Cracking Points of Finite Gelfand Pairs

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Abstract

Consider the Gelfand pair (G_n, K_n) as constructed in [BR18]. It was proven in [BR18] that for a finite, non-abelian group Γ, there is some integer $2 < N(\Gamma) \leq |\Gamma|$ for which (G_n, K_n) is a Gelfand pair for all $n < N(\Gamma)$, but (G_n, K_n) fails to be a Gelfand pair for all $n \geq N(\Gamma)$. In this paper we find a new smaller upper bound for $N(\Gamma)$ for certain Γ. We also showed that for $\Gamma = S_k$, $N(\Gamma) \leq 4$.

1 Introduction

Let $L(G)$ be the space of complex-valued functions on a finite group G. This is an algebra under the convolution product

$$f * g(x) = \sum_{y \in G} f(xy^{-1})g(y).$$

Given a subgroup K in G, the set

$$L(K \backslash G / K) = \{f \in L(G) : f(k_1 x k_2) = f(x) \text{ for all } k_1, k_2 \text{ in } K\}$$

of K-bi-invariant functions on G forms a subalgebra of $L(G)$. If $L(K \backslash G/K)$ is commutative, then we say that the pair (G, K) is a **finite Gelfand Pair**.

There are two additional equivalent definitions of a finite Gelfand pair. These two will be the most useful in our setting.

Definition 1.1. The pair (G, K) is a *finite Gelfand pair* if the left quasi-regular representation $\mathrm{Ind}_K^G(triv_K)$ of G in $L(G/K)$ is multiplicity free.

Irreducible representations of G which occur in $L(G/K)$ are called *K-spherical*.

Definition 1.2. The pair (G, K) is a *finite Gelfand pair* if for each irreducible representation (π, V) of G, the space V^K of K-fixed vectors in V has dimension $\dim(V^K) \leq 1$.

There is an analogous definition of Gelfand pairs in the setting of Lie groups, and such Gelfand pairs have served as a valuable tool in studying the representation theory of complex semisimple Lie groups. In the setting of Lie groups with compact subgroups, Gelfand pairs are fully classified and their representation theory is well understood [GV88, Hel84]. There has also been extensive work studying Gelfand pairs in the setting of nilpotent Lie groups [FGJR+18]. In contrast, very little is known about their structure in the finite group setting, and a classification has not yet been obtained. This REU project examines a certain family of finite Gelfand pairs, which can be constructed in the following way.

For a finite group Γ, the symmetric group S_n acts by on Γ^n by permuting the factors. That is, for $\sigma \in S_n$,

$$\sigma \cdot (x_1, \ldots, x_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

The resulting semi-direct product $G_n := \Gamma^n \rtimes S_n$ is called the *wreath product* of Γ with S_n. Let Δ_n be the diagonal subgroup in Γ^n where

$$\Delta_n := \{(x, \ldots, x) : x \in \Gamma\}.$$

The direct product $K_n := \Delta_n \times S_n$, is a subgroup of $G_n = \Gamma^n \rtimes S_n$. From this point on we will consider the pairs of the form (G_n, K_n). If Γ is abelian, then the pair (G_n, K_n) is a finite Gelfand pair [AC12]; however, if Γ is non-abelian, the situation is more subtle. In particular, the following results have already been shown in [BR18].

Theorem 1.3. *If Γ is non-abelian, then $(G_{|\Gamma|}, K_{|\Gamma|})$ fails to be a Gelfand pair.*

Furthermore, Benson–Ratcliff established that for any non-abelian Γ, there exists some integer $N(\Gamma)$ between 2 and $|\Gamma|$ such that

1. (G_n, K_n) is a Gelfand pair for all $n < N(\Gamma)$, but
2. (G_n, K_n) fails to be a Gelfand pair for all $n \geq N(\Gamma)$.

We call this $N(\Gamma)$ the **cracking point**. The behavior of these cracking points for different finite groups has only recently begun to be studied, and there are many unanswered questions in this field. The goal of our summer REU project was to examine the following two broad questions:

Question 1: Can the cracking points of a Gelfand pair be arbitrarily large?

Question 2: Does the representation theory of a group have an impact on the cracking point?

The results we have accomplished are the following:

Result 1: We have found a new upper bound for the cracking points of groups Γ which admit representations of a certain form. (Precise conditions can be found in Proposition 5.1.)

Result 2: We have made significant progress in computing cracking points of all symmetric groups. In particular, we found that $N(S_3) = 6$, $N(S_4) = 4$, $N(S_5) = N(S_6) = N(S_7) = 3$, and $N(S_k) \leq 4$ for all $k \geq 7$.

The organization of this paper is as follows. First, we discuss the necessary background in group theory and representation theory. Then we describe the reductions and tools established in [BR18]. After this we give a detailed example using these reductions and tools. We conclude with a description of the results which are new to this paper. This paper should accessible to undergraduate readers with a basic background in group theory and representation theory.

2 Group Theory and Representation Theory Background

In order to continue, we must learn some necessary group theory and representation theory tools. Recall in our Gelfand pairs of interest, (G_n, K_n), we defined $G_n = \Gamma^n \rtimes S_n$. This is to say that G_n is the **semidirect product** of Γ^n and S_n. Semidirect products of groups can be defined as follows.

Let H and K be groups and let ϕ be a homomorphism from K into $\mathrm{Aut}(H)$. Let $\cdot$ denote the (left) action of K on H determined by ϕ. Let G be the set of ordered pairs (h, k) with $h \in H$ and $k \in K$ and define the following multiplication on G:

$$(h_1, k_1)(h_2, k_2) = (h_1\, k_1 \cdot h_2,\ k_1 k_2).$$

Definition 2.1. The group described above is called the **semidirect product** of H and K with respect to ϕ and will be denoted by $H \rtimes K$.

Example 2.2. We claim that the dihedral group D_3 is the semidirect product of the cyclic groups C_3 and C_2, where C_2 acts on C_3 by inversion (this is an automorphism since C_3 is abelian). We denote this $D_3 = C_3 \rtimes C_2$. Define a map from D_6 to $C_3 \rtimes C_2$ by "flip" := $f \mapsto (0_3, -1)$, and "rotate" := $r \mapsto (1, 0_2)$ in $C_3 \rtimes C_2$ (we are using 0 and -1 to denote the elements of C_2, while 0, 1, and 2 denote the elements of C_3). We know we can present

$$D_3 = \langle f, r |\ f^2 = e,\ r^3 = e,\ rf = fr^{-1} \rangle,$$

so to check that this is a homomorphism, we must check that these relations still hold in $C_3 \rtimes C_2$. First, we show $(1, 0_2)^3 = (0_3, 0_2)$ under mulitplication in the semidirect product. We have

$$(1, 0_2)(1, 0_2)(1, 0_2) = (2, 0_2 + 0_2)(1, 0_2) = (3, 0_2) = (0_3, 0_2),$$

since 0_2 acts as the identity on C_3. Next, we must show $(0_3, -1)^2 = (0_3, 0_2)$ under mulitplication in the semidirect product. We have

$$(0_3, -1)(0_3, -1) = (0_3 - 0_3, -1 - 1) = (0_3, -2) = (0_3, 0_2).$$

Finally, we must show $(1, 0_2)(0_3, -1) = (0_3, -1)(1, 0_2)^{-1}$. We have

$$(1, 0_2)(0_3, -1) = (1 + 0_2 \cdot 0_3, 0_2 - 1) = (1, -1) \text{ and,}$$

$$(0_3, -1)(-1, 0_2) = (0_3 + (-1_2 \cdot -1_3), -1 + 0_2) = (1, -1).$$

Hence, the function in question is indeed a homomorphism. In particular, since it is also a bijection, it is an isomorphism.

In the next section we create a new representation from two representations in the following way.

Definition 2.3. If V is a representation of a group G, and W is a representation of a group H, then the **external tensor product** $V \hat{\otimes} W$ is a representation of $G \times H$ by $(g \times h) \cdot (v \otimes w) = g \cdot v \otimes h \cdot w$.

Next, recall that one of the definitions of finite Gelfand pairs (Definition 1.1) relied on the notion of an *induced representation*. Next we will describe this construction. Let H be a subgroup of a group G. Suppose V is a representation of a group G, and $W \subset V$ is a subspace which is H-invariant. For any g in G, the subspace $g \cdot W = \{g \cdot w : w \in W\}$ depends only on the left coset gH of g modulo H, since $gh \cdot W = g \cdot (h \cdot W) = g \cdot W$; for a coset σ in G/H, we write $\sigma \cdot W$ for this subspace of V.

Definition 2.4. We say that V is **induced** by W if every element in V can be written uniquely as a sum of elements in such translates of W, that is,

$$V = \bigoplus_{\sigma \in G/H} \sigma \cdot W.$$

In this case, we write $V = \operatorname{Ind}_H^G W = \operatorname{Ind} W$.

Example 2.5. Let $H = S_2$, $G = S_3$, and let $\mathbb{C}$ be the trivial representation of S_2. The cosets of S_2 are

$$c_0 = \{e,\ (12)\}$$
$$c_1 = \{(13), (123) = (13)(12)\}$$
$$c_2 = \{(23),\ (132) = (23)(12)\}.$$

So the vector space of the induced representation is

$$\mathbb{C} \oplus (13) \cdot \mathbb{C} \oplus (23) \cdot \mathbb{C}$$

with the action of S_3 given by its permutation action on the cosets. For example, let us take the verctor $v = (1, 4, -2)$ and act on it by (123). Then we have

$$(123) \cdot e = (123) \sim (13)(12) \text{ which is in the same coset as } (13),$$

$$(123) \cdot (13) = (132) \sim (23)(12) \text{ which is in the same coset as } (23),$$

$$(123) \cdot (23) = (12) \text{ which is in the same coset as } e.$$

Thus, $(123) \cdot v = (-2, 1, 4)$. Note that the values of the entries do not change since the action of S_2 is trivial.

Suppoese we instead use the sign representation of S_2, that is (with $\mathbb{C}$ as the underlying vector space)

$$\mathrm{sgn}(g) = \begin{cases} \text{multiplication by } 1, & \text{if } g = e\text{; the even permutation} \\ \text{multiplication by } -1, & \text{if } g = (12)\text{; the odd permutation.} \end{cases}$$

In this case, since $(123) \cdot g_i$ is a left multiple of (12) for each representative g_i, the action of (123) on a vector changes the sign on each entry as well as permuting them. In particular, the above example would become $(123) \cdot v = (2, -1, -4)$.

Alternatively, let $f \in L(S_3/S_2)$. In other words, f is a function $f : \{c_0, c_1, c_2\} \to \mathbb{C}$ where c_i denote the cosets of S_2 in S_3 in the order listed in the first example. Then the function corresponding to the vector v in the previous example is

$$\tilde{f} = \begin{cases} c_0 \mapsto 1 \\ c_1 \mapsto 4 \\ c_2 \mapsto -2. \end{cases}$$

Computing the quasi-left-regular action of (123) on $\tilde{f}$ we have

$$(123) \cdot \tilde{f}(x) = \tilde{f}((123)^{-1} \cdot x).$$

Since $(123)^{-1} = (132)$, we have

$$(132) \cdot e = (132) \in c_2,$$

$$(132) \cdot (13) = (12) \in c_0,$$

$$(132) \cdot (23) = (13) \in c_1$$

So we have

$$\tilde{f}((123)^{-1} \cdot c_0) = \tilde{f}(c_2) = -2,$$
$$\tilde{f}((123)^{-1} \cdot c_1) = \tilde{f}(c_0) = 1,$$
$$\tilde{f}((123)^{-1} \cdot c_2) = \tilde{f}(c_1) = 4.$$

Hence we see that the action of (123) on $\tilde{f}$ corresponds with the action of (123) on v from the previous example.

Another crucial concept in this paper is character theory. In particular, we need to understand character tables.

Definition 2.6. If V is a representation of G, its **character** χ_V is the complex-valued function on the group defined by

$$\chi_V(g) = \mathrm{Tr}(g|_V),$$

the trace of g on V.

A **character table** is a way of expressing the basic information about the irreducible representations of a group G. This is a table with the conjugacy classes $[g]$ of G listed across the top, usually given by a representative g, with the number of elements in each conjugacy class over it; the irreducible representations V of G listed on the left; and, in the appropriate box, the value of the character χ_V on the conjugacy class $[g]$.

Example 2.7. **Character table of S_3:** Our conjugacy classes are listed across the top, with the number of elements they contain listed above them. There is the identity element 1 with one element, a two-cycle (12) which has 3 elements, and a three-cycle (123) which has three elements. Our three representations, the trivial representation, U, the alternating representation, U', and another two dimensional representation, V. We will explicitly find all of the characters in example 2.8 once we define the orthogonality relations in Theorem 2.7.

	(1)	(3)	(2)
	1	(12)	(123)
U	1	1	1
U'	1	1	-1
V	2	-1	0

In the following theorem we use the Hermitian inner product. Let ϕ, ψ be two complex-valued functions on G. Then

$$\langle \phi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\phi(g)}\, \psi(g).$$

Theorem 2.8. *[FH91] Character tables display the following orthogonality relations.*

1. *The irreducible characters of a finite group G are orthonormal; that is, if χ_V is the character of an irreducible representation V, then $\langle \chi_V, \chi_V \rangle = 1$, and if χ_V and $\chi_{V'}$ are the characters of two nonisomorphic irreducible representations V and V', then $\langle \chi_V, \chi_{V'} \rangle = 0$.*

2. *The number of irreducible representations of G is equal to the number of conjugacy classes.*

3. *Let $V_1, \ldots, V_r$ represent the isomorphism classes of irreducible representations of G, and let $\chi_1, \ldots, \chi_r$ be their characters. The dimension d_i of V_i (or of χ_i) divides the order $|G|$ of the group, and $|G| = d_1^2 + \cdots + d_r^2$.*

Now that we are armed with the tools of Theorem 2.7, we can give a detailed example of computing the character table of S_4.

Example 2.9. **Character table of S_4:** In S_4, we have the conjugacy classes of the identity element 1 which has one element, a two-cycle (12) which has 6 elements, a three-cycle (123) which has 8 elements, a four cycle (1234) which has 6 elements, and the product of two disjoint two cycles (12)(34) which has 3 elements. The trivial representation, U, takes on the values $(1, 1, 1, 1, 1)$ on the five conjugacy classes, and the alternating representation, U', takes on the values $(1, -1, 1, -1, 1)$.

Another representation that we need to find is the standard representation, V, which is the quotient of the permutation representation associated to the standard action of S_4 on a set of 4 elements by the trivial subrepresentation. In exercise 2.5 of [FH91] (*the original fixed point formula*), it says that if V is the permutation representation associated to the action of a group G on a finite set X, the character $\chi_V(g)$ is the number of elements of X fixed by g. This tells us that the character of the permutation representation on $\mathbb{C}^4$ is $\chi_{\mathbb{C}^4} = (4, 2, 1, 0, 0)$, and so $\chi_V = \chi_{\mathbb{C}^4} - \chi_U = (3, 1, 0, -1, -1)$, which we can see is irreducible since $|\chi_V| = 1$. So far, our character table looks like the following:

	(1)	(6)	(8)	(6)	(3)
	1	(12)	(123)	(1234)	(12)(34)
U	1	1	1	1	1
U'	1	-1	1	-1	1
V	3	1	0	-1	-1

However, we are not finished since the sum of the squares of the dimensions of U, U', and V is $1 + 1 + 9 = 11$, but by the orthogonality relations, the sum of the squares of all of the irreducible representations must add up to 24. So there must be more irreducible representations of S_4 such that the squares of the dimensions add up to $24 - 11 = 13$. Since the number of irreducible representations of the symmetric group is equal to the number of its conjugacy classes, this tells us that we have two more irreducible representations left to find in our character table. Once again, by the orthogonality relations, one will have dimension 2 and the other will have dimension 3. If we take the tensor of V and U', we get a 3 dimensional representation V' where $\chi_{V'} = \chi_V \cdot \chi_{U'} = (3, -1, 01, -1)$, which is irreducible since $|\chi_{V'}| = 1$. Finally, we can find our final representation, W, by using the orthogonality relations. So our completed character table of S_4 is

	(1)	(6)	(8)	(6)	(3)
	1	(12)	(123)	(1234)	(12)(34)
U	1	1	1	1	1
U'	1	-1	1	-1	1
V	3	1	0	-1	-1
V'	3	-1	0	1	-1
W	2	0	-1	0	2

Character tables will become very important in sections 4 and 5.

3 Reductions

The goal of this section is to decompose our Gelfand pairs, and state the propositions from [BR18] that will simplify the process of finding cracking points of Gelfand pairs. Detailed examples of this will be given in section 5 if the reader wishes to read that section first.

We know from [BR18], that if either (G_n, S_n) or (Γ^n, Δ_n) is a Gelfand pair, then so is (G_n, K_n). Furthermore, (G_n, K_n) is always a Gelfand pair for $n = 2$. Next, the decomposition of $L(\Gamma^n/\Delta_n)$ is shown in [BR18].

Let $\widehat{\Gamma^n}$ denote the set of equivalence classes of irreducible representations of Γ^n. For a representation $\pi_1, \ldots, \pi_n \in \widehat{\Gamma}$, let $m(\pi_1, \ldots, \pi_{n-1}|\pi_n)$ denote the multiplicity of π_n in $\pi_1 \otimes \cdots \otimes \pi_{n-1}$, that is, how many times π_n appears in the direct sum decomposition of $\pi_1 \otimes \cdots \otimes \pi_{n-1}$ into its irreducible factors. Since (Γ^n, Δ_n) is a Gelfand pair if and only if $L(\Gamma^n/\Delta_n)$ is multiplicity free, this proves the following.

Proposition 3.1. *[BR18, §3 Prop. 3.1]* (Γ^n, Δ_n) *is a Gelfand pair if and only if the interior tensor product representation* $\pi_1 \otimes \cdots \otimes \pi_{n-1}$ *is multiplicity free for all irreducible representations* $\pi_1, \ldots, \pi_{n-1} \in \widehat{\Gamma}$.

Next, the decomposition of $L(G_n/K_n)$ is shown in [BR18] as well. Let π be an irreducible representation of Γ^n, where $\pi = \pi_1 \hat{\otimes} \cdots \hat{\otimes} \pi_n$ is the external tensor product of irreducible representations $(\pi_j, V_j) \in \widehat{\Gamma}$. The stabilizer of π in S_n is

$$S_\pi = \{\sigma \in S_n : \pi_{\sigma(j)} = \pi_j \text{ for } j = 1, \ldots, n\}$$

which acts on $V_1 \otimes \cdots V_n$ via the intertwining representation

$$\omega : S_\pi \to GL(V_1 \otimes \cdots \otimes V_n), \quad \omega(\sigma)(v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}.$$

Now, given any $\rho \in \widehat{S_\pi}$, the induced representation

$$R_{\pi,\rho} = \mathrm{Ind}_{\Gamma^n \rtimes S_\pi}^{G_n}((\pi \circ \omega) \hat{\otimes} \rho)$$

is irreducible and every irreducible representation of G_n is of this form.

Remember from Definition 1.2 that $(G_n, K_n) = (\Gamma^n \rtimes S_n, \Delta_n \times S_n)$ is a Gelfand pair if and only if the space R^{K_n} of K_n-fixed vectors in R has $\dim(R^{K_n}) \leq 1$ for every irreducible representation $R \in \hat{G}_n$. Let $K_\pi = \Delta_n \times S_\pi$. Then we can state the following.

Lemma 3.2. *[BR18, §3 Lemma. 3.3] The number of* K_n*-fixed vectors in* $R_{\pi,\rho}$ *is equal to the number of* K_π*-fixed vectors in* $(\pi \circ \omega)\hat{\otimes}\rho$.

Now that we know the required propositions, we can walk through the general method of how these decompositions help us decided whether or not (G_n, K_n) is a Gelfand pair for some n. In section 5 we will show a detailed example of this.

To begin our decomposition, recall that if (Γ^n, Δ_n) is a Gelfand pair, then so is (G_n, K_n). This reduction is useful because (Γ^n, Δ_n) is a much simpler pair to consider than (G_n, K_n). From Definition 1.1, we know that (Γ^n, Δ_n) is a Gelfand pair if $\mathrm{Ind}_{\Delta_n}^{\Gamma^n}(triv_{\Delta_n})$ in $L(\Gamma^n/\Delta_n)$ is multiplicity free. So we must find all of the representations which occur in $L(\Gamma^n/\Delta_n)$, that is, the spherical representations. If there is no multiplicity, then (Γ^n, Δ_n) is a Gelfand pair, and therefore so is (G_n, K_n). However, if there is multiplicity, we must try a different reduction of (G_n, K_n).

Recall that if (G_n, S_n) is a Gelfand pair, then so is (G_n, K_n). From Definition 1.2 we know that (G_n, S_n) is a Gelfand pair if for each irreducible representation $(\pi, R_{\pi,\rho})$ of G_n, the space R^{K_n} of K_n-fixed vectors in $R_{\pi,\rho}$ has dimension $\dim(R^{K_n}) \leq 1$. (Abusing terminology we call $\dim(R^{K_n})$ the "number of K_n-fixed vectors in R". But remember from Lemma 3.2 that the number of K_n-fixed vectors in $R_{\pi,\rho}$ is equal to the number of K_π-fixed vectors in

$(\pi \circ \omega)\hat{\otimes}\rho$. So then we just need to check if (G_n, K_π) is a Gelfand pair, which once again is much simpler than (G_n, K_n).

To begin, let $\pi = \pi_1 \hat{\otimes} \cdots \hat{\otimes} \pi_n$ be a representation of Γ^n, let S_π be the stabilizer of π in S_n, and let ω be the intertwining representation of S_π. Let χ_π be the character for $\pi \circ \omega$. We need to find the spherical representations of the form $R_{\pi,\rho}$ for $\rho \in \widehat{S_\pi}$ which, by Lemma 3.2, amounts to finding the representations which contain K_π-fixed, where $K_\pi = \Delta_n \times S_\pi$. By [BR18], equation (5.1), the number of such vectors is:

$$\frac{1}{|S_\pi|} \sum_\sigma \left(\frac{1}{|\Delta_n|} \sum_\delta \chi_\pi(\delta, \sigma) \right) \chi_\rho(\sigma)$$

where we identify $\delta \in \Gamma$ with an element of Δ_n.
For each fixed $\sigma \in S_\pi$, we define the function

$$m_{(\pi,\sigma)}(\delta) = \chi_\pi(\delta, \sigma)$$

which is a class function on Δ. This $m_{(\pi,\sigma)}(\delta)$ can be expressed as a linear combination of irreducible characters. So define

$$M_\pi(\sigma) = \frac{1}{|\Delta|} \sum_\delta \chi_\pi(\delta, \sigma) = \langle m_{(\pi,\sigma)}(\delta), 1 \rangle_\Delta$$

which is a class function on S_π. Let $\rho_1, \ldots, \rho_\ell$ be the irreducible representations of S_π. Then M_π be expressed uniquely as a linear combination $M_\pi = \sum_{i=1}^{\ell} a_i \chi_{\rho_i}$ of irreducible characters χ_{ρ_i} of S_π, for some a_i. This tells us the multiplicity of the K_π-spherical representations, and therefore, whether or not (G_n, K_n) is a Gelfand pair.

4 Example: Symmetric Groups

In this section, we will illustrate the reduction techniques of section 3 using the example of the symmetric group. Since $\Gamma = S_2$ is abelian, the pair (G_n, K_n) is a Gelfand pair by [AC12]. It was shown in [BR18] that for $\Gamma = D_p$ with p an odd prime, that $N(\Gamma) = 6$. Since $D_3 \cong S_3$, that implies $N(S_3) = 6$.

Example 4.1. $\Gamma = S_4$: We begin by observing the character table of S_4 which we found in section 2:

	(1)	(6)	(8)	(6)	(3)
	1	(12)	(123)	(1234)	(12)(34)
U	1	1	1	1	1
U'	1	-1	1	-1	1
V	3	1	0	-1	-1
V'	3	-1	0	1	-1
W	2	0	-1	0	2

The case $\mathbf{n = 3}$: We will first see whether or not (Γ^3, Δ_3) is a Gelfand pair. Our strategy is to examine the tensor products of representations of Γ, as described in the last section, and we check to see if any of their irreducible subrepresentations have multiplicity greater than one. Anything of the form $U \otimes J$ or $U' \otimes J$ is clearly multiplicity free, so we will explicitly show the less trivial cases:

$$\chi_{(V \otimes V)} = \chi_{(V' \otimes V')} = \begin{pmatrix} 9 & 1 & 0 & 1 & 1 \end{pmatrix} = \chi_U + \chi_V + \chi_{V'} + \chi_W$$

$$\chi_{(V \otimes V')} = \begin{pmatrix} 9 & -1 & 0 & -1 & 1 \end{pmatrix} = \chi_{U'} + \chi_V + \chi_{V'} + \chi_W$$

$$\chi_{(V \otimes W)} = \chi_{(V' \otimes W)} = \begin{pmatrix} 6 & 0 & 0 & 0 & -2 \end{pmatrix} = \chi_V + \chi_{V'}$$

$$\chi_{(W \otimes W)} = \begin{pmatrix} 4 & 0 & 1 & 0 & 4 \end{pmatrix} = \chi_U + \chi_V + \chi_W$$

Since there are no cases of multiplicity, (Γ^3, Δ_3) is a Gelfand pair.

The case $\mathbf{n = 4}$: Using the same strategy, consider the representation $V^{(4)} = V \hat{\otimes} V \hat{\otimes} V \hat{\otimes} V$. Then

$$\chi_{V^{(4)}} = \begin{pmatrix} 81 & 1 & 0 & 1 & 1 \end{pmatrix}$$

Note that we consider this representation because it has the highest dimension. This is important because, in this case, it tells us that $V^{(4)}$ must have multiplicity since there is no way to write $\dim(V) = 81$, as the sum of $\dim(U) = 1$, $\dim(U') = 1$, $\dim(V) = 3$, $\dim(V') = 3$, $\dim(W) = 2$ without multiplicity. Therefore (Γ^4, Δ_4) is not a Gelfand pair. Now we can check the second reduction, that is, if (G_n, K_π) is a Gelfand pair using the techniques of section 3. Also note that to exhaustively check that we still have a Gelfand pair for $n = 4$, we would need to find all cases of multiplicity of the representations of Γ^4 which occur in $L(\Gamma^4, \Delta_4)$ and check whether or not the second reduction holds.

So once again, using the techniques of section 4, let $\pi = V^{(4)}$ be a representation of Γ^4. Here, our stabilizer $S_\pi = S_4$. Then for $\delta \in S_4$, we must compute all $m_{\pi,\sigma}(\delta)$ which are the following:

$$m_{\pi,e}(\delta) = \chi_\pi(\delta, e) = \begin{pmatrix} 81 & 1 & 0 & 1 & 1 \end{pmatrix}$$

$$m_{\pi,(12)}(\delta) = \chi_\pi(\delta^2)\chi_\pi(\delta)^2 = \begin{pmatrix} 3 & 3 & 0 & -1 & 3 \end{pmatrix} \begin{pmatrix} 9 & 1 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 27 & 3 & 0 & -1 & 3 \end{pmatrix}$$

$$m_{\pi,(123)}(\delta) = \chi_\pi(\delta^3)\chi_\pi(\delta) = \begin{pmatrix} 3 & 1 & 3 & -1 & -1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 03 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 9 & 1 & 0 & 1 & 1 \end{pmatrix}$$

$$m_{\pi,(1234)}(\delta) = \chi_\pi(\delta^4) = \begin{pmatrix} 3 & 3 & 0 & 3 & 3 \end{pmatrix}$$

$$m_{\pi,(12)(34)}(\delta) = \chi_\pi(\delta^2)\chi_\pi(\delta^2) = \begin{pmatrix} 3 & 3 & 0 & -1 & 3 \end{pmatrix} \begin{pmatrix} 3 & 3 & 0 & -1 & 3 \end{pmatrix} = \begin{pmatrix} 9 & 9 & 0 & 1 & 9 \end{pmatrix}$$

Next, we compute all $M_\pi(\sigma)$:

$$M_\pi(e) = \langle m_{\pi,e}, 1 \rangle_{S_4} = 4$$

$$M_\pi(12) = \langle m_{\pi,(12)}, 1 \rangle_{S_4} = 2$$

$$M_\pi(123) = \langle m_{\pi,(123)}, 1 \rangle_{S_4} = 1$$

$$M_\pi(1234) = \langle m_{\pi,(1234)}, 1 \rangle_{S_4} = 2$$

$$M_\pi(12)(34) = \langle m_{\pi,(12)(34)}, 1 \rangle_{S_4} = 4$$

Therefore, $M_\pi = \begin{pmatrix} 4 & 2 & 1 & 2 & 4 \end{pmatrix}$. Since $\langle M_\pi, \rho_\circ \rangle = 2$, (where $\rho_\circ$ is the trivial representation of S_π), then $R_{\pi,\rho_\circ} = \mathrm{Ind}_{\Gamma^4 \rtimes S_\pi}^{G_4}((\pi \circ \omega) \hat{\otimes} \rho_\circ)$ has multiplicity 2 in $L(G_4/K_4)$. Hence, the cracking point is $N(S_4) = 4$.

Example 4.2. $\Gamma = S_5$

The case $\mathbf{n = 3}$: Notice that we only needed our highest dimensional representation in computing the cracking point in our last example. So consider the partial character table of S_5 with the two highest dimensional representations:

	(1)	(10)	(15)	(20)	(20)	(24)	(30)
	I	2	2, 2	3	3, 2	5	4
W	5	1	1	−1	1	0	1
Q	6	0	−2	0	0	1	0

Then $\chi_{(Q\otimes Q)} = \begin{pmatrix} 36 & 0 & 4 & 0 & 0 & 1 & 0 \end{pmatrix}$. By computing $\langle \chi_{(Q\otimes Q)}, \chi_W \rangle = 2$, we see that W has multiplicity 2. Since there is multiplicity, (Γ^3, Δ_3) is not Gelfand pair. Let $\pi = Q\hat{\otimes}Q\hat{\otimes}W$ be a representation of Γ^3 and $S_\pi = S_2 \times S_1$. Then,

$$m_{\pi,e}(\delta) = \chi_Q(\delta)^2 \cdot \chi_W(\delta) = 180 \quad 0 \quad 4 \quad 0 \quad 0 \quad 0 \quad 0$$

$$m_{\pi,(1\,2)}(\delta) = \chi_Q(\delta^2) \cdot \chi_W(\delta) = 30 \quad 6 \quad 6 \quad 0 \quad 0 \quad 0 \quad 2$$

$$M_\pi(e) = \langle m_{(e)}(\delta), 1 \rangle = \frac{1}{120}(180 \cdot 1 + 4 \cdot 15) = 2$$

$$M_\pi(1\,2) = \langle m_{(1\,2)}(\delta), 1 \rangle = \frac{1}{120}(30 \cdot 1 + 6 \cdot 10 + 6 \cdot 15 + 2 \cdot 30) = 2$$

One can see that $M_\pi = 2\chi_{\rho_\circ}$, and hence $R_{\pi,\rho_\circ} = \mathrm{Ind}_{\Gamma^3 \rtimes S_\pi}^{G_3}((\pi \circ \omega)\hat{\otimes}\rho_\circ)$ has multiplicity 2 in $L(G_3/K_3)$. Therefore (G_3, K_3) is not a Gelfand Pair, and thus we have the cracking point $N(S_5) = 3$.

We continued computing the cracking point of S_k up to $k = 7$, and found that $N(S_k) = 3$ for $5 \leq k \leq 7$. This led us to believe the following.

Conjecture 4.3. *For $\Gamma = S_k$, the cracking point $N(S_k) = 3$ for all $k \geq 5$.*

We have yet to prove this, but we have proved the following.

Proposition 4.4.

Proof. Let $\Gamma = S_k$, and $\{\pi_i\}_i \in \widehat{\Gamma}$. Then let $\pi = \pi_1^{(1)} \in \widehat{\Gamma^4}$. Then

$$M_\pi(e) = \langle m_e(\delta), \mathrm{triv}_\Gamma \rangle_\Gamma = \frac{1}{k!}\bigg(\dim(V_\pi)^4 + \sum(\text{positive terms}) \bigg)$$

where we know that the terms in the sum are positive since they result from raising the values of χ_π to an even power. Since the stabilizer for π is $S_\pi = S_4$, by Proposition 5.1, it is enough to show that $M_\pi(e) > \sum_{\rho \in \hat{S_4}} \dim(V_\rho) = 10$. Thus, we will show that $\frac{1}{k!}\dim(V_\pi)^4 > 10$ for some $\pi = \pi_1^{(4)}$. That is, we will show that this holds when π_1 is the highest dimensional irreducible representation of S_k.

Suppose π_1 is such a representation of S_k. It was proven in [VK85] that

$$e^{-\frac{c}{2}\sqrt{k}}\sqrt{k!} < \dim(V_{\pi_1}).$$

where $c = 2.5651$ and, in particular, does not depend on k. So we want to show that

$$\frac{1}{k!}\dim\left(V_\pi\right)^4 > e^{-2c\sqrt{k}}k! > 10.$$

A direct calculation shows that this is true for $k = 13$. Moreover, $e^{-2c\sqrt{k}}k!$ is increasing since

$$\frac{e^{-2c\sqrt{k+1}}(k+1)!}{e^{-2c\sqrt{k}}k!} = e^{-2c(\sqrt{k+1}-\sqrt{k})}(k+1).$$

Evaluating this for $k = 13$, and using the fact that $\sqrt{k+1} - \sqrt{k}$ is decreasing, we see that $e^{-2c(\sqrt{k+1}-\sqrt{k})} > .49$ for all $k \geq 13$. Therefore, the ratio of consecutive terms is greater than 1. Thus, this proves the result for $k \geq 13$. The remaining $5 \leq k \leq 12$ can be directly calculated, and the following table contains those results.

k	$\dim(V_\pi)$	$\frac{1}{k!}\dim(V_\pi)^4$
5	6	10.8
6	16	$91.0\bar{2}$
7	35	297.74...
8	90	1627.232...
9	216	5998.62...
10	768	95869.80...
11	2310	713332.81...
12	7700	7338814.94...

□

The computations of the symmetric group led us to the following proposition.

5 A New Reduction

As we computed the examples in the previous section, we noticed that the cracking points seemed to depend more on the dimensions of the irreducible representations of the stabilizers. This observation evolved into the following proposition.

Proposition 5.1. *Let $\pi = \pi_1\hat{\otimes}\cdots\hat{\otimes}\pi_k$ be a representation of Γ^k, and let S_π be its stabilizer. Let $\rho_1, \ldots, \rho_\ell$ be the irreducible representations of S_π. If*

$$M_\pi(e) > \sum_{i=1}^{\ell} dim(\rho_i),$$

then in the decomposition $M_\pi = \sum_{i=1}^{\ell} a_i\chi_{\rho_i}$ of the class function M_π into irreducible characters of S_π, there exists some integer coefficient $a_i > 1$. In particular, in this case, (G_k, K_k) is not a Gelfand pair.

Proof. We know that M_π is a class function on S_π, which can be expressed uniquely as a linear combination $M_\pi = \sum_{i=1}^{\ell} a_i \chi_{\rho_i}$ of irreducible characters χ_{ρ_i} of S_π, for some a_i. Since $a_i = \langle M_\pi, \chi_{\rho_i} \rangle$ counts the number of K_π-fixed vectors in $R_{\pi,\rho}$, then $a_i \in \mathbb{Z}$ for all $1 \leq i \leq \ell$. Therefore, if

$$M_\pi(e) > \sum_{i=1}^{\ell} \dim(\rho_i),$$

there must be some $1 \leq i \leq \ell$ such that $a_i > 1$. Hence, (G_k, K_k) is not a Gelfand pair. □

Then k is an upper bound for cracking points of an arbitrary group Γ in which we can find a representation of the form $\pi = \pi_1 \hat{\otimes} \cdots \hat{\otimes} \pi_k$ of Γ^k. This k depends on the dimensions of irreducible representations of stabilizers, and in some cases, may be a smaller upper bound than the order of the group. For instance, this is true in the case of the symmetric group S_k.

6 Conclusion

When heading toward future directions, we hope to prove the following conjecture that arose from our symmetric group computations in section 4.

Conjecture 6.1. *For $\Gamma = S_k$, the cracking point $N(\Gamma) = 3$ for all $k \geq 5$.*

Following from Proposition 5.1, we know that if we have a representation of S_k^3 of the form $\pi = \pi_1 \hat{\otimes} \pi_1 \hat{\otimes} \pi_2$ with the property that $M_\pi(e) \geq 3$, that (G_3, K_3) is not a Gelfand pair. At first glance this seems practicable to prove since the stabilizer S_π will always equal S_2. However, the situation becomes more complicated because it is not the case that we know all of the terms in $M_\pi(e)$ are positive as we knew in Proposition 4.3. So there is still work to be done on this conjecture.

Furthermore, we hope to still answer the question of whether or not $N(\Gamma)$ can be arbitrarily large. It may be helpful to compute examples of more finite groups such as groups of Lie type and Coxeter groups.

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