\pi(e) > \sum_{\rho \in \widehat{S}_\pi} \dim \rho$, then (G_n, K_n) is not a Gelfand pair.*

Proof. We know that M_π is a class function on S_π, which can be expressed uniquely as a linear combination $M_\pi = \sum_{\rho \in \widehat{S}_\pi} a_\rho \chi_\rho$ of irreducible characters χ_ρ of S_π, for some $\{a_\rho\}_{\rho \in \widehat{S}_\pi}$. Since $a_\rho = \langle M_\pi, \chi_\rho \rangle$ counts the number of K_π-fixed vectors in $(\pi \circ \omega) \hat{\otimes} \rho$, we see that actually $a_\rho \in \mathbb{Z}^+$ for all $\rho \in \widehat{S}_\pi$. Therefore, if

$$M_\pi(e) = \sum_{\rho \in \widehat{S}_\pi} a_\rho \dim \rho > \sum_{\rho \in \widehat{S}_\pi} \dim \rho$$

there must be some $\rho \in \widehat{S}_\pi$ such that $a_\rho > 1$. Hence, (G_n, K_n) is not a Gelfand pair. □

Proposition 4.2. *The symmetric groups S_k crack at 3 for $k \geq 5$.*

Proof. Fix $k \geq 5$, and let π_m be the highest dimensional irreducible representation of S_k. We claim that there is an irreducible representation ψ of S_k such that for $\pi = \pi_m \hat{\otimes} \pi_m \hat{\otimes} \psi$, there is some irreducible character of S_π which has a coefficient greater than 1 in the decomposition of M_π, and hence (G_3, K_3) is not a Gelfand pair for $\Gamma = S_k$.

Now, we see that if $\psi = \pi_m$, then $S_\pi = S_3$. Otherwise, $S_\pi = S_2 \times S_1 \simeq S_2$. We will prove that (G_3, K_3) is not a Gelfand pair by showing that if $\psi = \pi_m$ then $M_\pi(e) > 4$, and if $\psi \neq \pi_m$ then $M_\pi(e) > 2$, then applying Lemma 3.1. By Proposition 3.1, this is equivalent to showing that the coefficient of π_m in $\pi_m \otimes \pi_m$ is greater than 4, or the coefficient of π_i in $\pi_m \otimes \pi_m$ is greater than 2 for some $\pi_i \neq \pi_m$ where $\pi_i \in \widehat{S}_k$. It then suffices to show that

$$\dim {V_{\pi_m}}^2 > 4 \dim V_{\pi_m} + \sum_{\pi \in \widehat{S}_k,\ \pi \neq \pi_m} 2 \dim V_\pi.$$

Since π_m is of maximal dimension in $\widehat{S}_k$, it is enough to show that

$$\dim {V_{\pi_m}}^2 \geq 4 \dim V_{\pi_m} + 2(p(k) - 1) \dim V_{\pi_m}$$

where $p(k)$ is the number of partitions of k, which is equal to the number of irreducible representations of S_k. Simplifying, this amounts to establishing the inequality

$$\dim V_{\pi_m} \geq 2p(k) + 2.$$

An asymptotic lower bound is given for $\dim V_{\pi_m}$ in [12, Thm.1], namely

$$\dim V_{\pi_m} \geq e^{-c\sqrt{k}} \sqrt{k!}$$

where $c = \frac{\pi}{\sqrt{6}}$. Similarly, an asymptotic upper bound was found for $p(k)$ in [7]:

$$p(k) \leq \frac{1}{4k\sqrt{3}} e^{\pi\sqrt{\frac{2k}{3}}}.$$

Combining these results in our above inequality, we see that (G_3, K_3) fails to be a Gelfand pair if the following holds:

$$e^{-c\sqrt{k}}\sqrt{k!} \geq \frac{1}{2k\sqrt{3}} e^{\pi\sqrt{\frac{2k}{3}}} + 2.$$

This is equivalent to the condition that the ratio

$$r(k) := \frac{2k\sqrt{3}e^{-c\sqrt{k}}\sqrt{k!} - 4k\sqrt{3}}{e^{\pi\sqrt{\frac{2k}{3}}}}$$

is greater than or equal to 1. A direct calculation shows that this holds for $k = 12$. For $k > 12$ note that, after replacing $k!$ with the Gamma function $\Gamma(k+1)$, the derivative $\frac{d}{dk}r(k)$ is positive and hence $r(k)$ is increasing. Thus, $r(k) \geq 1$ for $k \geq 12$ and $N(S_k) = 3$ in that case. The rest we calculate through a case by case analysis.

In our previous REU project, we computed the cracking points of S_k for $k = 5, 6$, and 7. Here, we show $N(S_5) = 3$ as an example. To do this, we will calculate M_π directly for a specific choice of π in $\widehat{S_5^3}$. Consider the following partial character table of S_5, which contains the characters of the highest dimensional and second highest dimensional irreducible representations:

	(1)	(10)	(15)	(20)	(20)	(24)	(30)
	I	2	2, 2	3	3, 2	5	4
W	5	1	1	-1	1	0	1
Q	6	0	-2	0	0	1	0

Now let $\pi = Q\hat{\otimes}Q\hat{\otimes}W$, which has a stabilizer of $S_\pi \simeq S_2$ in S_3. Then calculating $M_\pi(\sigma)$ directly from (2.4), we see that $M_\pi(\sigma) = 2$ for both σ in S_π. Hence, $M_\pi = 2\chi_{triv}$, and (G_3, K_3) is not a Gelfand pair for $\Gamma = S_5$. The result follows for S_6 and S_7 analogously. More precisely, the representation π consisting of two copies of the highest dimensional irreducible representation and one copy of the second highest suffices in both cases.

For S_8 through S_{11}, we will show directly that

$$\dim V_{\pi_m} \geq 2p(k) + 2.$$

The computations are contained in the table below. The values for $\dim V_{\pi_m}$ are given in [5].

k	$\dim V_{\pi_m}$	$2p(k)+2$
8	90	46
9	216	62
10	768	86
11	2310	114

□

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